Development of Discrete Mechanics for Distributed Parameter Mechanical Systems and Its Application to Vibration Suppression Control of a String

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Abstract: In this study, a new stabilization method by blending discrete mechanics and nonlinear optimization for 1-dimensional distributed parameter mechanical systems is developed. Discrete mechanics is a kind of numerical solutions for distributed parameter mechanical systems and it is known that it has some advantages in terms of numerical errors and preserving property of the original systems. First, for discrete Euler-Lagrange equations with control inputs, we formulate a nonlinear optimal control problem with constraints by setting an objective function, and initial and boundary conditions. Then, it is shown that the problem is represented as a finite-dimensional nonlinear optimal problem with constraints and it can be solved by the sequential quadratic programming method. After that, a vibration suppression control problem for a string is dealt with as a physical example. As a result, it can be confirmed that vibration of the string is suppressed and the whole of the system is stabilized by the proposed new method.

1 INTRODUCTION

In general, when we control a given system, we first derive its mathematical model represented by continuous-time differential equations. Next, we analyze the features of the model and then design a continuous-time controller which can achieve a given control purpose. Since computers deal with only digital signals, we have to consider "discretization" of the mathematical model or the controller for the use of computers. However, the discretization process causes various problems such as loss of properties of the original continuous-time model and controller, debasement of control performances, and destabilization of the system. Therefore, for controller design and synthesis with computers, we have to think a great deal of the relationship between continuous and discrete signals.

During recent years, for concentrated constant systems, a new discretizing method called "discrete mechanics" has been developed (Marsden et al., 1998; Kane et al., 2000; Marsden and West, 2001; Junge et al., 2005). In discrete mechanics, first, some fundamental concepts and principles such as Lagrangians, Hamiltonians, Hamilton's principle, and Lagrange-d'Alembert's principle are discretized, and then discrete equations of motion for systems are derive and called "discrete Euler-Lagrange equations." It is known that discrete mechanics has some remarkable advantages in comparison with other methods, and thus it has great potential as a powerful numerical solution. The authors have researched applications of discrete mechanics to control theory and derived some results, for example, swing-up control of the cart-pendulum system (Kai, 2012; Kai et al., 2012; Kai and Shintani, 2014), and stable gait generation and obstacle avoidance control for biped robots (Kai and Shintani, 2011; Kai and Shibata, 2015; Kai, 2015). It is expected that discrete mechanics has application potentiality to not concentrated constant systems but distributed parameter systems.

In this study, discrete mechanics for 1-dimensional distributed parameter mechanical systems is developed and its application to control theory is considered. First, Section 2 describes details on discrete mechanics for distributed parameter mechanical systems. Next, Section 3 shows a new control method based on discrete mechanics and nonlinear optimization. Then, in Section 4, we treat the vibration suppression control of a string as a

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physical example, and some numerical simulations are shown in order to confirm the new method.

2 DISCRETE MECHANICS FOR DISTRIBUTED PARAMETER MECHANICAL SYSTEMS

In this section, discrete mechanics for 1-dimensional distributed parameter mechanical systems are presented.

Let us denote the time variable as $t \in \mathrm{R}$ and in the position of the 1-dimensional space as $x \in \mathrm{R}$. We also refer a displacement of the system at the time t and the position x as $u(t,x) \in \mathrm{R}$, and $u(t,x)$ with a subscript indicates partial derivative of $u(t,x)$ with respect to the subscript, e.g. u_t, u_x, u_{tt}, u_{tx}, u_{xx}. In this paper, we deal with a continuous Lagrangian density which includes through second-order partial derivative of $u(t,x)$ as

$$L^c(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}) \tag{1}$$

Next, we consider discretization of variables. As shown in Fig. , the time variable t and the position x are discretized with sampling intervals h and d as as

$$\begin{aligned} t &\approx hk \ \ (k = 1,2,\cdots,K-1,K), \\ x &\approx dl \ \ (l = 1,2,\cdots,L-1,L), \end{aligned} \tag{2}$$

respectively, where $k \in \mathrm{Z}$ $(1 \leq k \leq K)$ and $l \in \mathrm{Z}$ $(1 \leq l \leq L)$ are indices of t and x, respectively.

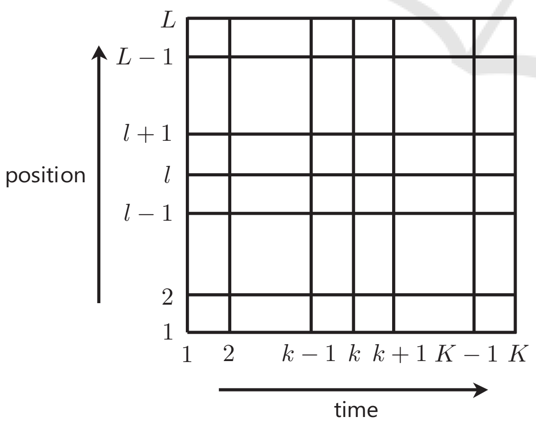

Figure 1: Discretization of Time and Position.

Now, we use a new notation $U_{k,l} \in \mathrm{R}$ as a discrete version of the displacement of the system at the time step k and the position l. Then, as shown in Fig. 2, the displacement of the system at the time t and the position x: $u(t,x)$ is represented as

$$\begin{aligned} u(t,x) \approx{} & (1-\alpha)(1-\beta)U_{k,l} + (1-\alpha)\beta U_{k,l+1} \\ & + \alpha(1-\beta)U_{k+1,l} + \alpha\beta U_{k+1,l+1} \end{aligned} \tag{3}$$

with four data $U_{k,l}$, $U_{k,l+1}$, $U_{k+1,l}$, $U_{k+1,l+1}$, where α, $\beta \in \mathrm{R}$ are dividing parameters $(0 < \alpha,\ \beta < 1)$. Partial derivatives of $u(t,x)$ are also represented by

$$\begin{aligned} u_t(t,x) &\approx \frac{U_{k+1,l} - U_{k,l}}{h}, \\ u_x(t,x) &\approx \frac{U_{k,l+1} - U_{k,l}}{d}, \\ u_{tt}(t,x) &\approx \frac{U_{k+1,l} - 2U_{k+1,l} + U_{k-1,l}}{h^2}, \\ u_{tx}(t,x) &\approx \frac{U_{k+1,l+1} - U_{k,l+1} - U_{k+1,l} + U_{k,l}}{hd}, \\ u_{xx}(t,x) &\approx \frac{U_{k,l+1} - 2U_{k,l} + U_{k,l-1}}{d^2}. \end{aligned} \tag{4}$$

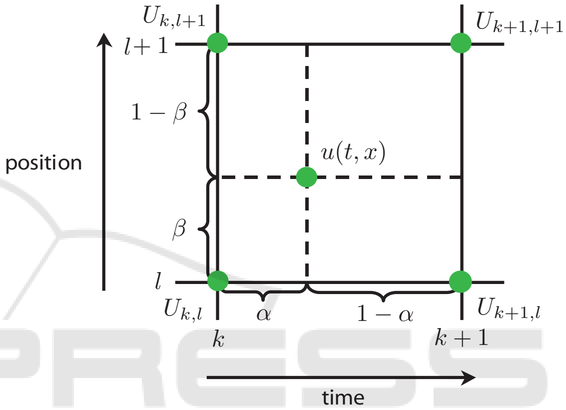

Figure 2: Discretization of $u(t,x)$.

By substituting (2)–(4) into (1) and multiplying it by hd, we here define "a discrete Lagrangian density" as

$$L^d_{k,l} = L^d(k,l,U_{k-1,l},U_{k,l-1},U_{k,l},U_{k,l+1},U_{k+1,l},U_{k+1,l+1}). \tag{5}$$

We also define "a discrete action sum" as

$$S^d(U) := \sum_{k=2}^{K-1}\sum_{l=2}^{L-1} L^d_{k,l}, \tag{6}$$

and consider "a discrete variation" as

$$\delta S^d(U) := S^d(U + \delta U) - S^d(U), \tag{7}$$

where δU is a variation of U and satisfies the boundary conditions:

$$\begin{aligned} \delta U_{1,l} = \delta U_{2,l} = \delta U_{K-1,l} = \delta U_{K,l} = 0, \\ \delta U_{k,1} = \delta U_{k,2} = \delta U_{k,L-1} = \delta U_{k,L} = 0. \\ (k = 1,\cdots,K;\ l = 1,\cdots,L) \end{aligned} \tag{8}$$

As a analogy of Hamilton's principle in the continuous-time version, we consider "discrete

Hamilton's principle" and it states that "only a motion such that the discrete action sum (6) is stationary, that is, $S^d(U) = 0$, can be realized." By applying discrete Hamilton's principle to the discrete action sum (6), and calculating in details, we can derive "discrete Euler-Lagrange equations" as the following (due to limitations of space, the proof is omitted).

Theorem 1 : For the discrete Lagrangian density $L^d_{k,l}$ (5), the discrete Euler-Lagrange equations that satisfy discrete Hamilton's principle is given by

$$\frac{\partial L^d_{k-1,l-1}}{\partial U_{k,l}} + \frac{\partial L^d_{k-1,l}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l-1}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l+1}}{\partial U_{k,l}} + \frac{\partial L^d_{k+1,l}}{\partial U_{k,l}} = 0. \quad (k = 3,4,\cdots,K-2;\ l = 3,4,\cdots,L-2) \tag{9}$$

$\square$

It is noted that the discrete Euler-Lagrange equations (9) are represented as a set of difference equations that include 17 variables: $U_{k-2,l-1}$, $U_{k-2,l}$, $U_{k-1,l-2}$, $U_{k-1,l-1}$, $U_{k-1,l}$, $U_{k-1,l+1}$, $U_{k,l-2}$, $U_{k,l-1}$, $U_{k,l+1}$, $U_{k,l+1}$, $U_{k,l+2}$, $U_{k+1,l-1}$, $U_{k+1,l}$, $U_{k+1,l+1}$, $U_{k+1,l+2}$, $U_{k+2,l}$, $U_{k+2,l+1}$ as shown in Fig. 3, and we calculate all the KL displacements $U_{k,l}$ $(1 \leq k \leq K;\ 1 \leq l \leq L)$ by using the discrete Euler-Lagrange equations (9) under suitable initial and boundary conditions. In addition, the discrete Euler-Lagrange equations (9) are generally nonlinear and implicit, and hence we need some numerical solutions for nonlinear equations such as Newton's method in order to calculate all the displacements of the system.

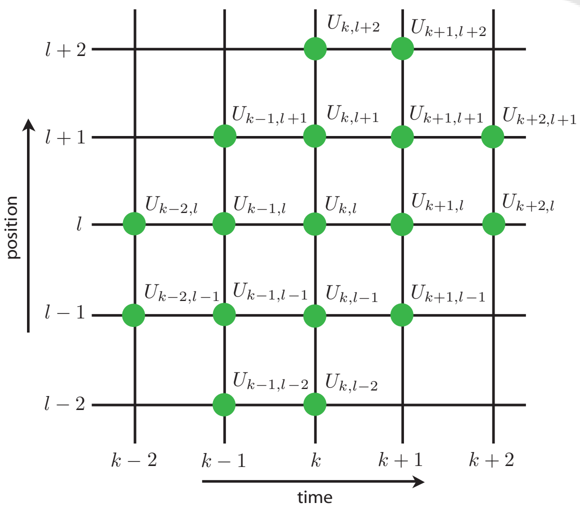

Figure 3: Discrete Euler-Lagrange Equation.

3 OPTIMAL CONTROL PROBLEM FOR DISCRETE MECHANICS MODEL

In this section, a nonlinear control problem for a mathematical model derived by discrete mechanics is formulated, and a solution method of the problem is considered. First, the setting on control inputs is shown. Denote a control input at the time step k and the position l as $F_{k,l} \in \mathrm{R}$. If an actuator is not installed at the position l, we set $F_{k,l} == 0$ $(k = 1,\cdots,K)$. We also denote and a set of indices l such that actuators are installed as Δ. Thus, the discrete discrete Euler-Lagrange equations with control inputs are given by

$$\frac{\partial L^d_{k-1,l-1}}{\partial U_{k,l}} + \frac{\partial L^d_{k-1,l}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l-1}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l}}{\partial U_{k,l}} + \frac{\partial L^d_{k,l+1}}{\partial U_{k,l}} + \frac{\partial L^d_{k+1,l}}{\partial U_{k,l}} = F_{k,l}. \quad (k = 3,4,\cdots,K-2;\ l = 3,4,\cdots,L-2) \tag{10}$$

In this study, the next control problem is dealt with for the discrete discrete Euler-Lagrange equations with control inputs (10)

Problem 1: For the discrete Lagrangian density (5) and the discrete Euler-Lagrange equation with control inputs (10), find control inputs $F_{k,l}$ $(k = 2,\cdots,K-1;\ l \in \Delta)$ that make all the specified displacements $U_{k,l}$ $(k = \kappa,\cdots,K;\ l = 1,\cdots,L)$ converge to 0. $\square$

In order to solve Problem 1, we consider an optimal control approach. Using weight parameters a, b, c, we set an evaluation function as

$$J(U,F) = a\sum_{k=1}^{\kappa-1}\sum_{l=1}^{L} U_{k,l}^2 + b\sum_{k=\kappa}^{K}\sum_{l=1}^{L} U_{k,l}^2 + c\sum_{k=3}^{K-2}\sum_{l\in\Delta} F_{k,l}^2, \tag{11}$$

where the first and second terms evaluate the displacements from $k = 1$ to $k = \kappa - 1$ and ones from $k = \kappa$ to $k = K$, respectively, and the third term evaluates the values of control inputs. It can be expect that we can make all the specified displacements converge to 0. by minimizing the evaluation function (11). The optimal control problem for the discrete Euler-Lagrange equation with control inputs (10) can be formulated as

$$\begin{aligned} &\min_{U,F}\ (11), \\ &\text{subject to } (10), \\ &\text{given initial conditions, boundary conditions.} \end{aligned} \tag{12}$$

The optimal control problem (12) can be referred as a finite-dimensional nonlinear optimization problem with constraints, and hence we can solve it by numerical solutions such as "the sequential quadratic programming method" (Nocedal and Wright, 2006; Gurwitz, 2015). It is known that the sequential quadratic programming method can be applied to a relatively large-scale problems and effectively obtain an optimal or near-optimal solution.

4 APPLICATION TO VIBRATION SUPPRESSION CONTROL OF STRING

4.1 Problem Setting

This section treats an application to a physical system: a string and confirms the proposed control method via numerical simulations.

We deal with a string clamped at both ends as illustrated in Fig. 4. Denote the position of the string as x and the displacement of the string at time t and the position x as $u(t,x)$. physical parameters of the string are set as ρ: a energy density of the string, N: tension of the string. Then, the continuous Lagrangian density of the string is given by

$$L^c = \frac{1}{2}\rho u_t^2 - \frac{1}{2}N u_x^2. \tag{13}$$

Note that the continuous Lagrangian density (13) contains through first-order partial derivative u_t, u_x.

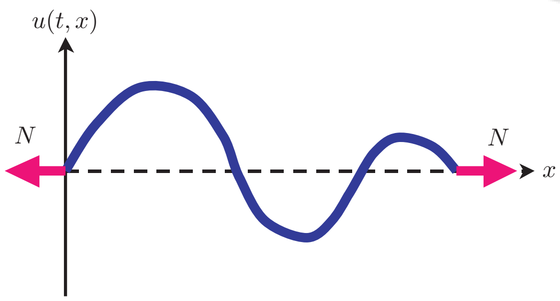

Figure 4: String.

Discretization setting are the same as the one in the previous section. From (13), we have the discrete Lagrangian density:

$$L^d_{k,l} = \frac{hd}{2}\left\{\rho\left(\frac{U_{k+1,l}-U_{k,l}}{h}\right)^2 - N\left(\frac{U_{k,l+1}-U_{k,l}}{d}\right)^2\right\}, \tag{14}$$

and hence from (9) we obtain the discrete Euler-Lagrange equation of the string as

$$\begin{aligned} -\frac{\rho d}{h}U_{k-1,l} + \frac{Nh}{d}U_{k,l-1} \\ + \left(\frac{2\rho d}{h} - \frac{2Nh}{d}\right)U_{k,l} \\ + \frac{Nh}{d}U_{k,l+1} - \frac{\rho d}{h}U_{k+1,l} = 0. \end{aligned} \tag{15}$$

We see that (15) contains 5 displacement variables $U_{k-1,l}$, $U_{k,l-1}$, $U_{k,l}$, $U_{k,l+1}$, $U_{k,l+1}$ as depicted in Fig. 5.

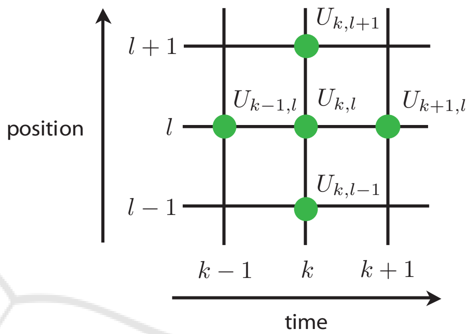

Figure 5: Discrete String Model.

Now, we shall investigate numerical stability of (15). In computation of partial differential equations by computers with numerical solutions, the concept "numerical stability" is quite essential. Let us denote a solution $u(t,x)$ of the distributed parameter mechanical system in the complex form:

$$u(t,x) = u(t)e^{imx}, \tag{16}$$

where m is the number of waves and $i = \sqrt{-1}$ is the imaginary unit. That is to say, (16) shows a wave whose amplitude is $u(t)$ and wave number is m. Discretizing (16), we have

$$U_{k,l} = U_k e^{imld}. \tag{17}$$

If the amplitude U_k is intensifying over time, it becomes numerically instable. A numerical stability condition focused on amplification degrees is called "a von Neumann condition" (Thomas, 1998; Quarteroni and Valli, 2008). The next proposition gives a von Neumann condition for the discrete Euler-Lagrange equation of the string (due to limitations of space, the proof is omitted).

Proposition 1: A von Neumann condition such that the discrete Euler-Lagrange equation of the string (15) is numerically stable is given by

$$0 < \frac{N}{\rho}\frac{h^2}{d^2} \leq 1. \tag{18}$$

□

From the result of Proposition 1, It turns out that for given a string with physical parameters ρ, N, we set sampling intervals h, d such that (18) satisfies, then numerical stability is guaranteed. In next subsections, some numerical simulations will be performed, and physical parameters are set as $\rho = 0.1$, $N = 1$. For this setting, we set sampling intervals as $h = 0.01$, $d = 0.1$, and these parameters satisfies the von Neumann condition (18):

$$\frac{N}{\rho}\frac{h^2}{d^2} = 0.1. \tag{19}$$

In addition, we consider initial conditions of the string as a sine curve:

$$\begin{aligned} U_{1,l} &= \sin\left(3\pi\frac{l-1}{L-1}\right) \quad (l = 1,\cdots,L), \\ U_{2,l} &= 0.99\sin\left(3\pi\frac{l-1}{L-1}\right) \quad (l = 1,\cdots,L), \end{aligned} \tag{20}$$

and boundary conditions on clamp at both ends:

$$U_{k,1} = U_{k,L} = 0 \quad (k = 1,\cdots,K). \tag{21}$$

Therefore, for given initial and boundary conditions (20), (21), all the displacement are calculated by the discrete Euler-Lagrange equation of the string (15) as shown in Fig. 6.

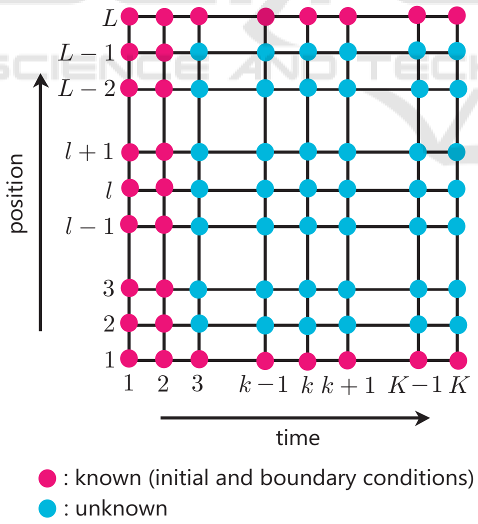

Figure 6: Calculation of Discrete String Model.

In Fig.7, free vibration of the string with out control inputs is illustrated with $K = 200$, $L = 50$. From this figure, we can see that the string is periodically vibrating with the maximum amplitude 1 and this is consistent with actual behavior of the string.

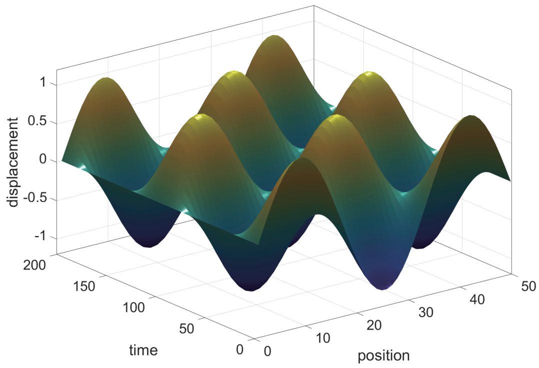

Figure 7: Free Vibration of String with No Control.

4.2 Simulation I

In this subsection, a numerical simulation is carried out by the proposed control method in order to check the effectiveness. We now assume that the number of control inputs is 1, that is to say, the actuator that can generate a control input is installed at only the extreme left of the string as illustrated in Fig. 8. Parameters are set as steps: $K = 400$, $L = 50$, the set of actuator indices: $\Delta = \{2\}$, the start time step of stabilization: $\kappa = 300$, the weight parameters of evaluation function: $a = 1$, $b = 1000$, $c = 1$.

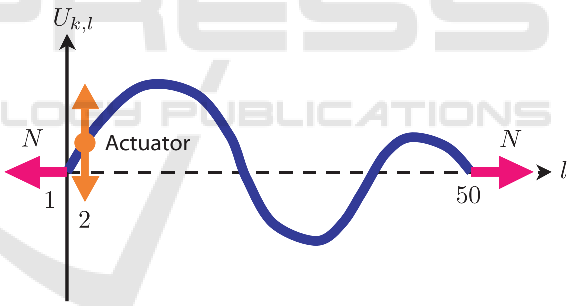

Figure 8: Setting of Simulation I.

Figs. 9 and 10 shows simulation results. Fig. 9 shows a 3D plot of the displacements of the string $U_{k,l}$, and fig. 10 illustrates a time series on average of the absolute value of $U_{k,l}$:

$$\frac{1}{L}\sum_{l=1}^{L}|U_{k,l}|. \tag{22}$$

From these results, it can be confirmed that all the displacements of the string in the desired time step $k = 300-400$ converge to 0, and hence vibration suppression control is achieved. However, since the number of control inputs is 1, if the start time of stabilization is set as a smaller one, the value of the control input rises and the control performance is degraded.

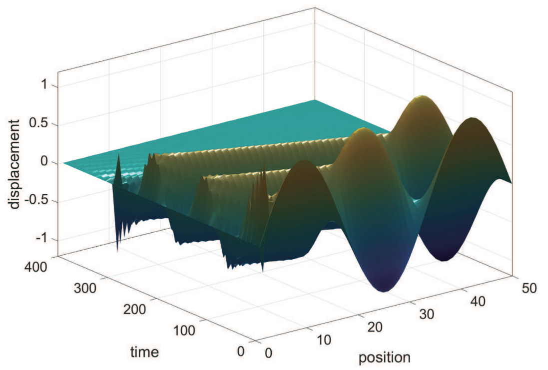

Figure 9: 3D Plot for Displacement of String (Simulation I).

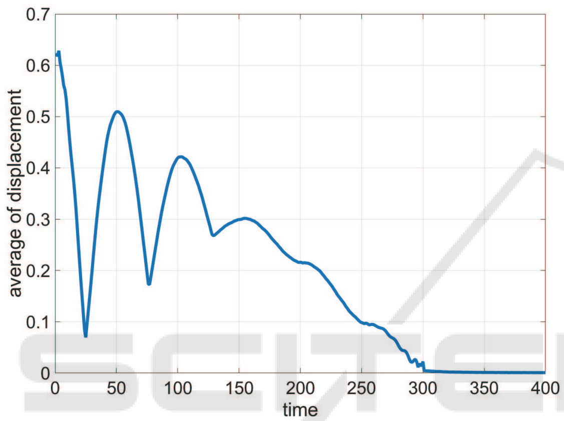

Figure 10: Time Series for Average of Displacement (Simulation I).

4.3 Simulation II

Next, this subsection shows another numerical simulation. It is assumed that the number of control inputs is 2, that is to say, the actuator that can generate a control input is installed at both ends of the string as illustrated in Fig. 11. Parameters are set as steps: $K = 300$, $L = 50$, the set of actuator indices: $\Delta = \{2, 49\}$, the start time step of stabilization: $\kappa = 200$, the weight parameters of evaluation function: $a = 1$, $b = 1000$, $c = 1$.

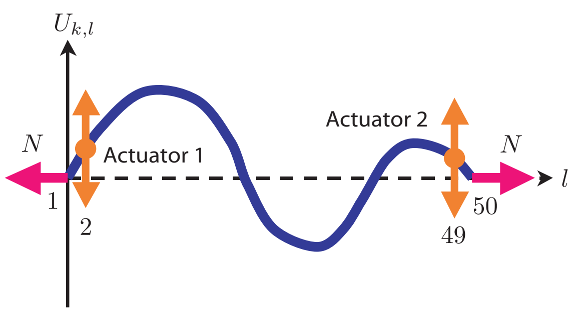

Figure 11: Setting of Simulation II.

In Figs. 12 and 13, simulation results are depicted. Fig. 12 illustrates a 3D plot of the displacements of the string $U_{k,l}$, and fig. 13 shows a time series on average of the absolute value of $U_{k,l}$ (22). From these results, we can see that all the displacements of the string converge to 0 in the desired time step $k = 200 - 300$, and hence vibration suppression control is achieved in common with Simulation I. Moreover, since the number of control inputs is 2, the system is stabilized at earlier time step ($k = 200$) than the one in Simulation I ($k = 300$). It is also confirmed that by setting the weight parameter of the evaluation function b as a larger value, stabilization can be started at earlier time step.

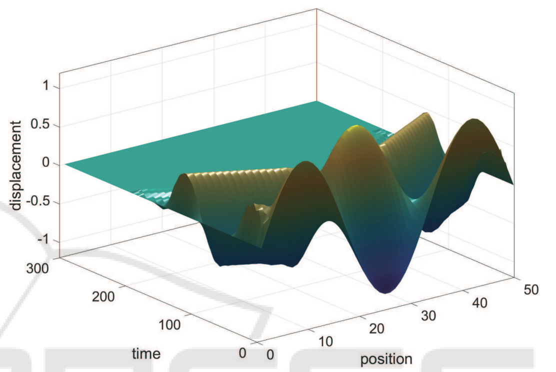

Figure 12: 3D Plot for Displacement of String (Simulation II).

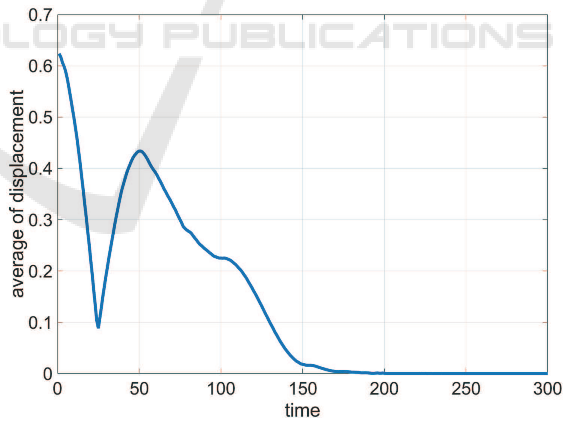

Figure 13: Time Series for Average of Displacement (Simulation II).

5 CONCLUSIONS

In this study, discrete mechanics for 1-dimensional distributed parameter mechanical systems has been developed and its application to control theory has been considered. Vibration suppression control of a string as an example of physical systems has been also shown in order to verify the effectiveness of the proposed method. As a result, this study derives a new control approach to distributed parameter mechanical systems.

The future work includes the following topics; theoretical analysis on discrete Euler-Lagrange equations, Extension to 2-dimensional distributed parameter mechanical systems, and development of feedback-type controllers.

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