Threshold Concepts Vs. Tricky Topics
Exploring the Causes of Student´s Misunderstandings with the Problem Distiller Tool

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Abstract: This paper presents a study developed within the international project JuxtaLearn. This project aims to improve student understanding of threshold concepts by promoting student curiosity and creativity through video creation. The math concept of 'Division', widely referred in the literature as problematic for students, was recognised as a 'Tricky Topic' by teachers with the support of the Tricky Topic Tool and the Problem Distiller tool, two apps developed under the JuxtaLearn project. The methodology was based on qualitative data collected through Think Aloud protocol from a group of teachers of a public Elementary school as they used these tools. Results show that the Problem Distiller tool fostered the teachers to reflect more deeply on the causes of the students' misunderstandings of that complex math concept. This process enabled them to develop appropriate strategies to help the students overcome these misunderstandings. The results also suggest that the stumbling blocks associated to the Tricky Topic 'Division' are similar to the difficulties reported in the literature describing Threshold Concepts. This conclusion is the key issue discussed in this paper and a contribution to the state of the art.

1 INTRODUCTION

This paper presents a study conducted in the scope of the JuxtaLearn project. This European project focuses on helping students understanding 'threshold concepts' in science and technology with the help of technological tools and collaborative high-level reflections. The idea of threshold concept came from a national survey conducted in the United Kingdom by Meyer and Land in 2003, and since then it has been a buzz in the Academia (Cousin, 2006). According to these authors a threshold concept is a complex concept of high level that the student has difficulty in understanding and overcoming, sometimes taking refuge in memorisation without understanding. Because of this insuperable barrier to comprehension, the student cannot progress (Meyer and Land, 2006), and often gives up studying. Understanding the causes of the students' barriers helps the teacher to adopt appropriate teaching strategies to support the student in overcoming these barriers to understanding the threshold concept.

This study presents the complex concept 'Division' through the perspective of two Math teachers and compares that with the related Academic literature. To support teachers identifying the barriers to the concept of 'Division' we used a tool designed and developed in the JuxtaLearn project entitled 'Problem Distiller'. The Problem Distiller displays a set of tabbed panes 'prompting teachers to reflect on and select possible reasons why their students might be having a particular problem, connecting all the information entered to the appropriate tricky topic and stumbling block or blocks' (Adams and Clough, 2015, p. 6). In the JuxtaLearn project, 'Tricky Topic' was the name suggested by teachers to refer to the threshold concepts identified by their students (Adams and Clough, 2015). Teachers said that this term relates better to their practice, and 'threshold concept was a formalised academic term that was a threshold concept in itself' (Adams and Clough, 2015, p. 41). According to these two authors, the tricky topics identified by teachers in their practice may not always correspond to the threshold concepts already documented in the literature. This statement is a key issue we explore in this paper.

In section 2, we present a framing for threshold

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concepts and we introduce the term 'tricky topic'. In Section 3 we present the methods of the data collection process. The Section 4, we present the content analysis of the interviews with the two Math teachers, the curricular concept of Division and the Problem Distiller tool. In Section 5, we present our main results and reflections. We conclude in Section 6 with a synthesis and proposals for future work.

2 BACKGROUND

2.1 What is a “Threshold Concept”

Meyer and Land (2003) introduced the notion of ‘threshold concept’ as learning barriers inhibiting the students’ deeper understanding of a concept. They are said to be more than just ‘key’ or ‘core’ concepts (Harlow et al., 2011; Lucas and Mladenovic, 2007). A threshold concept is able to create in students a state of uncertainty, anxiety, confusion, doubt, or even a sense of surprise (Meyer and Land 2006). The barriers presented by threshold concept can be so great, they may cause students to fail or give up a subject altogether (Machiocha, 2014).

According to Meyer and Land (2003), a concept is likely to be threshold if it has one or more of the following criteria:

- **Transformative** – once understood, it potentially causes a significant shift in the perception of a subject (or part thereof); sometimes it may even transform one’s personal identity
- **Irreversible** – it is unlikely that a Threshold Concept is forgotten or unlearned once acquired due to transformation
- **Integrative** – a Threshold Concept is able to expose “the previously hidden interrelatedness of something”
- **Bounded** – a Threshold Concept can have borders with other Threshold Concept which help to define disciplinary areas
- **Troublesome** – they may be counterintuitive (common sense understanding vs. expert understanding)

Nevertheless, the authors emphasize that once understood and overcomed, the ‘threshold concept’ opens up a new understanding of the concept (Meyer and Land, 2003), and allows the student to be able to solve problems with degree of advanced difficulty (Meyer, Knight, Callaghan & Baldock, 2015).

Loertscher, Green, Lewis, Lin and Minderhout (2014) conducted a study involving 75 teachers and 50 students, where involved an iterative process intended to identify threshold concepts in biochemistry. These authors used a process to identify threshold concepts that consists of five phases. Using this process, they were able to identify threshold concepts that are fundamental to the deeper understanding of the biochemistry but are also strongly related to fundamental concepts of the discipline of chemistry and biology discipline. Meyer, Knight Callaghan and Baldock (2015) conducted a case study which used a data triangulation approach to identify threshold concepts that students should understand before solving specific problems of a civil engineering course. For collection purposes teachers took part in dialogue on understanding and conceptual capacity enabling learning for all participants in the process. They concluded that involving the various course stakeholders in an analysis about conceptual understanding and capacity makes learning achievable to all process participants. It also provides a basis for pedagogies and evaluations to facilitate advanced results in students. Also Barradell and Kennedy-Jones (2013) introduced a conceptual model that integrates three components: the students learning, the threshold concepts and curriculum. According to this holistic model, when talking about the threshold concepts can meet various ideas and these ideas when understood as part of a whole provide a more systematic way of thinking about how to improve educational practice.

2.2 What is a “Tricky Topic”

The JuxtaLearn project created an interactive online tool called ‘Tricky Topic Tool' (TTT) to help the teacher identifying a tricky topic (Figure 1).

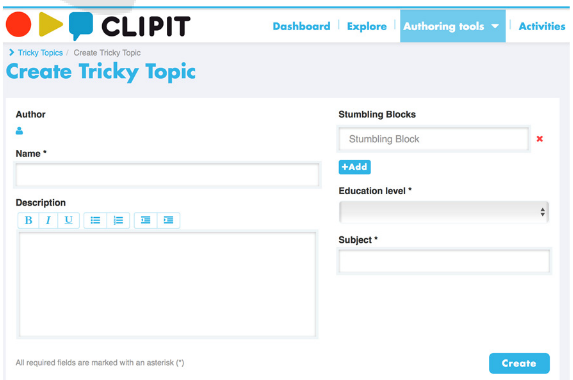

Figure 1: Tricky Topic Tool.

Co-developed with teachers, and included in the CLIPIT - the Web Space for the JuxtaLearn project - the TTT is an online database with a catalogue of tricky topics created by teachers from their perspective and based on their practice. If a tricky topic does not already exist in the TTT, identified by other teachers, the teacher can add one that fits their students' learning problems.

Once the teacher recognises (or adds) a main tricky topic, (s)he can link into some 'stumbling blocks' commonly found by the students with another feature of the TTT: the 'Problem Distiller'. The Problem Distiller has a key role in the JuxtaLearn process (Figure 2).

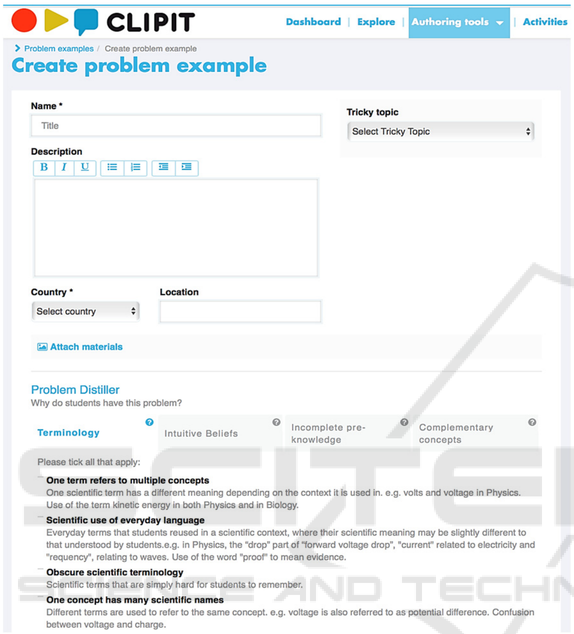

Figure 2: Problem Distiller Tool.

According to Clough et al. (2015), Problem Distiller helps the teacher 'to focus on not just *what* the students have problems understanding, but on *why* they are having these problems'. Student problems and their associated stumbling blocks will be used to give to the teacher guiding cues to create quizzes that address these specific problems. After several trials done in the United Kingdom and Portugal, the CLIPIT has a database of tricky topics, and their related stumbling blocks, examples of student problems, quizzes, and teaching materials.

The teacher creates the quizzes in CLIPIT (or reuse one of the quizzes made previously by another teacher) to assess whether his students have these difficulties. As the teacher creates the quiz, they link each question to one or more related stumbling blocks, selecting the question type (multiple choices, checkboxes, true/false or numeric), the possible options, and the correct answer.

When the students take the quizzes, their results are presented as a visualisation that shows where the gaps in their understanding exist. These results highlight the problem areas and support the teacher in the design of a proper classroom intervention.

2.3 The Tricky Topic "Division"

According to some authors, many children have problems on division (Correa, Nunes, & Bryant, 1998; Kornilaki and Nunes, 2005; Nunes et al., 2015; Fernandes and Martins, 2014), and there is consensus on the fact that children's understanding of division depend on experiences with sharing (Squire and Bryant, 2002a, 2002b). A global understanding, in terms of procedures and in conceptual terms, is essential for the success of the teaching process and learning of the division operation. When we add a procedural understanding to a conceptual understanding, students will be able to understand the division and use it in their day-to-day life with ease (Fernandes and Martins, 2014). The division becomes even more complicated when the dividend is not evenly divided by the divisor (Montague, 2003). The multiplication operation and the division operation are first presented to students from pre-school. From the 3.º grade to 5.º grade, students will develop the meaning of multiplication and division of whole numbers (NCTM, 2008). A constructivist approach to teaching division uses problematic situations to develop in students a conceptual understanding of the process of division (Montague, 2003). Zhao et al., (2014) in their research on the differences in the field of the four basic arithmetic operations (addition, subtraction, multiplication and division) between Flemish and Chinese children between 8 and 11 years old, show that the Chinese students outweigh the Flemish students in each year analyzed. However, this difference diminishes as the grade increases. Their results also indicate that the levels of mastery of the four skills varies between Chinese and Flemish students, but that multiplication was easier for Chinese students. Multiplication is the inverse operation of division. For Greer (2012) inverting is a relational fundamental building block in mathematics and within the purely formal arithmetic, the inverse relationship between addition and subtraction, and multiplication and division, have important implications for the assessment of conceptual understanding of students. Unlu and Ertekin (2012) conducted a study to investigate knowledge of a group of mathematics teachers on the division in the form of fraction. Their results showed that the understanding of the problems raised by fractions, some students applied the multiplication of fractions instead of the fractional division, using reverse algorithm. This was compelling evidence that the students did not have an adequate understanding of fractions.

3 METHOD

Data collection involved interviews with two math teachers from elementary school (5th and 6th grades). The first teacher (T1) is a male, in his fifties, and teaches in a school in Marco de Canaveses, near the city of Porto. The other teacher (T2) is a female with forty-six years old, and teaches in a school in the city of Braga. Both teachers dedicated themselves to teaching for their entire working career.

Data was collected through structured interviews (20 minutes each) with the support of the Problem Distiller tool and Think Aloud protocol (Van et al., 1994). Based on their teaching practice they identified the math tricky topics that are problematic for their students, and checked if the tricky topics were already listed in the database. Next, we explained how to generate a new tricky topic and corresponding stumbling blocks. Then, with the guidance of the Problem Distiller tool, they divided each tricky topic into stumbling blocks, and wrote a brief description of students' specific problems. The aim was to ensure that each interview presented the teachers with exactly the same questions in the same order (the JuxtaLearn taxonomy). This guarantees that answers can be reliably aggregated and that comparisons can be made with confidence between the two teachers.

For the processing and analysis of the obtained data, content analysis was performed (Bardin, 2013), as it allows for logical deductions based on the data obtained. The teachers' utterances were recorded and transcribed for the analysis. During the process, set of dimensions and categories emerged from data: (i) algorithm, (ii) basic operations, (iii) teaching method in the 1st level of education, (iv) reasoning and (v) use the calculator. It is interesting to notice that dimensions ii and iv are also reported in the literature of 'Division' (Fernandes and Martins, 2014; Montague, 2003; Zhao et al., 2014). In the dimension 'a' (algorithm), we analysed the relationship between the difficulty in the division operation and knowledge that students have the division algorithm. In this dimension, we represent the speeches of teachers by "T1.a" or "T2.a". In dimension 'o' (operations), we analyse the relationship between the difficulty in the division operation and the students' knowledge of basic operations and we represent the speeches of teachers by "T1.o" or "T2.o". In dimension 'm' (method), we analyse the relationship between the difficulties in operating with diagnosed division in students and the teaching method in the 1st level of education. In this dimension, we represent the speeches of teachers by "T1.m" or "T2.m". In dimension 'r' (reasoning), we analyse the relationship between the difficulties in operating with the division and thinking capacity demonstrated by students. In this dimension we represent the speeches of teachers by "T1.r" or "T2.r". In dimension 'c' (calculator), we analyse the relationship between the difficulties in operating with the division and the use of calculators by students. In this dimension, we represent the utterances of teachers by "T1.c" or "T2.c". The utterances were numbered according to their occurrences in the text.

4 RESULTS

The content analysis was developed according to the phases suggested by Bardin (2013). Table 1, presents teachers´ voices according to the five categories considered in the analysis.

There appear to be a greater number of evidences in the dimension "Algorithm" and "Reasoning". However, it turns out that there is only one evidence for the dimension teaching method in the 1st level of education. Teachers see the lack of knowledge in the algorithm as a deterrent for students to perform division operations. They point to students' "*difficulty in applying the divide operation algorithm and the location of elements: divider, rest, quotient and divisor*" (T2.a1), and claim that in "*the division operation students have many difficulties*" (T1.a3). In their view, students need to spend more time learning the algorithm, realizing that they "*do not know the algorithm implementation rules and do not know decompose a number*" (T1.a1). In a subject such as the Euclidean algorithm, taught in 5.º grade, teachers recommend "*obliging students to do successive divisions*" (T1.a2), pointing out that students have great difficulties in doing this. Students also have a lot of difficulties on "*the organization of values in the process of division*"(T2.a2) and on "*organization of calculations*"(T2.a3) when they are making the division operation.

Teachers see the lack of knowledge of basic operations as an issue that prevents the students

Table 1: Category of analysis.

Category of analysis	N	Evidences
Algorithm	6	"Students do not know the algorithm implementation rules and do not know how to decompose a number" (T1.a1) "The Euclidean algorithm requires students to do successive divisions. The difficulties for them are huge. They can apply the algorithm realize the algorithm because it forces you to do successive divisions" (T1.a2) "In the division operation, students have many difficulties, mainly because most students cannot understand the division by two numbers"(T1.a3). "Students have difficulty in applying the divide operation algorithm and the location of elements: divider, rest, quotient and divisor" (T2.a1). "In the algorithm, students also have difficulty in organizing values in the process of division"(T2.a2). "Difficulty in organizing calculations when they are split"(T2.a3).
Basic Operations	5	"Students have more difficulties in what we call the basic prerequisites, this is, the level of basic operations: addition, subtraction, multiplication and division. Of these four operations, where they appear the greatest difficulties is the division" (T1.o1) "the main difficulties of them: calculation, basic operations. We may say so, students know add, they know subtract, but if we multiply there are already great difficulties. So if we are talking in the room, mainly by two numbers, mainly by two numbers I say that most students can not do" (T1.o2) "I think mainly, the great difficulty is their basic operations, they confuse the signs of rules of multiplication or division. In mathematics master who does not add up, subtract, multiply and divide, how will dominate powers? how will dominate the other things?" (T1.o3). "Few can convert fractions to decimals, They have many difficulties" (T1.o4). "Students need to learn to add, subtract, multiply, are concepts and procedures that have many difficulties and if they have difficulties, not having the basic knowledge required, these difficulties still will aggravate"(T2.o1).
Teaching method in the 1st level of education	1	"Students come in different primary schools accustomed to different methods, some learn through successive subtractions others by adding the reverse" (T2.m1)
Reasoning	6	"They have to use the implicit reasoning in the division operation they fail to do." (T1.r1) "Them difficulties appear, for example, conversions of fractions to decimals" (T1.r2) "Mathematics is a discipline that requires training, this is, students do exercises and give up the first difficulty of the exercises. And the difficulties begin to be increasing. If the student fails to follow the matter in 5.º grade, how will you get there ahead? The difficulties are increasing and not only gets what the student learns in school." (T1.r3) "Can apply to real life situations and they see that is materializable for them, and with these real-life situations carrying her later for more complicated mathematical concepts and more difficult for them to understand" (T1.r4). They can not perceive, and the difficulty of abstraction combined with the lack of prerequisites to make the division is a problem that can not overcome this difficulty (T2.r1). Students have a hard mental calculation, especially in multiplication and division" (T2.r2). "I notice that students not able to find the successive divisions and do not know the multiplication table" (T2.r3).
Using Calculator	3	"The problem here is often the use of calculating machine or non-use of the adding machine (T1.c1). "If you have difficulties, with the use of the machine, these difficulties will still worsen because they do not have why not use the calculator." (T1.c2) "Then they get used to using the machine and forget what they previously learned" (T2.c1)

from performing division operations. Students present "*difficulties in terms of basic knowledge: addition, subtraction, multiplication and division*" (T1.o1). The development of skills in the basic operations is seen as essential if the student can work with division, because "*in mathematics, for students who do not master the add, subtract, multiply and divide, how will they master powers?, how will they overcome the other things?*" (T1.o3). Teachers said that students had difficulties to converting a minute into seconds or to convert an hour into minutes. They noticed also that if they ask students to do any form of division "*mainly by two numbers, most students can not*"(T1.o2). Students also have many difficulties in "*converting fractions to decimals*" (T1.o4). The competence of using an algorithm is compulsory according to the Portuguese educational policies, but students are not prepared or able to learn them and so difficulties rise: "*if they [the students] have difficulties, not having the basic knowledge required, these difficulties still will aggravate*" (T2.o1).

Only in the category "*teaching method in the 1st level of education*", one of the teachers pointed out that the learning division using didactic methods can leads to later difficulties when working with division. Also, the fact that students often come from "*different primary schools, accustomed to different methods*" (T2.m1) are also problems associated with the Tricky Topic.

Teachers understand that "*the difficulty of abstraction coupled with a lack of basic knowledge*" (T2.r1) presents a problem of understanding when students attempt to acquire new knowledge. The students "*have to use the implicit reasoning in the division operation and they fail to do so*" (T1.r1). The need for the student to remember the notion of a multiple number and know how to apply the division algorithm are factors that hinder students' ability to perform the division operation. According to the teachers, students present "*difficulty in mental calculation, especially in multiplication and division*" (T2.r2) and "*are not able to find the successive divisions*" (T2.r3). The fact that the students "*do not know thirr multiplication tables*"(T2.r3) is also a pointed problem for students unable to do a division. The discipline of Mathematics "*requires training, this is, students do exercises and give up the first difficulty of the exercises. And the difficulties begin to be increasing. If the student can not understand the content in 5.º grade, how will they move forward? The difficulties increase and not only gets what the student learns in school.*" (T1.r3). To improve understanding and visualization, teachers call for situations where students: "*can apply maths to real life situations and develop a sound understand in context, building on this understanding to learn more complicated mathematical concepts*" (T1.r4).

Teachers see the use of calculators in 5.º grade to 6.º grade as an easier alternative adopted by students to perform division. They find that the "*use of calculator or non-use of the adding machine*" (T1.c1), can lead students to forget the algorithm. The students that use the calculator a lot "*forget what they previously learned about the algorithm*" (T2.c1). According to participant teachers, if students have difficulties and use the calculator, their understanding of the fundamental concepts in division will diminish and their ability to perform division without the aid of a calculator will get worse.

4.1 Problem Distiller Tool

We used the Problem Distiller tool to help the teachers reflect on the causes of the student problems they had identified. When teachers expressed problems explaining why their students had difficulty understanding the Tricky Topic, they were guided by Problem Distiller tool to identify the Stumbling blocks. To Tricky Topic "division operation", T1 identified the following Stumbling blocks: (1) organize calculations, (2) adding notion, (3) multiplication and (4) subtraction. We present below the mindmap created with Tricky Topic and Stumbling blocks identified by this teacher:

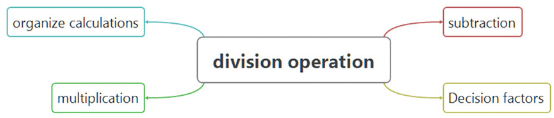

Figure 3: Tricky Topic and their Stumbling blocks to T1.

For the Tricky Topic "division operation", T2 identified the following Stumbling blocks: (1) subtraction, (2) multiplication tables and (3) multiplication. We present below the mindmap created with Tricky Topic and Stumbling blocks.

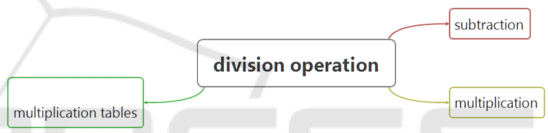

Figure 4: Tricky Topic and their Stumbling blocks to T2.

The Problem Distiller tool guides the teacher in identifying the difficulties of understanding of their students, adding particular examples of student problems based on the teacher's experience with students.

Figure 5: CLIPIT with info gathered from T1.

Figure 6: CLIPIT with info gathered from T2.

As they made selections from Problem Distiller tool, teachers were identifying problems that students typically encounter in understanding the concept of division and were also able to reflect on why these problems occur and how they can might be solved in the classroom.

5 DISCUSSION

Throughout the first years of school, students will develop a sense of number, but only in their 3rd, 4th and 5th grade, more emphasis is given on the development of skills in multiplication and division. The learning of the division operation and the calculation of a division is often associated with several students' difficulties (Mendes, 2013). Understanding the implicit thinking in a division operation, from a mathematical point of view, involves knowledge of other simple operations such as addition and multiplication skills. The division and multiplication operations, although simple, reveal some complexity at cognitive level when presented in problematic situations, because the values have new meanings and the figures presented are sometimes differently exploited (Montague, 2003). One of the fundamental knowledge in the teaching of mathematics is the calculation of the four basic operations: addition, subtraction, multiplication and division. As the student develops the sense of number, (s)he should be able to establish a rationale involving numbers (NCTM, 2008). By using the Tricky Topic Tool we identified together with these two teachers the concept of division as complex concept for students.

To work with the division operation at the start of the 2nd cycle of basic education, it is assumed that students recall some concepts such as the concept of multiple of a number, the division algorithm and algebraic expressions. In general, the data collected from these teachers demonstrates the importance of student's understanding of division in order to solve problems, knowing how to use the division algorithm to keep pace with some of the topics covered in the Curricular Goals for 5th grade. Students tend to use the existing knowledge or related concepts when they learn a new concept and therefore the problems and errors made by the students tend to be systematic. Thus, when doing division students often rely on knowledge about multiplication and division that may well be wrong (Montague, 2003). This data reinforces the importance of giving students a solid understanding of this concept in the 1st cycle.

In Portugal, the concept of division is covered for the first time in the curricular goals in the 2nd year of primary school (Bívar, Grosso, Oliveira, Timóteo, 2012). The concept of division is once again addressed in the 3rd, 4th and 5th grade where other concepts will be combined relating to this operation. According to Professor T1 on the four operations addressed, "*the greatest difficulties arise in the division, I'm talking about students who are in the fifth year*" (T1). Adding that from his experience teaching in the 5th year of primary school, "*90% of students have difficulty in the division operation*" (T1) and the "*division of two numbers, 99% of students can't do it*" (P1). For the teachers involved in our study, sometimes the division algorithm "*have difficulty in identify the elements*" (T1), the dividend, the divisor, the quotient, the rest and the "*organization of the elements when making the division algorithm*" (T2). That is, when using the algorithm to work with the division, sometimes they "*switch between the dividend the divisor*" (T2). According to Professor P1 as the students not always understand the division, "*they do not recognize the process of division and forget the value that is carried*" (T2). The division algorithm, is a set of processes that follow the same order in similar situations (Brocardo and Serrazina, 2008) and it's not always understood by the students.

The fact that they do not know their multiplication tables and are not able to perform a multiplication limits the students' ability to work on concepts and procedures (e.g. division) that need those auxiliary calculations. The poor performance of students not only in understanding necessary strategies, but also in using them to solve a problem leads them to give up. Therefore it is essential to teach students these important processes and strategies that help them solve the problems in a more effective and efficient way (Montague, 2003). Zhao et. al. (2012) in a study which looked at Chinese and Flemish students to know what it takes to master the four basic arithmetic operations (addition, subtraction, multiplication and division), identified that students demonstrated gaps in the four basic operations.

The division operation involves dividing a given number of equal parts. During the early years of school students learn the meaning of the division, understand the effects of dividing by integers, use and understand the notion that the division operation is the inverse operation of multiplication (NCTM, 2008). According to the results, the fact that students cannot resolve a task or problem involving a division appears to discourage students and prevent them from progressing. Also Montague (2003) states that the division operation is a mathematical procedure with some complexity and understanding division therefore involves understanding the other mathematical operations. Many children have difficulties in using the traditional division algorithm. And when the operation is necessary in mathematical problems, many students give up. Unlike the addition operation, multiplication or subtraction, the division algorithm involves the knowledge and identification of four terms dividend, divisor, quotient and rest. These terms can also cause difficulties for the students as the teachers stated in tricky topic tool when they list the understanding issues that are commonly found in students. From the point of view of these teachers "*the great difficulty of the students is the basic operation*" (T1). To develop the competence of calculation through division operation, students need to have knowledge in terms of counting and arithmetic operations such as multiplication tables. Arguments were put forward by both teachers when identifying the difficulties that students have when performing division. They mention that students sometimes fail to "*identify the elements in the division*" (P2) and on the 5th year students are expected to "*work with conversions and the Euclidean algorithm.*" (T1). According to Arends (2008), an effective teacher must in addition to other duties, be able to list a set of good practice and be able to think about the process of teaching. The mathematical knowledge of the teacher is essential to teach the division operation in order to be able to identify students' difficulties and realize in which algorithm stage is this difficulty (Fernandes and Martins, 2014). The teacher plays a fundamental role so that students can understand the mathematical meaning of the division, the procedures involved in the operation, using the correct terminology and an appropriate mathematical language. By using Tricky Topic Tool we promote thinking moments on teachers around the Tricky Topic, the ability to recall moments of work between students and difficulties in the construction of knowledge about the concept of division.

Students' problems often identified by these teachers refer to difficulties in terms of successive subtraction to solve tasks associated with the division; including "*not able to find the successive divisions*" (T2) and "*Euclidean's algorithm requires to do successive divisions.*" (T1). For Montague (2003) the use of additions and successive subtraction is a strategy used by children who learn division and which is based on pre-existing knowledge about addition, subtraction and multiplication. The teachers also mentioned the fact that students are not aware to the inverse relationship between multiplication and division, can also be a problem to the understanding of division operation. They also report that students usually manifest difficulty operating between numbers written in the form of fraction, because "*do not realize the meaning of the elements in the fraction*" (T2), have difficulty to "*identify the dividend and the divisor*" (T1). To suit the results obtained by Unlu and Ertekin (2012) who investigated the knowledge of a group of mathematics teachers on the division between numbers written in the form of fraction, they realized that the knowledge about the division operation with fractions does not go beyond functional knowledge. These teachers were able to apply the rules and the process inherent in the division, but were unable to explain its meaning.

Through the use of Problem Distiller tool with teachers, we realized that the understanding of essential concepts around the Tricky Topic division "*sometimes it depends on a badly learned concept*" (T2). Presuppose the use of "*already acquired knowledge of division*" (T1) as new knowledge is being developed. The lack of essential concepts, fundamental knowledge that is related to the Tricky Topic, without which the student can not understand, was pointed out on Problem Distiller tool as one of the causes for the difficulties in the division operation. Teachers mentioned the lack of knowledge about the scientific method and the lack of support and understanding prior knowledge that the student needs to improve to understand the Tricky Topic. The lack of complementary knowledge to the division operation from the point of view of these teachers can also be a problem. They noted also that some imperfect reasoning around the division and intuitive ways of thinking about the division process can evenly become an obstacle to the understanding of division. The reflection upon the causes for the understanding of problems detected in students, allowed teachers to increase the level of awareness about the knowledge of the student.

The teaching of division operation not only involves knowing how to use the traditional algorithm but also understand the division operation in different situations, understand the relationship between division and multiplication and simultaneously develop a network of numerical relationships around this operation. Even the teachers who teach mathematics in the 1st and on the 2nd cycles admit that the division is a difficult operation to teach to their students and their learning process is sometimes confused with the mechanization of rules associated with the algorithm instead of understanding the division operation (Mendes, 2013). The acquisition of mathematical knowledge allows us to develop reasoning, structure thinking and help future students to think and to decide. Understanding how students learn and how teachers teach mathematical concepts is of fundamental importance for the individual student progress and the organizations to which he belongs. The Tricky Topic tool guided the teachers in the identification of the tricky topics, and corresponding stumbling blocks. The Problem Distiller tool supports them in thinking through the students' difficulties, reflecting on possible causes for those difficulties, and on ways to overcome them. This was because the connections of each Tricky Topic in the Problem Distiller tool allowed teachers to dissect the concept into simpler parts (the stumbling blocks), and establish a critical and reflective look at the teaching and learning of division operation based in the four areas identified as problematical for students: i) Terminology, ii) Incomplete Pre-Knowledge, iii) Essential Concepts, and iv) Intuitive Beliefs. From our perspective, this process was essential to find ways to enable an effective and consolidated teaching about the tricky topic. The difficulties listed by the teachers match the data in the literature, particularly those obtained by Montague (2003), by Zhao et. al. (2012) and Fernandes and Martins (2014). Also the NCTM (2008) states that from the 3rd to 5th grade, students need to understand in greater depth the multiplicative nature of the number system. The results suggest that the obstacles associated with Tricky Topic identified by teachers are similar to the difficulties described in the literature about learning the division operation. The results also showed that the thinking achieved among teachers with the use of Problem Distiller prompted them to think outside their comfort zone. From the perception of teachers we can say that the division operation is a Tricky Topic for the students, and the data obtained so far allow us to conclude that it is a threshold concept according to the criteria listed by Meyer and Land (2003). Linking the perception of teachers with the criteria listed by Meyer and Land (2003) for which a concept is a threshold, we found out upon teachers voices:

- Can be seen as **Transformative**, given that by understanding the division operation students will be able to "*use in everyday situations*" (T2) and "*to make conversions for example*" (T1);
- It is **Irreversible** once learned is difficult to be forgotten; however teachers recognise that "*the abusive use of calculator*" (T1) can lead to loss of an algorithm learned in the first cycle;
- Being the division operation a key operation to for example "*do successive divisions in Euclidean algorithm*" (T1), to respond to "*problematic situations of everyday life*" (T2), it is suggested that it is **Integrative**;
- When the division operation is used to as the basis for understanding of other mathematical concepts. The misunderstanding in division can "*compound the difficulties*" (T1), because if students "*do not have the necessary base knowledge, their difficulties in learning related concepts will increase*" (T2), suggesting that the division operation may be **Bounded**.
- Failure to understand the concept or "*confusion problems with the multiplication operation*" (T2) for example may indicate that it is a **Troublesome**, an incorrect understanding can lead to counterintuitive relations.

6 CONCLUSIONS

This paper compares the process of identifying a complex math concept 'Division' from the pedagogical practice of two teachers, with the way it is reported in the literature. The data collected demonstrates the importance of students acquiring skills of mental calculation, specifically for multiplication and division. The data also shows us that although teachers find it easy to identify the

Tricky Topics and associated stumbling blocks that their students have problems with, they benefit from support in reflecting on 'why' the students had these problems. In this particular, the Problem Distiller tool proved to be essential in scaffolding teachers reasoning on students´ difficulties. The technology supports the teacher's brainstorming process, guiding them in the identification of the causes for student´s misunderstandings, once the possible reasons appear already listed in a catalogue of options provided by the system. By identifying the roots of student misunderstandings of a stumbling block, the teachers became aware of the student's difficulties and could prepare and adopt appropriate teaching strategies. The teachers were able to identify the operation of 'Division' as a Tricky Topic. As the teachers used the Tricky Topic Tool and Problem Distiller to break down the complexities of division, we were able to evaluate it against the characteristics of a threshold concept as specified by Meyer and Land (2003). We found that the teacher-identified topic 'Division' matches the definition of a Threshold Concept as defined in Meyer and Land (2003).

It also seems appropriate to analyse in future research if the level of reflection achieved with the use of the Problem Distiller tool contributes to change the teachers' professional practice.

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