On Duality with Support Functions for a Multiobjective Fractional Programming Problem

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Abstract: In this article, a different class of function called $(K \times Q)$-F-type I has been introduced. Further, we have formulated a problem over cones and appropriate duality results have been established taking the concerned functions to be $(K \times Q)$- F-type I. The results which we have put forward in the paper generalizes some of the known results appeared in the literature.

1 INTRODUCTION

The mathematical programming problems involving ratio of two functions in the objective function is called fractional programming problems. Many types of optimization problems involves this fascinating subject. Fractional programming problem emerges in several types of optimization problems (as for example, portfolio selection, production, information theory and numerous decision making problems in management science). It has also been extensively used in business and economics situation. Further, these type of problems are also useful in engineering and economics where it is often used to maximize a fractional function for measuring the efficiency or productivity of a system. (Bector and Chandra, 1987), (Bector et al., 1993), (Schaible, 1995) etc. have given some applications of fractional programming problems.

Duality theory happens to be the central concept in optimization. This theory aids us to experience and develop new algorithms (numerical algorithms) because it gives us appropriate stopping rules for a pair of primal-dual problems. (Dorn, 1960) considered a convex minimization problem of a differentiable function subject to some linear constraints and studied its duality relations. Over the past few years, many researchers generalized these results to the case of nondifferentiable convex problem (Schechter, 1979) and differentiable convex problem (Hanson, 1981). Assuming the functions to be invex, (Hanson, 1981) established that KKT conditions are sufficient for optimality. Two different types of functions (namely Type I and Type II) were first presented by (Hanson and Mond, 1987) for a scalar optimization problem. (Rueda and Hanson, 1988) extended these functions to pseudo-Type I and quasi-Type I.

The Type I function for a single objective was extended to a MOPP (multiobjective programming problem) by (Kaul et al., 1994), where they defined the Type I, its different generalizations, thus established duality relations for the Wolfe and the Mond-Weir type model. (Kuk and Tanino, 2003) considered nonsmooth programming problem and derived the duality results considering the functions to be generalized Type I. On the other hand, (Suneja et al., 2008) presented (F, ρ, σ)-type I functions for the case of higher order. Further, they considered two dual models (one Mond-Weir and the other Schaible type both are in higher order case) and obtained their corresponding dual relations (for multiobjective fractional programs in nondifferentiable case).

Recently, fractional programming duality has become an interesting topic of research. For a convex nondifferentiable fractional problem, (Bector et al., 1993) established some optimality conditions (namely Fritz John and KKT necessary and sufficient optimality criteria) and proved some duality results. Considering a vectorial optimization problem over cones, (Bhatia, 2012) discussed the sufficient optimality conditions and proved some results (duality theorems) using cone convex and its generalizations for the case of higher order. (Slimani and Mishra, 2014) introduced a nonlinear multiple objective fractional programming with inequality constraints and proved duality results for a Mond-Weir type model using semilocally V-type I-preinvex functions.

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(Kim and Lee, 2009) introduced the nondifferentiable multiobjective problem involving cone constraints and hence studied their duality relations using higher order invexity assumptions. On the other hand, considering the same problem as in (Kim and Lee, 2009), (Ahmad, 2012) formulated a dual program (unified and higher order) and discussed some results in duality (under higher order generalized type-I functions). In recent times, (Debnath et al., 2015) constructed a pair of higher order Wolfe type multiobjective nondifferentiable symmetric dual program over arbitrary cones and studied duality relations under higher-order K-F convexity assumption.

The paper is arranged as we describe. In section 2, a class of $(K\times Q)$-F-type I function has been put forward and further some definitions and terminologies have been given. In the section 3, we have formulated a dual model (Mond-Weir model) for a fractional problem over arbitrary cones (for nondifferentiable multiobjective case) and proved duality relations considering the concerned functions as higher order $(K\times Q)$-F-type I. Section 3 contains the conclusion.

2 SOME BASIC DEFINITIONS

Consider the following MOPP (multiobjective programming problem):

(VP) K-minimize $\phi(x)$

subject to $-\psi(x)\in Q$,
$x\in X\subseteq R^n$,

where $X\subset R^n$ be open and $\phi: X\to R^k$, $\psi: X\to R^m$ defines vector valued differentiable functions, $K\subseteq R^k$ and $Q\subseteq R^m$ denotes the closed convex pointed cones having non-null interiors. Let $X_0=\{x\in X: -\psi(x)\in Q\}$ denotes the feasible set.

Definition 2.1 (Agarwal et al., 2010)**.** A point $\bar{x}\in X^0$ is a weak efficient solution of (VP) if there exists no $x\in X^0$ such that

$$\phi(\bar{x})-\phi(x)\in \text{int}K.$$

Definition 2.2 (Agarwal et al., 2010)**.** A point $\bar{x}\in X^0$ is an efficient solution of (VP) if there exists no $x\in X^0$ such that

$$\phi(\bar{x})-\phi(x)\in K\backslash\{0\}.$$

Definition 2.3 (Gupta et al., 2012)**.** The positive dual cone K^* of K is defined by

$$K^*=\{y: x^Ty\geq 0,\ \text{ for all }\ x\in K\}.$$

Definition 2.4. For all $(x,u)\in X\times X$, a functional $H: X\times X\times R^n\to R$ is called sublinear in respect with the third component, if

(i) $H(x,u;b_1+b_2)\leq H(x,u;b_1)+H(x,u;b_2)$ for all $b_1,b_2\in R^n$,

(ii) $H(x,u;\beta b)=\beta H(x,u;b)$, for all $\beta\in R_+$ and for all $b\in R^n$.

Clearly, $H(x,u;0)=0$.

Definition 2.5. Let $F: X\times X\times R^n\to R$ be called a functional which is sublinear with respect to the third variable. Also, let $H: X\times R^n\to R^k$, $G: X\times R^n\to R^m$ be differentiable functions. Then the function (ϕ,ψ) will be called higher-order $(K\times Q)$-F type I at $u\in R^n$ in respect with the functions H and G, if for each $x\in X_0$, $p_i,\ q_j\in R^n$, $(i=1,2,...,k,\ j=1,2,...,m)$, we have

$$\Big(\phi_1(x)-\phi_1(u)-F(x,u;\nabla_x\phi_1(u)+\nabla_{p_1}H_1(u,p_1))-H_1(u,p_1)+p_1^T[\nabla_{p_1}H_1(u,p_1)],...,\phi_k(x)-\phi_k(u)-F(x,u;\nabla_x\phi_k(u)+\nabla_{p_k}H_k(u,p_k))-H_k(u,p_k)+p_k^T[\nabla_{p_k}H_k(u,p_k)]\Big)\in K.$$

and

$$\Big(-\psi_1(u)-F(x,u;\nabla_x\psi_1(u)+\nabla_{q_1}G_1(u,q_1))-G_1(u,q_1)+q_1^T\nabla_{q_1}G_1(u,q_1),...,-\psi_m(u)-F(x,u;\nabla_x\psi_m(u)+\nabla_{q_m}G_m(u,q_m))-G_m(u,q_m)+q_m^T\nabla_{q_m}G_m(u,q_m)\Big)\in Q.$$

Definition 2.6 (Gupta et al., 2012)**.** Let φ be a convex set in R^n which is also compact. The support function of φ is given as

$$\tau(x|\varphi)=\max\{x^Ty: y\in\varphi\}.$$

The subdifferentiable of $\tau(x|\varphi)$ is defined by

$$\partial\tau(x|\varphi)=\{z\in\varphi: z^Tx=\tau(x|\varphi)\}.$$

We now present the following problem (KP) (multiobjective fractional programming problem) over arbitrary cones containing support functions.

(KP) K-minimize $\Big[\dfrac{\phi_1(x)+\tau(x|C_1)}{\psi_1(x)-\tau(x|D_1)},...,\dfrac{\phi_k(x)+\tau(x|C_k)}{\psi_k(x)-\tau(x|D_k)}\Big]$

subject to

$$-\Big[\varphi_j(x)+\tau(x|M_j)\Big]\in Q,\ j=1,2,...,m.$$

where

$\phi : R^n \to R^k$, $\psi : R^n \to R^k$ and $\varphi : R^n \to R^m$ are continuously differentiable functions. Assume that $\phi_i(.) + \tau(.|C_i) \geq 0$ and $\psi_i(.) - \tau(.|D_i) > 0$. C_i, D_i and M_j denotes the compact convex sets in R^n and their respective support functions are denoted by $\tau(x|C_i)$, $\tau(x|D_i)$ and $\tau(x|M_j)$.

Throughout the paper, the following notations have been used:

$$\frac{P(.)+(.)^T z}{Q(.)-(.)^T v} = \Big\{ \frac{\phi_1(.)+(.)^T z_1}{\psi_1(.)-(.)^T v_1}, \frac{\phi_2(.)+(.)^T z_2}{\psi_2(.)-(.)^T v_2}, \ldots, \frac{\phi_k(.)+(.)^T z_k}{\psi_k(.)-(.)^T v_k} \Big\}$$

$$R(.)+(.)^T w = \Big\{ \varphi_1(.)+(.)^T w_1, \varphi_2(.)+(.)^T w_2, \ldots, \varphi_m(.)+(.)^T w_m \Big\}.$$

$$H = \Big(H_1(u,p_1), H_2(u,p_2), \ldots, H_k(u,p_k)\Big).$$

$$G = \Big(G_1(u,q_1), G_2(u,q_2), \ldots, G_k(u,q_m)\Big).$$

In the following section, we associate dual model for the primal problem (KP) and establish duality relations between them.

3 MOND-WEIR TYPE DUAL MODEL

Consider the following Mond-Weir type higher order dual program of the problem (KP):

(MPD) K-maximize

$$\Big[\frac{\phi_1(u)+u^T z_1}{\psi_1(u)-u^T v_1}, \ldots, \frac{\phi_k(u)+u^T z_k}{\psi_k(u)-u^T v_k}\Big]$$

subject to

$$\sum_{i=1}^{k} \lambda_i \Big[\nabla \frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\Big] + \sum_{j=1}^{m} \mu_j \Big[\nabla(\varphi_j(u)+u^T w_j)\Big] + \sum_{i=1}^{k} \lambda_i \nabla_{p_i} H_i(u,p_i) + \sum_{j=1}^{m} \mu_j \nabla_{q_j} G_j(u,q_j) = 0. \quad (1)$$

$$\sum_{j=1}^{m} \mu_j \Big[(\varphi_j(u)+u^T w_j) + G_j(u,q_j) - q_j^T \nabla_{q_j} G_j(u,q_j)\Big] \geq 0. \quad (2)$$

$$\sum_{i=1}^{k} \lambda_i \Big[H_i(u,p_i) - p_i^T \nabla_{p_i} H_i(u,p_i)\Big] \geq 0. \quad (3)$$

$z_i \in C_i$, $v_i \in D_i$, $w_j \in M_j$, $(i = 1,2,\ldots,k$, $j = 1,2,\ldots,m)$,

$\lambda \in \text{int}K^*$, $\mu \in \text{int}Q^*$, $(\lambda,\mu) \neq (0,0)$.

Remark 3.1. If we consider $K = R^k_+$ and $Q = R^m_+$, the above discussed Mond-Weir model reduces to the model studied in (Suneja et al., 2008).

Next, we will prove duality results between (KP) and (MPD).

Theorem 3.1 (Weak Duality Theorem)**.** Consider x be a member of the feasible set for (KP) and $(u,v,w,\lambda,\mu,z,p,q)$ belongs to the feasible set for (MPD). Suppose that

(i) $\Big[\frac{P(.)+(.)^T z}{Q(.)-(.)^T v}, R(.)+(.)^T w\Big]$ is higher order $(K \times Q)$-F-type I at u in respect with functions H and G, where $H : X \times R^n \to R^k$ and $G : X \times R^n \to R^m$ are differentiable functions,

(ii) $R^k_+ \subseteq K$ and $R^m_+ \subseteq Q$.

Then

$$\Big(\frac{\phi_1(u)+u^T z_1}{\psi_1(u)-u^T v_1}, \ldots, \frac{\phi_k(u)+u^T z_k}{\psi_k(u)-u^T v_k}\Big) - \Big(\frac{\phi_1(x)+\tau(x|C_1)}{\psi_1(x)-\tau(x|D_1)}, \ldots, \frac{\phi_k(x)+\tau(x|C_k)}{\psi_k(x)-\tau(x|D_k)}\Big) \notin K\backslash\{0\}.$$

Proof. We will prove the result by contradiction. Suppose,

$$\Big(\frac{\phi_1(u)+u^T z_1}{\psi_1(u)-u^T v_1}, \ldots, \frac{\phi_k(u)+u^T z_k}{\psi_k(u)-u^T v_k}\Big) - \Big(\frac{\phi_1(x)+\tau(x|C_1)}{\psi_1(x)-\tau(x|D_1)}, \ldots, \frac{\phi_k(x)+\tau(x|C_k)}{\psi_k(x)-\tau(x|D_k)}\Big) \in K\backslash\{0\}.$$

Since $\lambda \in \text{int}K^*$, we have

$$\sum_{i=1}^{k} \lambda_i \Big[\Big(\frac{\phi_i(u)+u^T z_i}{\psi_i(v)-u^T v_i}\Big) - \Big(\frac{\phi_i(x)+\tau(x|C_i)}{\psi_i(x)-\tau(x|D_i)}\Big)\Big] > 0. \quad (4)$$

Now, since $x^T z_i \leq \tau(x|C_i)$, $x^T v_i \leq \tau(x|D_i)$, we obtain

$$\Big[\frac{\phi_i(x)+\tau(x|C_i)}{\psi_i(x)-\tau(x|D_i)} - \frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i}\Big] \geq 0.$$

Using hypothesis (ii), we have $K^* \subseteq R^k_+ \Rightarrow \lambda \in \text{int}K^* \subseteq \text{int}R^k_+$ which yields $\lambda > 0$. Therefore, the above inequality implies,

$$\sum_{i=1}^{k} \lambda_i \Big(\frac{\phi_i(x)+\tau(x|C_i)}{\psi_i(x)-\tau(x|D_i)} - \frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i}\Big) \geq 0. \quad (5)$$

Further, on adding (4) and (5), we have

$$\sum_{i=1}^{k} \lambda_i \Big[\Big(\frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i}\Big) - \Big(\frac{\phi_i(u)+u^T z_i}{\psi_i(v)-u^T v_i}\Big)\Big] < 0. \quad (6)$$

By assumption (i) (since $\left[\frac{P(.)+(.)^T z}{Q(.)-(.)^T v}, R(.)+(.)^T w\right]$ is higher order $(K\times Q)$-F-type I at u in respect with functions H and G), we thus have

$$\Big(\frac{\phi_1(x)+x^Tz_1}{\psi_1(x)-x^Tv_1}-\frac{\phi_1(u)+u^Tz_1}{\psi_1(u)-u^Tv_1}-F\Big(x,u;\nabla\Big(\frac{\phi_1(u)+u^Tz_1}{\psi_1(u)-u^Tv_1}\Big)+\nabla H_1(u,p_1)\Big)-H_1(u,p_1)+p_1^T\nabla_{p_1}H_1(u,p_1),\dots,\frac{\phi_k(x)+x^Tz_k}{\psi_k(x)-x^Tv_k}-\frac{\phi_k(u)+u^Tz_k}{\psi_k(u)-u^Tv_k}-F\Big(x,u;\nabla\Big(\frac{\phi_k(u)+u^Tz_k}{\psi_k(u)-u^Tv_k}\Big)+\nabla H_k(u,p_k)\Big)-H_k(u,p_k)+p_k^T\nabla_{p_k}H_k(u,p_k)\Big)\in K, \tag{7}$$

and

$$\Big(-\varphi_1(u)-u^Tw_1-F\Big(x,u;\nabla(\varphi_1(u)+u^Tw_1)+\nabla_{q_1}G_1(u,q_1)\Big)-G_1(u,q_1)+q_1^T\nabla_{q_1}G_1(u,q_1),\dots,-\varphi_m(u)-u^Tw_m-F\Big(x,u;\nabla(\varphi_m(u)+u^Tw_m)+\nabla_{q_m}G_m(u,q_m)\Big)-G_m(u,q_m)+q_m^T\nabla_{q_m}G_m(u,q_m)\Big)\in Q. \tag{8}$$

From (7) and $\lambda\in \mathrm{int}K^*$, we obtain

$$\sum_{i=1}^{k}\lambda_i\Big[\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i}-\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}-F\Big(x,u;\nabla\Big(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)+\nabla_{p_i}H_i(u,p_i)\Big)-H_i(u,p_i)+p_i^T\nabla_{p_i}H_i(u,p_i)\Big]\geq 0.$$

Using $\lambda>0$ and the fact that F is sublinear, we get

$$\sum_{i=1}^{k}\lambda_i\Big(\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i}-\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)\geq\sum_{i=1}^{k}\lambda_i\Big[H_i(u,p_i)-p_i^T\nabla_{p_i}H_i(u,p_i)\Big]+F\Big(x,u;\sum_{i=1}^{k}\lambda_i\Big(\nabla\Big(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)+\nabla_{p_i}H_i(u,p_i)\Big)\Big). \tag{9}$$

Again from (8) and $\mu\in \mathrm{int}Q^*$, we have

$$\sum_{j=1}^{m}\mu_j\Big(-(\varphi_j(u)+u^Tw_j)-F(x,u;\nabla(\varphi_j(u)+u^Tw_j)+\nabla_{q_j}G_j(u,q_j))-G_j(u,q_j)+q_j^T\nabla_{q_j}G_j(u,q_j)\Big)\geq 0.$$

Now, by assumption (ii), $Q^*\subseteq R^m_+\Rightarrow\mu\in \mathrm{int}Q^*\subseteq \mathrm{int}R^m_+$ which yields $\mu>0$. Therefore, the above inequality and the sublinearity of F together imply,

$$\sum_{j=1}^{m}\mu_j\Big[-(\varphi_j(u)+u^Tw_j)-G_j(u,q_j)+q_j^T\nabla_{q_j}G_j(u,q_j)\Big]\geq F\Big(x,u;\sum_{j=1}^{m}\mu_j\Big(\nabla(\varphi_j(u)+u^Tw_j)+\nabla_{q_j}G_j(u,q_j)\Big)\Big). \tag{10}$$

It follows from the sublinearity of F, (9) and (10) that

$$\sum_{i=1}^{k}\lambda_i\Big[\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i}-\Big(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)\Big]\geq\sum_{i=1}^{k}\Big[H_i(u,p_i)-p_i^T\nabla_{p_i}H_i(u,p_i)\Big]+F\Big[x,u;\sum_{i=1}^{k}\lambda_i\Big(\nabla\Big(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)+\nabla_{p_i}H_i(u,p_i)\Big)+\sum_{j=1}^{m}\mu_j\Big(\nabla(\varphi_j(u)+u^Tw_j)+\nabla_{q_j}G_j(u,q_j)\Big)\Big]-\sum_{j=1}^{m}\mu_j\Big[-(\varphi_j(u)+u^Tw_j)-G_j(u,q_j)+q_j^T\nabla_{q_j}G_j(u,q_j)\Big].$$

Finally, using dual constraint (1)-(3), we get

$$\sum_{i=1}^{k}\lambda_i\Big[\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i}-\Big(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\Big)\Big]\geq 0,$$

which contradicts (6). Hence the result. □

Definition 3.1 (Clarke, 1983)**.** The function $g:R^n\to R$ will be called a locally Lipschitz at $x_0\in R^n$ if $\exists\, k\geq 0$ and a neighbourhood $\delta(x_0)$ of x_0 s. t.

$$||g(x)-g(y)||\leq k||x-y||,\ \forall\, x,\ y\in\delta(x_0).$$

Definition 3.2 (Husain and Zabeen, 2005)**.** A locally Lipschitz function $g:R^n\times R^n\to R$ at $x_0\in R^n$ in the direction $t\in R^n$ is said to have generalized directional derivative if

$$g'(x_0;t)=\lim_{y\to x_0}\sup_{p\downarrow 0^+}\frac{g(y+pt)-g(y)}{p},$$

where $y\in R^n$ and $p>0$.

Definition 3.3 (Wang, 2005)**.** A function g which is also locally Lipschitz at $x_0 \in R^n$ is said to have generalized gradient or the subdifferential if

$$\partial g(x_0) = \{\zeta \in R^n : g'(x_0;t) \geq \langle \zeta, t\rangle,\ \forall\, t \in R^n\}.$$

Remark 3.2.

(i) For a convex function ϕ, it will be called locally Lipschitz at $x_0 \in R^n$, if

$$\partial\phi(x_0) = \{\zeta \in R^n : \phi(x) - \phi(x_0) \geq (x - x_0)^T\zeta,\ \forall\, x \in R^n\}.$$

(ii) If ϕ at x_0 is continuously differentiable, then ϕ at x_0 will be locally Lipschitz, and therefore, $\partial\phi(x_0) = \{\nabla\phi(x_0)\}$.

Lemma 3.1. Let us deal with the problem:

$$\text{(P)}\quad K-\text{minimize}\quad \theta(x)$$

$$\text{subject to}\quad -\omega(x) \in Q,$$

where $\theta : R^n \to R^k$ and $\omega : R^n \to R^m$ are locally Lipschitz functions. For this problem, suppose $\bar{x}$ be weak efficient solution, then $\exists\ (0,0) \neq (\lambda, \mu) \in \text{int}K^* \times \text{int}Q^*$ in such a way that

$$0 \in \partial^c(\lambda^T\theta + \mu^T\omega)(\bar{x}),\ (\mu^T\omega)(\bar{x}) = 0.$$

Proof. It follows on the lines of (Craven, 1989) and (Wang et al., 2008). □

Proposition 3.1 (Husain and Zabeen, 2005)**.** Let us suppose that $\phi_1 : R^n \to R$ and $\phi_2 : R^n \to R$ at $x_0 \in R^n$ be locally Lipschitz with $\phi_2(x) \neq 0$. Then $\dfrac{\phi_1}{\phi_2}$ will be locally Lipschitz at $x_0 \in R^n$.

With the help of the above lemma and the proposition, we will now prove the following result, between (KP) and (MPD).

Theorem 3.2 (Strong Duality)**.** If $\bar{x}$ is a weak efficient of (KP), then $\exists\ (0,0) \neq (\bar{\lambda}, \bar{\mu}) \in \text{int}K^* \times \text{int}Q^*$, $\bar{z}_i \in C_i$, $v_i \in D_i$, $w_j \in M_j$; i belongs to the index set $\{1,2,\ldots,k\}$, and j belongs to $\{1,2,\ldots,m\}$ in such a way that, $(\bar{x}, \bar{v}, \bar{w}, \bar{\lambda}, \bar{\mu}, \bar{z}, \bar{p} = 0, \bar{q} = 0)$ belongs to the feasible set for (MPD) and their respective objective functions possesses same values provided $H(\bar{x},0) = 0$, $G(\bar{x},0) = 0$, $\nabla_{p_i}H_i(\bar{x},0) = 0$, $\nabla_{q_j}G_j(\bar{x},0) = 0$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,m$.

Moreover, if the supposition of Theorem 3.1 are fulfilled for every feasible solution x of (KP) and $(u,v,w,\lambda,\mu,z,p,q)$ of (MPD), then $(\bar{x}, \bar{v}, \bar{w}, \lambda, \bar{\mu}, \bar{z}, \bar{p} = 0, \bar{q} = 0)$ is an efficient for (MPD).

Proof. Let $\bar{x} \in X_0$ be a weak efficient of (KP) and suppose that $\theta : R^n \to R^k$, $\varphi : R^n \to R^m$ be taken as

$$\theta(x) = \Big(\frac{\phi_1(x) + \tau(x|C_1)}{\psi_1(x) - \tau(x|D_1)}, \ldots, \frac{\phi_k(x) + \tau(x|C_k)}{\psi_k(x) - \tau(x|D_k)}\Big)$$

and $\omega(x) = \Big(\varphi_1(x) + \tau(x|M_1), \ldots, \varphi_m(x) + \tau(x|M_m)\Big)$.

The functions $\tau(x|C_i)$, $\tau(x|D_i)$ and $\tau(x|M_j)$, $(i = 1,2,\ldots,k,\ \ j = 1,2,\ldots,m)$ are locally Lipschitz, since each of them are convex. Also, ϕ , ψ and φ are functions which are continuously differentiable, hence the above functions are also locally Lipschitz and as a result, $\phi_i(x) + \tau(x|C_i)$, $\psi_i(x) - \tau(x|D_i)$, i takes values from $1,2,\ldots,k$ and $\varphi_j(x) + \tau(x|M_j)$, j takes values from $1,2,\ldots,m$ are locally Lipschitz.

Using the Proposition 3.1, we thus conclude that $\theta(x)$ and $\omega(x)$ are also locally Lipschitz.

Following the Lemma 3.1, $\exists\ \bar{\lambda} \in \text{int}K^*$ and $\bar{\mu} \in \text{int}Q^*$, $(\bar{\lambda}, \bar{\mu})$ not equal to $(0,0)$ in such a way that

$$0 \in \partial^c\Big[\sum_{i=1}^{k}\bar{\lambda}_i\Big(\frac{\phi_i(\bar{x}) + \tau(\bar{x}|C_i)}{\psi_i(\bar{x}) - \tau(\bar{x}|D_i)}\Big) + \sum_{j=1}^{m}\bar{\mu}_j\Big(\varphi_j(\bar{x}) + \tau(\bar{x}|M_j)\Big)\Big]$$

$$\text{and}\quad \sum_{j=1}^{m}\bar{\mu}_j\Big(\varphi_j(\bar{x}) + \tau(\bar{x}|M_j)\Big) = 0,$$

which implies

$$0 \in \sum_{i=1}^{k}\bar{\lambda}_i\Big(\partial^c\Big(\frac{\phi_i(\bar{x}) + \tau(\bar{x}|C_i)}{\psi_i(\bar{x}) - \tau(\bar{x}|D_i)}\Big)\Big) + \sum_{j=1}^{m}\bar{\mu}_j(\nabla\varphi_j(\bar{x})) + \sum_{j=1}^{m}\bar{\mu}_j\partial^c S(\bar{x}|M_j).$$

As we know that the support functions are convex, we have

$$\partial^c\tau(\bar{x}|C_i) = \partial\tau(\bar{x}|C_i), \partial^c\tau(\bar{x}|D_i) = \partial\tau(\bar{x}|D_i) \text{ and}$$
$$\partial^c\tau(\bar{x}|M_j) = \partial\tau(\bar{x}|M_j)$$

Therefore, there exist $\bar{z}_i \in \partial\tau(\bar{x}|C_i)$, $\bar{v}_i \in \partial\tau(\bar{x}|D_i)$ and $\bar{w}_j \in \partial\tau(\bar{x}|M_j)$ just as if

$$\bar{x}^T\bar{z}_i = \tau(\bar{x}|C_i),\ \bar{x}^T\bar{v}_i = \tau(\bar{x}|D_i) \text{ and } \bar{x}^T\bar{w}_j = \tau(\bar{x}|M_j), \tag{11}$$

Hence,

$$\sum_{i=1}^{k}\bar{\lambda}_i\Big(\nabla\Big(\frac{\phi_i(\bar{x}) + \bar{x}^T\bar{z}_i}{\psi_i(\bar{x}) - \bar{x}^T\bar{v}_i}\Big)\Big) + \sum_{j=1}^{m}\bar{\mu}_j(\nabla\varphi_j(\bar{x}) + \bar{x}^T\bar{w}_j) = 0,$$

$$\text{and}\quad \sum_{j=1}^{m}\bar{\mu}_j(\nabla\varphi_j(\bar{x}) + \bar{x}^T\bar{w}_j) = 0.$$

Using $H(\bar{x},0) = G(\bar{x},0) = 0$, $\nabla_{p_i} H_i(\bar{x},0) = 0$ and $\nabla_{q_j} G_j(\bar{x},0) = \underline{0}$, $(i = 1,2,...,k,\ \ j = 1,2,...,m)$, we obtain $(\bar{x},\bar{v},\bar{w},\bar{\lambda},\bar{\mu},\bar{z},\bar{p} = 0,\bar{q} = 0)$ belongs to the domain feasible for (MPD) and also the respective values of the objectives are equivalent.

We now claim that for (MPD) $(\bar{x},\bar{v},\bar{w},\bar{\lambda},\bar{\mu},\bar{z},\bar{p} = 0,\bar{q} = 0)$ is efficient.

On the contrary, let us assume that $(\bar{x},\bar{v},\bar{w},\bar{\lambda},\bar{\mu},\bar{z},\bar{p} = 0,\bar{q} = 0)$ be efficient for (MPD), therefore $\exists\ (u,v,w,\lambda,\mu,z,p,q)$, which is in the feasible domain for (MPD) such that

$$\Big[\frac{\phi_1(u)+u^T z_1}{\psi_1(u)-u^T v_1},...,\frac{\phi_k(u)+u^T z_k}{\psi_k(u)-u^T v_k}\Big] - \Big[\frac{\phi_1(\bar{x})+\bar{x}^T \bar{z}_1}{\psi_1(\bar{x})-\bar{x}^T \bar{v}_1},...,\frac{\phi_k(\bar{x})+\bar{x}^T \bar{z}_k}{\psi_k(\bar{x})-\bar{x}^T \bar{v}_k}\Big] \in K\backslash\{0\}$$

which using (11) imply

$$\Big[\frac{\phi_1(u)+u^T z_1}{\psi_1(u)-u^T v_1},...,\frac{\phi_k(u)+u^T z_k}{\psi_k(u)-u^T v_k}\Big] - \Big[\frac{\phi_1(\bar{x})+\tau(\bar{x}|C_1)}{\psi_1(\bar{x})-\tau(\bar{x}|D_1)},...,\frac{\phi_k(\bar{x})+\tau(\bar{x}|C_k)}{\psi_k(\bar{x})-\tau(\bar{x}|D_k)}\Big] \in K\backslash\{0\},$$

a contradiction to the Theorem 3.1. Therefore, the required result. □

4 CONCLUSIONS

In this paper, we have presented a current class of higher order $(K \times Q)$- F-type I function. A Mond-Weir type higher order multiobjective fractional problem (which is also nondiifferentiable) over cone has been constructed. Considering this dual program, we have established the corresponding duality relation under higher order $(K \times Q)$- F-type I function. The results which we have put forward in this paper are extension of some previously studied results appearing in the literature. It is to be noted that, researchers can further extend our work for different types of duality problems for fractional problems, such as, mixed type duality etc.

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