Game Theoretic Models for Competition in Public Transit Services

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Abstract: As metropolitan areas grow, the need to travel by the populace has increased the burden on the transport systems, leading to increased traffic congestion and environmental concerns. In this paper, we discuss some game-theoretic models that can be used to investigate the competitive situation when several service providers offer public transit services. The competition among the operators can be modelled by a class of games called potential games, and we discuss mathematical programmes that can be used to find the Nash equilibria for these games. By examining the equilibrium solutions, we can investigate the relative merits and tradeoffs for different structures of the transit networks, and the interplay between the services offered and the overall ridership of the system.

1 INTRODUCTION

Metropolitan areas have accounted for the majority of increases in population and economic growth in recent decades. China's phenomenal economic development has been fuelled by growth in the major cities, many of which has over 5 million in population. Metropolitan areas account for over half of the population, and a significant majority of the GDP, of the United States. As the geographical size and population of major metropolitan areas have increased, much economic activity remain focussed in the central business districts of the metropolises, thus the average travel distances for work have not decreased as expected. The average commuting distance for London is over 10 kilometres. The need to travel by the populace has placed significant burden on the transport systems of metropolitan areas, leading to increased traffic congestion and attendant safety and environmental concerns.

Development of transport infrastructure and public transit services have not kept apace with the swell and sprawl of metropolitan areas, with serious congestion occurring in central business districts and insufficient coverage in peripheral areas. In metropolises where public transit services are provided by private firms in a relatively free market, operators tend to focus on high-profit routes and outlying smaller communities are under-served. In Hong Kong, the already congested Central business district is often jammed with half-empty double-decker buses from all the bus operators, while bus services to satellite communities in the New Territories are very infrequent and expensive.

In this paper, we discuss some game-theoretic models that can be used to investigate the competitive situation when several service providers offer public transit services, and study the impact on the total set of services offered to the public and the resultant level of ridership of the system. The competition among the operators can be modelled by a class of games called potential games. We discuss mathematical programmes that can be used to find the Nash equilibria for these games. By examining the equilibria solutions, we can examine the relative merits and tradeoffs for different structures of the transit networks, and the interplay between the services offered and the overall ridership of the system. We hope that our modelling and analysis may provide some insight on the types and bundling of routes being offered by operators, and the locations for transportation interchanges and hubs.

2 BACKGROUND

Whilst not explicitly acknowledged, concepts of game theory have been pervasively used in traffic studies. As Fisk (1984) pointed out, the famous

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Wardrop's (1952) user-equilibrium principle is essentially the condition for a Nash (1950) game-theoretic equilibrium among road-users, since no driver can reduce his/her travel time by switching to a different route choice. Wardrop's principle has been a cornerstone in road traffic research for decades. For an overview of traffic equilibrium models, see Patriksson and Labbe (2004).

Other researchers have developed specific game-theoretic models for transport-related problems. Bell (2000) investigates network reliability by studying a zero-sum game between a cost-minimising driver and a *demon* that sets the link costs. This game is a *concept* game in the sense that the demon is not a real player, and is used to explore the worst-case scenarios faced by the driver. Other researchers have also explored concept games among road users. James (1998) studies a game among n road-users where any player's utility of using the road segment decreases when there are more users. Levinson (2005) also studies congestion by investigating a game where the players (drivers) choose their departure times. Pedersen (2003) investigates road safety by a game where players choose the behavioural level of driving aggression. All these games study the competition among road users. Holland and Prashker (2006) give an excellent review of recent literature on non-cooperative games in transport research.

Surprisingly, studies on the competitive situation amongst public transit operators have received little attention from transport researchers. Castell et al. (2004) modelled a Stackelberg game between two authorities (one determining flow, and the other capacities) in a freight transport network, which is different to a passenger transit network since the route choice is not determined by the transportee (freight). According to Holland and Prashker (2006), the "small number of such games is surprising, considering that NCGT [non-cooperative game theory] seems a natural tool for analysing relations between authorities. ... Trends such as tendering and privatisation, that have a vital role on today's transport agenda, also seem apt to be modelled through games between authorities". Surprisingly, there has been very little research along this line.

Some researchers have studied games between authorities and travellers. Fisk (1984) investigates a Stackelberg game between the authority that sets traffic signals and all travellers who then finds the user-equilibrium solution. Chen and Ben-Akiva (1998) investigates a similar game in a dynamic setting. Reyniers (1992) studies a game between the railway operators who sets the capacities for different fare classes and the passengers who chooses which class to use. Hollander et al. (2006) studies a game between the parking authority and travellers to explore the incentives for public transport ridership. All these games, however, only involve one operator/authority. Only few researchers have investigated games with several operators and passengers. Van Zuylen and Taale (2004) studies a game with two authorities (one for urban roads and one for ring roads) and the driving public. Fernandez and Marcotte (1992) and Fernandez et al. (1993) investigated traffic equilibrium models involving car users, bus users and bus operators and presented algorithms for finding the equilibrium solutions; their models are general and considered traffic congestion effects. The focus of our paper is on the strategic competition among public transit operators.

3 MODELS

We have made some preliminary investigation into the strategic gaming situation among competing public transit service providers. In our first-cut model, we assume that all the operators have the same cost and price structure, and that the total ridership between each origin-destination pair is equally divided among all the operators that service that particular route. In this setting, a player of the game is the service provider, and its strategy is the set of routes that it chooses to offer service. Each player tries to maximize its total profit, and a Nash equilibrium occurs when no player can improve its profit by unilaterally changing the set of routes it services.

We can show that this game can be modelled as what is known as a potential game (first introduced by Rosenthal, 1973) where the equilibrium can be computed by solving an auxiliary mathematical programme with a *potential* function as a surrogate objective. The solvability for the Nash equilibrium allows us to make some comparisons between the competitive equilibrium and a centralised monopolistic profit-maximizing operator and draw some insights.

3.1 Basic Competitive Model

We consider a game among n players (service providers) and m possible routes (origin-destination pairs). Player i's strategy consists of a subset of the routes $S_i \subseteq M = \{1, 2, \cdots, m\}$. For each route j, let k_j be the number of players who choose to offer service on the route, i.e., $k_j = |\{i : j \in S_i\}|$. Note that k_j is endogenously determined. Let a_j denote the revenue from the total ridership on route j, assumed exogenously determined, and let δ_j be the cost of operating the route. Each player (service provider) that offers service on this route earns an identical payoff of $p_j(k_j)$ which depends on the total number of players (k_j) who choose to serve that route:

$$p_j(k_j) = \begin{cases} \frac{a_j}{k_j} - \delta_j, & k > 0, \\ 0, & k = 0. \end{cases}$$

Each player i selects a strategy to maximise his total profit $\pi_i(S_1, S_2, \cdots, S_n) = \sum_{j \in S_i} p_j(k_j)$. A pure Nash (1950) equilibrium is a set of strategies $\{S_1^*, S_2^*, \cdots, S_n^*\}$ such that each player cannot unilaterally improve his total profit, that is,

$$\pi_i(S_1^*, \cdots, S_{i-1}^*, S_i^*, S_{i+1}^*, \cdots, S_n^*) \geq \pi_i(S_1^*, \cdots, S_{i-1}^*, S_i, S_{i+1}^*, \cdots, S_n^*), \forall\, S_i$$

Following Rosenthal (1973), we can show that the Nash equilibrium for this game can be obtained by solving the following auxiliary mathematical programme:

$$(G0): \qquad Max \sum_{j=1}^{m} \sum_{y=1}^{k_j} p_j(y)$$

$$\text{s.t.} \sum_{i=1}^{n} x_i^j = k_j, \qquad \forall\, j = 1, \ldots, m, \tag{1}$$

$$x_i^j \in \{0,1\}, \qquad \forall i = 1, \ldots, n, j = 1, \ldots, m, \tag{2}$$

where x_i^j is 1 if player i offers service on route j. We note that this is **not a** straightforward binary linear programme, since the k_j values are also variables, so the objective is not a linear function.

Proof. We will show, by contradiction, that the solution of $(G0)$ yields a Nash equilibrium. Let $\{x^*, k^*\}$ be an optimal solution to $(G0)$ such that the associated strategy combination is not a Nash equilibrium. Then for some player l, there is a strategy $\hat{S}_l$ such that

$$\sum_{j \in \hat{S}_l \setminus S_l^*} p_j(k_j^* + 1) > \sum_{j \in S_l^* \setminus \hat{S}_l} p_j(k_j^*)$$

where S_l^* is the strategy used by l as indicated by the values of x_i^{*j}. Consider the new values $\{\hat{x}_i^j, \hat{k}_j\}$ associated with player l changing to the pure-strategy $\hat{S}_l$ (with the rest of the players not changing their strategies). The objective value of $(G0)$ for this solution is:

$$\begin{aligned} & \sum_{j=1}^{m} \sum_{y=1}^{\hat{k}_j} p_j(y) \\ = & \sum_{j=1}^{m} \sum_{y=1}^{k_j^*} p_j(y) + \underbrace{\sum_{j \in \hat{S}_l \setminus S_l^*} p_j(k_j^* + 1) - \sum_{j \in S_l^* \setminus \hat{S}_l} p_j(k_j^*)}_{>0} \\ > & \sum_{j=1}^{m} \sum_{y=1}^{k_j^*} p_j(y) \end{aligned}$$

which contradict with the optimality of $\{x_i^{*j}, k_j^*\}$. $\square$

We note that the result also holds when there is a limit to the number of routes offered by the operators.

Equivalently, a solution of $(G0)$ can be found by solving the following integer programme $(G1)$. Let

$$y_{jk} = \begin{cases} 1, & \text{if } k_j = k, \\ 0, & \text{otherwise.} \end{cases}$$

We also define

$$P_j(k) = \begin{cases} \sum_{z=1}^{k} p_j(z), & \text{for } k > 0, \\ 0, & \text{if } k = 0. \end{cases}$$

$$(G1): \qquad Max \sum_{j=1}^{m} \sum_{k=0}^{n} P_j(k) y_{jk}$$

$$\text{s.t.} \sum_{i=1}^{n} x_i^j = \sum_{k=0}^{n} k y_{jk}, \ \forall\, j = 1, \cdots, m, \tag{3}$$

$$\sum_{k=0}^{n} y_{jk} = 1, \ \forall\, j = 1, \cdots, m, \tag{4}$$

$$x_i^j \in \{0,1\}, \qquad \forall\, i = 1, \ldots, n, \ j = 1, \cdots, m. \tag{5}$$

Note that

$$\begin{aligned} P_j(k) &= p_j(1) + p_j(2) + \cdots + p_j(k) \\ &= (a_j - \delta_j) + (a_j/2 - \delta_j) + \cdots + (a_j/k - \delta_j) \end{aligned}$$

Therefore, $P_j(k)$ is at a maximum at

$$k_j^* = \arg\max\{h : p_j(h) > 0\}.$$

Observation 1:
Operators will continue to "enter the market" for a route until it is no longer profitable.

Therefore, as long as the marginal profit is nonnegative, additional operators will offer service on a route. This leads to better service for the customers, but reduces the profit for each operator. We compare this to a monopolistic setting below.

3.2 Basic Profit-Maximising Model

Consider the case when there is only one operator which wants to maximize the total profit. The optimal choice of routes to offer service is given by the solution to:

$$(G2): \qquad Max \sum_{j=1}^{m} \sum_{k=0}^{n} k p_j(k) y_{jk}$$

such that

$$\sum_{i=1}^{n} x_i^j = \sum_{k=0}^{n} k y_{jk}, \qquad \forall\ j = 1,\cdots,m, \tag{6}$$

$$\sum_{k=0}^{n} y_{jk} = 1, \qquad \forall\ j = 1,\cdots,m, \tag{7}$$

$$x_i^j \in \{0,1\}, \qquad \forall\ i = 1,\ldots,n,\ j = 1,\ldots,m, \tag{8}$$

$$y_{jk} \in \{0,1\}, \qquad \forall\ k = 0,\ldots,n,\ j = 1,\ldots,m. \tag{9}$$

Observation 2:
To maximize total profit, the optimal solution is to have at most one operator per route.

Proof. Since for $k \neq 0$,

$$kp_j(k) = k(a_j/k - \delta_j) = a_j - k\delta_j$$

the objective is maximised when $k_j = 1$ as long as $a_j > \delta_j$. □

We note that while the monopolistic profit-maximising solution is to assign only one operator per route, to determine an equitable allocation of routes to operators is not a simple task but requires solving a non-linear optimisation problem.

Observation 3:
The competitive equilibrium solution has as many as possible operators offering service on each route until it is no longer profitable to do so, whereas the centrally-controlled (total profit maximizing) solution has at most one operator per route.

3.3 Model with Player - and Route - Dependent Operating Cost

In reality, each operator might have different operating costs on different routes. Hence we extend our model to a congestion game with player- and route-specific payoff functions which admit a potential. In contrast to Milchtaich's work [13], our model is not a singleton game; players can choose more than one route in their strategy sets.

As with the basic model, the revenue is shared among all operators of a route, that is, the revenue to each player i serving route j is $r_j(k_j) = a_j/k_j$ for $k_j > 0$, where k_j is the number of players offering service on route j. Because of the cost structure, the payoff to player i offering service on route j is $p(i,j,k_j) = a_j/k_j - \delta_{ij}$, which depends on player i, route j and the total number of player choosing that particular route k_j. We next show that a Nash equilibrium of this problem can be found by solving the following auxiliary problem:

$$(GD1): \qquad Max \sum_{j=1}^{m}\sum_{y=1}^{k_j} r_j(y) - \sum_{j=1}^{m}\sum_{i=1}^{n} \delta_{ij} x_i^j$$

$$\text{s.t. } \sum_{i=1}^{n} x_i^j = k_j, \qquad \forall\ j = 1,\ldots,m, \tag{10}$$

$$x_i^j \in \{0,1\}, \qquad \forall i = 1,\ldots,n, j = 1,\ldots,m, \tag{11}$$

Proof. Let x^* be an optimal solution of $(GD1)$, and let $(S_1^*, S_2^*, \cdots, S_n^*)$ be the corresponding strategies of the players. If this is not an equilibrium solution, then for some player l, there exist another strategy $\hat{S}_l$such that

$$\pi_i(S_1^*,\cdots,S_{l-1}^*,\hat{S}_l,S_{l+1}^*,\cdots,S_n^*) > \pi_i(S_1^*,\cdots,S_{l-1}^*,S_l^*,S_{l+1}^*,\cdots,S_n^*),$$

that is,

$$\sum_{j\in \hat{S}_l \setminus S_l^*} \left(\frac{a_j}{(k_j^*+1)} - \delta_{lj} \right) - \sum_{j\in S_l^* \setminus \hat{S}_l} \left(\frac{a_j}{k_j^*} - \delta_{lj} \right) > 0 \tag{**}$$

Let $(\hat{x},\hat{k})$ be the solution of $(GD1)$ corresponding to $(S_1^*,\cdots,S_{l-1}^*,\hat{S}_l,S_{l+1}^*,\cdots,S_n^*)$. Then

$$\begin{aligned} & \sum_{j=1}^{m}\sum_{y=1}^{\hat{k}_j} r_j(y) - \sum_{j=1}^{m}\sum_{i=1}^{n} \delta_{ij}\hat{x}_i^j \\ = \ & \sum_{j=1}^{m}\sum_{y=1}^{k_j^*} r_j(y) - \sum_{j=1}^{m}\sum_{i=1}^{n} \delta_{ij}x_i^{*j} + \\ & \underbrace{\sum_{j\in \hat{S}_l \setminus S_l^*} \left(\frac{a_j}{(k_j^*+1)} - \delta_{lj} \right) - \sum_{j\in S_l^* \setminus \hat{S}_l} \left(\frac{a_j}{(k_j^*)} - \delta_{lj} \right)}_{> 0 \text{ by } (**)} \\ > \ & \sum_{j=1}^{m}\sum_{y=1}^{k_j^*} r_j(y) - \sum_{j=1}^{m}\sum_{i=1}^{n} \delta_{ij}x_i^{*j} \end{aligned}$$

Hence, we have another solution to $(GD1)$ with a better objective value contradicting the optimality of x^*. □

Similar to the basic model, $(GD1)$ can be formulated as an equivalent integer programme:

$$(GD2): \quad Max \sum_{j=1}^{m}\sum_{k=0}^{n} R_j(k) y_{jk} - \sum_{j=1}^{m}\sum_{k=0}^{n} \delta_{ij} x_i^j$$

$$\text{s.t. } \sum_{i=1}^{n} x_i^j = \sum_{k=0}^{n} k y_{jk},\ \forall\ j = 1,\cdots,m, \tag{12}$$

$$\sum_{k=0}^{n} y_{jk} = 1,\ \forall\ j = 1,\cdots,m, \tag{13}$$

$$x_i^j \in \{0,1\}, \quad \forall \quad i = 1,\ldots,n,\ j = 1,\cdots,m, \tag{14}$$

$$y_{jk} \in \{0,1\}, \quad \forall \quad k = 0,1,\ldots,n,\ j = 1,...,m, \tag{15}$$

where $R_j(k) = \sum_{y=1}^{k} r_j(k) = a_j + a_j/2 + \cdots + a_j/k$.

Observation 4:

As for the basic model, operators will continue to enter the market for a route until it is no longer profitable.

The number of operators for route j is

$$k_j^* = \max\{h : a_j/h - \delta_{[h]j} > 0\},$$

where $[\cdot]$ is a permutation of $\{1, 2, \cdots, n\}$ such that $\delta_{[1]j} \leq \delta_{[2]j} \leq \cdots \leq \delta_{[n]j}$.

4 NETWORK DESIGN

Using this initial framework, we have also explored the impact of the network structure on the profit for the service providers. We consider a service area with t townships and compared the equilibrium solution for a network structure where direct point-to-point services are offered between every pair of townships, to the equilibrium for a hub-and-spoke network where every route between any two townships involves an interchange via a central hub. The first one is a complete network with t nodes and $\sum_{u=1}^{t-1} = t(t-1)/2$ links, one for each origin-destination pair. In the second network structure, the routes offered are between a township and the central hub, and the total ridership from each origin (to all destinations) is consolidated into the ridership from the origin to the central hub. That is, the second is a hub-and-spoke network with $t+1$ nodes and t links. Figure 1 illustrate these two networks with $t = 6$. For each service provider, the profit from a route depends on the operating cost of offering the service, the total revenue due to the ridership and the number of competitors also servicing that route.

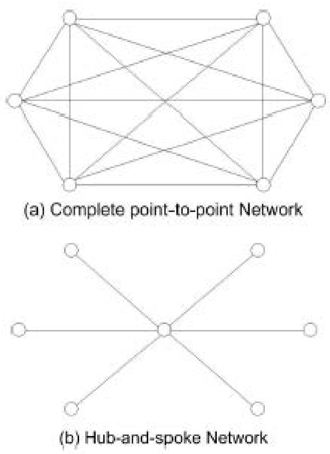

Figure 1: Network structures.

4.1 Hub-and-Spoke vs Complete Network

We consider a simple case where all origin-destination pairs have the same demand (that is, $a_j = a$ for all j) and all operators have the same cost for all routes (that is, $\delta_{ij} = \delta$ for all i and j). We assume there is no loss of ridership between i and j whether the route is direct or via a central hub. Therefore, all demand from a node i is aggregated to the demand on the link from the node to the hub. This is illustrated in Figure 2. For fair comparison between the

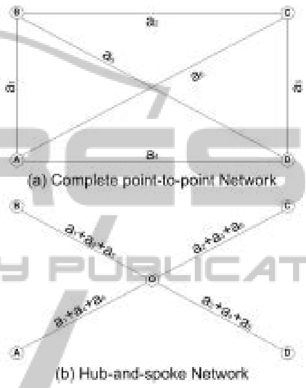

Figure 2: Demand on links.

two networks, the revenue on a route between a hub and an origin/destination is half that of a direct route between an origin and a destination. For the simplistic case where the ridership between every pair of townships are the same and all fixed operating costs are the same, the overall profits depends on the ratio of ridership to route operating cost. For a complete network on t nodes, the total profit in a competitive equilibrium is:

$$\pi_C = t(t-1)/2(a - \delta_k) \text{ where } k = \lfloor a/k \rfloor.$$

For the hub-and-spoke network, the total profit in a competitive equilibrium is:

$$\pi_H = t((t-1)a/2 - \delta h) \text{ where } h = \lfloor (t-1)a/2 \rfloor.$$

Therefore, π_H is greater than π_C if

$$\lfloor \frac{a(t-1)}{2\delta} \rfloor \leq \frac{t-1}{2} \lfloor \frac{a}{\delta} \rfloor.$$

It ca be observed that if the ratio a/δ is integer-valued and t is odd, we have $\pi_C = \pi_H$. When this is not the case, rounding has an impact, but not always favouring one network structure over the other, as illustrated in Figure 3.

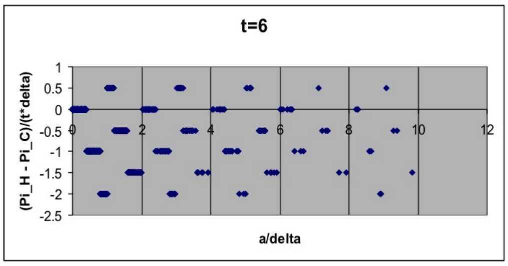

Figure 3: Rounding Effects.

5 SUMMARY AND FUTURE EXTENSIONS

In this paper, we investigate the competition among operators of public transit using the framework of congestion games. The Nash equilibria can be found by solving an auxiliary integer programme. Using this framework, we can draw some insights regarding competition vs. centralization, and the impact of network structure on the profit of the operators.

The assumption that the network structure and the total service bundle being offered will not affect the overall ridership is perhaps too restrictive and unrealistic. This assumption implies that all passenger who would travel from town X to town Y with direct service will still travel even if the trip involves an interchange via a central hub. Clearly, the convenience level, travel time and possibly travel cost will not be the same for the two trips. Also, apparently with wider service coverage and greater number of operators servicing a particular route (higher frequency), it will attract more ridership. A more realistic model would allow for ridership to depend on origin-destination pairs, on the network infrastructure and also on the set of transit services available. This may lead to a bi-level model where not only do operators compete with each other but the passengers preferences and patronage depends on the set of services offered by the transit operators. The *upper-level* model would represent the strategic game among the service providers as they select the services to be offered to maximise their individual profit. The *lower-level* is the game between the public and the operators as a group, in that the public may be diverted to other forms of transport (e.g. taxis, private vehicles) if the availability and service quality (e.g. interchanges required, circuitous routes, travel time) of the basket of services offered by the operators are too low. The two levels of the bi-level problem are interlinked since the choice of the public to utilise public transit or not would affect the potential ridership of the system and thus impact the potential profit of the operators.

The potential game framework is a non-cooperative game framework. Operators may consider to cooperate when their resources are limited (which is mostly true). One operator may offers services on a part of the network that serve as feeder links to the service provided by another operator, and vice versa. Developing a cooperative game-theoretic model may help us to compare and contrast the equilibrium solutions of both setting, the cooperative game and non-cooperative settings. In a cooperative game setting, it would also be interesting to investigate what is the appropriate profit-sharing scheme to induce higher profits or more comprehensive services for the public.

By investigating these extensions, we may obtain further insights into the relationship among the network infrastructure, competitive situation between operators and the impact on the type and level of services offered to the public. These relationships could further guide us in decision making on possible infrastructural investments and incentives to offer both operators and the riding public, which is very helpful for the government authorities and to ensure a public transit system that well-serves the public and benefits the community in terms of costs, convenience, quality, environmental impact and other concerns being designed.

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REFERENCES

Bell, M. G. H. (2000). A game theory approach to measuring the performance reliability of transport networks, *Transportation Research: Part B*, 34, 533-545.

Castelli, L., G. Longo, R. Pesenti and W. Ukovich (2004). Two-player noncooperative games over a freight transportation network, *Transportation Science*, 38(2), 149-159.

Chen, O. J. and M.E. Ben-Akiva (1998). Game-theoretic formulations of interaction between dynamic traffic control and dynamic traffic assignment, In *Transportation Research Record: Journal of the Transportation Research Board*, No. 1617, National Research Council, Washington, DC., pp. 179-188.

Fernandez, E. and P. Marcotte (1992) Operators-Users Equilibrium Model in a Partially Regulated Transit System, *Transportation Science*, 26(2): 93-105.

Fernandez L., J. E., P. Marcotte, S. Mondschein, J. Vera and A. Weintraub (1993). Solution Approaches to the Bus Operator Problem, *Transportation Research, Part B*, 27B(1): 1-11.

Fisk, C. S. (1984). Game theory and transportation systems modelling, *Transportation Research B*, 18B (4/5), 301-313.

James, T. (1998). A game theoretic model of road usage, *Proceedings of the 3rd IMA International Conference on Mathematics in Transport and Control*, 401-409.

Hollander, Y. and J. N. Prashker (2006). The applicability of non-cooperative game theory in transport analysis, *Transportation*, 33, 481-496.

Levinson, D. (2005). Micro-foundations of congestion and pricing: a game theory perspective, *Transportation Research: Part A*, 39, 691-704.

Monderer, D. and L. S. Shapley (1996). Potential Games, *Games and Economic Behaviour*, 14, 124-143.

Nash, J. (1950). Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences* 36(1):48-49.

Patriksson, M. and M. Labbé (eds.) (2004) *Transportation Planning: State of the Art*, Kluwer Academic Publishers.

Pedersen, P. A. (2003). Moral Hazard in traffic games, *Journal of Transport Economic Policy* 37 (1), 47-68.

Reyniers, D. (1992). Crowding levels and fare classes in public transport, In Griffiths, J. (ed.) *Mathematics in Transport planning and Control*, Oxford University Press, pp. 319-327.

Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria, *International Journal of Game Theory*, 2, 65-67.

Van Zuylen, H.J. and H. Taale (2004) . Urban network with ring roads: a two-level three player game, In: *Proceedings of the 83rd Annual Meeting of the Transportation Research Board*, Washington DC.

von Stackelberg, H. F. (1934). *Marktform und Gleichgewicht (Market and Equilibrium)*, Vienna.

Wardrop. J. G. (1952). Some theoretical aspects of road traffic research", *Proceedings of the Institution of Civil Engineers, Part II,* 1, 325-378.