A Model for Robotic Hand Based on Fibonacci Sequence

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Keywords: Robot Hand, Redundant Manipulator, Fibonacci Sequence, Discrete Control.

Abstract: We study a robot hand model involving Fibonacci sequence. Fingers are modeled via hyper-redundant planar manipulators. Binary controls rule the dynamics of the hand, in particular the extension and the rotation of each phalanx. By estabilishing a relation with Iterated Function Systems, we investigate the reachable workspace and its convex hull. Finally, we give an explicit characterization of the convex hull of the reachable workspace in a particular case.

1 INTRODUCTION

The aim of this paper is to give a model of robot hand whose links scale according to Fibonacci sequence and to develop a theoretical background (related to the theory of iterated function systems) in order to study some geometrical features of a such a manipulator. Fibonacci numbers attracted the interest of researchers due to their fascinating algebraic properties (e.g. the relation with Golden Mean) and due of their recurrence in natural phenomena. Examples of relations with Fibonacci sequence can be found in the branches of trees, in the arrangement of sunflowers seeds and, most interestingly for our model, in some human anatomic proportions (see (Hamilton and Dunsmuir, 2002)).

Self-similarity of configurations and an arbitrarily large number of fingers (including the opposable thumb) and phalanxes are the main features. Binary controls rule the dynamics of the hand, in particular the extension and the rotation of each phalanx. We assume that each finger moves on a plane; every plane is assumed to be parallel to the others, excepting the thumb and the index finger, that belong to the same plane. A discrete dynamical system models the position of the extremal junction of every finger. A *configuration* is a sequence of states of the system corresponding to a particular choice for the controls, while the union of all the possible states of the system is named *reachable workspace for the finger*. The closure of the reachable workspace is named *asymptotic reachable workspace*. Our model includes two binary control parameters on every phalanx of every finger of the robot hand. The first control parameter rules the lenght of the k-th phalanx, that can be either 0 or $f_k\rho^{-k}$, where f_k is the k-th Fibonacci number and ρ is a fixed scaling ratio, while the other control rules the angle between the current phalanx and the previous one. Such an angle can be either π, namely the phalanx is consecutive to the previous, or a fixed angle $\pi-\omega \in (0,\pi)$.

The structure of the finger ensures the set of possible configurations to be the projection of a particular self-similar set. This is the key idea underlying our model and our main tool of investigation.

We establish a connection between our model and the theory of iterated function systems. This yields several results describing the reachable workspace and its convex hull.

1.1 Previous Work and Motivations

The fingers of our robot hand are planar manipulators with rigid links and with a (arbitrarily) large number of degrees of freedom, that is they belong to the class of so-called macroscopically-serial hyper-redundant manipulators (the term was first introduced in (Chirikjian and Burdick, 1990)).

Hyper-redundant architecture was intensively studied back to the late 60's, when the first prototype of hyper-redundant robot arm was built (Anderson and Horn, 1967). The interest of researchers in devices with redundant controls was motivated, among others, by the ability to avoid obstacles and the ability to perform new forms of robot locomotion and grasping (see for instance (Chirikjian and Burdick, 1995)).

A large number of papers were devoted in the literature to both continuously and discretely controlled

Lai A., Loreti P. and Vellucci P..
A Model for Robotic Hand Based on Fibonacci Sequence.
DOI: 10.5220/0005115205770584
In *Proceedings of the 11th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2014), pages 577-584
ISBN: 978-989-758-040-6

hyper-redundant manipulators. Our approach, based on discrete actuators, is motivated by their precision with low cost compared to actuators with continuous range-of-motion. Moreover the resulting discrete space of configurations reduces the cost of position sensors and feedbacks. In (EbertUphoff and Chirikjian, 1996) the inverse kinematics of discrete hyper-redundant manipulators is investigated.

In general the number of points of the reachable workspace increases exponentially, the computational cost on the optimization of the density distribution of the workspace is investigated in (Lichter et al., 2002).

Note that the concept of a binary tree describing all the possible configurations underlies above mentioned approaches. In our method the self-similar structure of such a tree gives access to well-established results on fractal geometry and iterated function systems theory. Robotic devices with a similar fractal structure are described in (Moravec et al., 1996).

The relation between iterated function systems, expansions in non-integer bases and planar manipulator was investigated in (Lai and Loreti, 2011) and in (Lai and Loreti, 2013), by assuming the ratio between the lenghts of two consecutive links to be equal to a constant $\rho > 1$, so that the length l_k of the k-th link is equal to $1/\rho^k$. In the present paper we extend this investigation to the case $l_k = f_k/\rho^k$ where, as mentioned above, f_k is the k-th Fibonacci number.

This assumption yields a non-trivial generalization of the purely self-similar case $l_k = 1/\rho^k$ and, on the other hand, aims to mimic the recurrence of Fibonacci sequence in proportions of human limbs (Hamilton and Dunsmuir, 2002). In our model every link (phalanx) is controlled by a couple of binary controls. The control of the rotation at every joint is a common feature of all above mentioned manipulators. The study of a control ruling the extension of every link has twofold applications. In one hand it can be physically implemented by means of telescopic links, that are particularly efficient in constrained workspaces (see (Aghili and Parsa, 2006)). On the other hand, our model can be considered a discrete approximation of continuous snake-like manipulators - see for instance (Andersson, 2008).

1.2 Organization of Present Paper

The paper is organized as follows. In Section 2 we introduce the model. A characterization of the asymptotic reachable workspace via Iterated Function Systems is given Section 3. In Section 4 we describe the convex hull of the asymptotic reachable set and we explicitely characterize it in a particular case.

2 THE MODEL

In our model the robot hand is composed by H fingers, every finger has an arbitrary number of phalanxes. We assume junctions and phalanxes of each finger to be thin, so to be respectively approximated with their middle axes and barycentres and we also assume the junctions of every finger to be coplanar. Inspired by the human hand, we set the fingers of our robot as follows: the first two fingers are coplanar and they have in common their first junction (they are our robotic version of the thumb and the index finger of the human hand) while the remaining $H-1$ fingers belong to parallel planes. By choosing an appropriate coordinate system $oxyz$ we may assume that the first two fingers belong to the plane $p^{(1)} : z = 0$ while, for $h \geq 2$, h-th finger belongs to the plane $p^{(h)} : z = z_0^{(h)}$ for some $z_0^{(2)}, \ldots, z_0^{(H)} \in \mathbb{R}$.

We now describe in more detail the model of a robot finger. A configuration of a finger is the sequence $(\mathbf{x}_k)_{k=0}^K \subset \mathbb{R}^3$ of its junctions. The configurations of every finger are ruled by two phalanx-at-phalanx motions: extension and rotation. In particular, the length of k-th phalanx of the finger is either 0 or $\dfrac{f_k}{\rho^k}$, where f_k is the k-th fibonacci number, namely

$$\begin{cases} f_0 = f_1 = 1; \\ f_{k+2} = f_{k+1} + f_k \quad k \geq 0. \end{cases} \tag{1}$$

while $\rho > 1$ is a fixed ratio: this choice is ruled by a binary control we denote by using the symbol u_k, so that the lenght l_k of the k-th phalanx is

$$l_k := \| \mathbf{x}_k - \mathbf{x}_{k-1} \| = \frac{u_k f_k}{\rho^k}.$$

As all phalanxes of a finger belong to the same plane, say p, in order to describe the angle between two consecutive phalanxes, say the $k-1$-th and the k-th phalanx, we just need to consider a one-dimensional parameter, ω_k. Each phalanx can lay on the same line as the former or it can form with it a fixed planar angle $\omega \in (0, \pi)$, whose vertex is the $k-1$-th junction. In other words, two consecutive phalanxes form either the angle π or $\pi - \omega$. By introducing the binary control v_k we have that the angle between the $k-1$-th and k-th phalanx is $\pi - \omega_k$, where

$$\omega_k = v_k \omega.$$

To describe the kinematics of the finger we adopt the Denavit-Hartenberg (DH) convention. To this end, first of all recall that our base coordinate frame $oxyz$ is such that oxy is parallel to p (hence to every plane $p^{(h)}$) and we consider the finger coordinate frame

$o_I x_I y_I z_I$ associated to the 4×4 homogeneous transform

$$A_I=\begin{pmatrix}\cos\omega_I & -\sin\omega_I & 0 & x_I\\ \sin\omega_I & \cos\omega_I & 0 & y_I\\ 0 & 0 & 1 & z_0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

for some $\omega_I\in[0,2\pi)$. In particular if $\mathbf{x}$ and $\mathbf{x}_0$ are respectively coordinates of a point with respect to $oxyz$ and $o_I x_I y_I z_I$ then

$$\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}=A_I\begin{pmatrix}\mathbf{x}_0\\ 1\end{pmatrix}.$$

Remark 1. *When only one finger is considered one may assume the base coordinate frame to coincide with the finger coordinate frame: this reduces A_I to the identity and it could be omitted in the model. The need of a coordinate frame for the finger rises when more than one finger, especially in the case of coplanar, opposable fingers, is considered.*

Now, the (DH) method consists in attaching to every phalanx, say the k-th phalanx, a coordinate frame $o_k x_k y_k z_k$, so that $\mathbf{x}_k$ coincides with o_k and $\mathbf{x}_k-\mathbf{x}_{k-1}$ is parallel to $o_k x_k$. Note that the coordinates of $\mathbf{x}_{k+1}$ with respect to $o_k x_k y_k z_k$ are $(\frac{u_{k+1}f_{k+1}}{\rho^{k+1}}\cos\omega_{k+1},\frac{u_{k+1}f_{k+1}}{\rho^{k+1}}\sin\omega_{k+1},0)$.

Since we are considering a planar manipulator, for every $k>1$ the geometric relation between the coordinate systems the $k-1$-th and the k-th phalanx is expressed by the matrix

$$A_k:=\begin{pmatrix}\cos\omega_k & -\sin\omega_k & 0 & \frac{u_k f_k}{\rho^k}\cos\omega_k\\ \sin\omega_k & \cos\omega_k & 0 & -\frac{u_k f_k}{\rho^k}\sin\omega_k\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

where the rotation matrix

$$\begin{pmatrix}\cos\omega_k & -\sin\omega_k & 0\\ \sin\omega_k & \cos\omega_k & 0\\ 0 & 0 & 1\end{pmatrix}$$

represents the rotation of the coordinate frame $o_k x_k y_k z_k$ with respect to $o_{k-1}x_{k-1}y_{k-1}z_{k-1}$ and the vector $\left(\frac{u_k f_k}{\rho^k}\cos\omega_k,-\frac{u_k f_k}{\rho^k}\sin\omega_k,0\right)$ represent the position of o_k with respect to $o_{k-1}x_{k-1}y_{k-1}z_{k-1}$.

Set

$$T_k:=A_I\prod_{j=0}^{k}A_j.$$

By definition T_k is the composition the transforms $A_I,A_0,\dots,A_k$ and, consequently, it represents the relation between the base coordinate frame $oxyz$ and $o_k x_k y_k z_k$. In particular

$$T_k=\begin{pmatrix}R_k & P_k\\ 0 & 1\end{pmatrix}$$

where R_k is a 3×3 rotation matrix and the entries of the vector P_k are the coordinates of $o_k(=x_k)$ in the reference system $oxyz$. Expliciting T_k one has

$$R_k=$$
$$\begin{pmatrix}\cos\left(\omega_I+\sum_{j=0}^{k}\omega_j\right) & -\sin\left(\omega_I+\sum_{j=0}^{k}\omega_j\right) & 0\\ \sin\left(\omega_I+\sum_{j=0}^{k}\omega_j\right) & \cos\left(\omega_I+\sum_{j=0}^{k}\omega_j\right) & 0\\ 0 & 0 & 1\end{pmatrix}$$

and

$$P_k=P_I+\sum_{j=0}^{k}R_j\begin{pmatrix}\frac{u_j f_j}{\rho^j}\\ 0\\ 0\end{pmatrix}=$$
$$\begin{pmatrix}x_I+\sum_{j=0}^{k}\frac{u_j f_j}{\rho^j}\cos\left(\sum_{n=0}^{j}\omega_n\right)\\ y_I-\sum_{j=0}^{k}\frac{u_j f_j}{\rho^j}\sin\left(\sum_{n=0}^{j}\omega_n\right)\\ z_I\end{pmatrix}$$

Then, for every $k\geq 0$

$$\begin{pmatrix}x_k\\ y_k\\ z_k\end{pmatrix}=\begin{pmatrix}x_I+\sum_{j=0}^{k}\frac{u_j f_j}{\rho^j}\cos\left(\sum_{n=1}^{j}\omega_n\right)\\ y_I-\sum_{j=0}^{k}\frac{u_j f_j}{\rho^j}\sin\left(\sum_{n=1}^{j}\omega_n\right)\\ z_I\end{pmatrix}\qquad(2)$$

3 CHARACTERIZATION OF THE REACHABLE WORKSPACE VIA ITERATED FUNCTION SYSTEMS

We fix as initial state $(x_I,y_I,z_I)=(0,0,0)$ and assume $\omega_I=0$. By employing the isometry between $\mathbb{R}^2$ and $\mathbb{C}$ and by considering that our manipulator is essentially planar, we may rewrite (2) as

$$\begin{cases}x_k=\sum_{j=0}^{k}\frac{u_j f_j}{\rho^j}e^{-i\omega\sum_{n=0}^{j}v_n}\\ x_I=0.\end{cases}\qquad(3)$$

We aim to study the *asymptotic reachable workspace*

$$R_\infty\doteq\left\{\sum_{k=0}^{\infty}\frac{u_k f_k}{\rho^k}e^{-i\omega\sum_{j=0}^{k}v_j}|(v_j),(u_j)\in\{0,1\}^\infty\right\}$$

In order to have a more compact notation, infinite binary (control) sequences (u_j) and (v_j) are equivalently denoted by $\mathbf{u}$ and $\mathbf{v}$, respectively. We set

$$x(\mathbf{u},\mathbf{v})\doteq\sum_{k=0}^{\infty}\frac{u_k f_k}{\rho^k}e^{-i\omega\sum_{j=0}^{k}v_j}$$

and we define the *shift operator* on R_∞

$$\sigma : x(\mathbf{u},\mathbf{v}) \mapsto x(\sigma(\mathbf{u}),\sigma(\mathbf{v}))$$

so that if $x = x(\mathbf{u},\mathbf{v})$ then

$$\sigma(x) = \sum_{k=0}^{\infty} \frac{u_{k+1} f_k}{\rho^k} e^{-i\omega \sum_{j=0}^{k} v_{j+1}}.$$

Finally we define the auxiliary set

$$Q_\infty = \{(x,\sigma(x)) \mid x = x(\mathbf{u},\mathbf{v});\ \mathbf{u},\ \mathbf{v} \in \{0,1\}^\infty\}.$$

Note that $Q_\infty \in R_\infty \times R_\infty$ and $\pi(Q_\infty) = R_\infty$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection of a bidimensional complex vector on its first component.

We characterize Q_∞ and, consequently, R_∞ via the linear maps $F_{00}, F_{10}, F_{01}, F_{11} : \mathbb{C}^2 \to \mathbb{C}^2$ defined as follows

$$F_{uv}(z) = e^{-i\omega v}\left(A_\rho z + \begin{pmatrix} u \\ 0 \end{pmatrix}\right) \quad \text{for } u,v \in \{0,1\}$$

where $z \in \mathbb{C}^2$ and

$$A_\rho \doteq \begin{pmatrix} \frac{1}{\rho} & \frac{1}{\rho^2} \\ 1 & 0 \end{pmatrix}.$$

In order to describe the action of F_{uv}'s on Q_∞, for any $\mathbf{u},\mathbf{v} \in \{0,1\}^\infty$ set $\bar{\mathbf{u}}(u) \doteq u\mathbf{u}$ and $\bar{\mathbf{v}}(v) \doteq v\mathbf{v}$. In other words

$$\bar{u}_k(u) = \begin{cases} u & \text{if } k = 0 \\ u_{k-1} & \text{otherwise}. \end{cases}$$

$$\bar{v}_k(v) = \begin{cases} v & \text{if } k = 0 \\ v_{k-1} & \text{otherwise}. \end{cases}$$

Lemma 2. *Let $\mathbf{u},\mathbf{v} \in \{0,1\}^\infty$, $u,v \in \{0,1\}$. Set $x = x(\mathbf{u},\mathbf{v})$ and $\bar{x} = x(\bar{\mathbf{u}}(u),\bar{\mathbf{v}}(v))$. One has*

$$F_{uv}(x,\sigma(x)) = (\bar{x},\sigma(\bar{x})) = (\bar{x},x). \tag{4}$$

Remark 3. *F_{uv} acts on $x(\mathbf{u},\mathbf{v})$ by prepending to the control sequences $\mathbf{u}$ and $\mathbf{v}$ the controls u and v. Lemma 2 also implies that $F_{uv}(Q_\infty) \subset Q_\infty$ for every $u,v \in \{0,1\}$.*

Proof of Lemma 2. By definition of F_{uv} and of σ, and recalling $\sigma(\bar{\mathbf{u}}(u)) = \mathbf{u}$ and $\sigma(\bar{\mathbf{v}}(v)) = \mathbf{v}$, one has

$$F_{uv}((x,\sigma(x)) = \left(e^{-i\omega v}\left(\frac{1}{\rho}x + \frac{1}{\rho^2}\sigma(x) + u\right), \sigma(\bar{x})\right).$$

Then it is left to prove

$$e^{-i\omega v}\left(\frac{1}{\rho}x + \frac{1}{\rho^2}\sigma(x) + u\right) = \bar{x}.$$

Recalling $f_0 = f_1$, one has

$$\begin{aligned}
& e^{-i\omega v}\left(\frac{1}{\rho}x + \frac{1}{\rho^2}\sigma(x) + u\right) \\
&= ue^{-i\omega v} + \sum_{k=0}^{\infty} \frac{u_k f_k}{\rho^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&+ \sum_{k=0}^{\infty} \frac{u_{k+1} f_k}{\rho^{k+2}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= ue^{-i\omega v} + \frac{u_0 f_0}{\rho} e^{-i\omega(v_0 + v)} \\
&+ \sum_{k=1}^{\infty} \frac{u_k f_{k+1}}{\rho^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= uf_0 e^{-i\omega v} + \frac{u_0 f_1}{\rho} e^{-i\omega(v_0 + v)} \\
&+ \sum_{k=1}^{\infty} \frac{u_k f_{k+1}}{\rho^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= \frac{\bar{u}_0 f_0}{e}^{-i\omega \bar{v}_0} + \frac{\bar{u}_1 f_1}{\rho} e^{-i\omega \bar{v}_1} + \sum_{k=2}^{\infty} \frac{\bar{u}_k f_k}{\rho^k} e^{-i\omega \sum_{j=0}^{k} \bar{v}_j} \\
&= \sum_{k=0}^{\infty} \frac{\bar{u}_k f_k}{\rho^k} e^{-i\omega \sum_{j=0}^{k} \bar{v}_j} \\
&= \bar{x}.
\end{aligned}$$

□

Before stating next result, we recall that throughtout this paper the scaling ratio ρ is assumed greater than the Golden Mean.

Proposition 4. *Q_∞ is the unique compact subset of $\mathbb{C}^2$ satisfying*

$$\bigcup_{u,v \in \{0,1\}} F_{uv}(Q_\infty) = Q_\infty. \tag{5}$$

Proof. First of all we show (5) by double inclusion. The inclusion $\subseteq$ directly follows by Lemma 2 – see also Remark 3. Thus it suffices to show that for every $\bar{x} \in R_\infty$ there exist $x \in R_\infty$ and $u,v \in \{0,1\}$ such that

$$F_{uv}(x,\sigma(x)) = (\bar{x},\sigma(\bar{x})).$$

Let $\bar{\mathbf{u}},\bar{\mathbf{v}} \in \{0,1\}^\infty$ be a couple of the control sequences satisfying $\mathbf{x} = x(\mathbf{u},\mathbf{v})$. Then, again by Lemma 2,

$$F_{\bar{u}_1\bar{v}_1}(\sigma(\bar{x})),\sigma(\sigma(\bar{x})))) = (\bar{x},\sigma(\bar{x})).$$

Since R_∞ is closed with respect to σ, then $x \doteq \sigma(\bar{x}) \in R_\infty$ and this completes the proof of (5).

Now, let us prove the uniqueness of Q_∞. First of all we note that for every $u,v \in \{0,1\}$, F_{uv} is a linear map and consider its spectral radius $R(\rho)$. $R(\rho)$ is hence the greatest modulus of the eigenvalues of A_ρ. If $\rho > \varphi$, where φ is the Golden Mean, then $R(\rho) = \frac{\sqrt{5}\rho+\rho}{2\rho^2} < 1$. Consequently the induced norm of A_ρ^k

$$||A_\rho^k|| \doteq \max_{z \in \mathbb{C}^2, z \neq (0,0)-} ||A_\rho^k z|| \to 0 \quad \text{as } k \to \infty.$$

Then there exists k_ρ such that if $k \geq k_\rho$ then F_{uv}^k is a contraction. Since the quantity k_ρ is independent on u and v, one has that any concatenation of length $k \geq k_\rho$ of F_{uv}'s, say

$$G_{\mathbf{u}^k,\mathbf{v}^k} \doteq F_{u_1^k v_1^k} \circ \cdots \circ F_{u_k^k v_k^k},$$

is a contraction. Consequently one may consider the *Hutchinson operator*

$$\mathcal{G}(\cdot) \doteq \bigcup_{\mathbf{u}^k,\mathbf{v}^k \in \{0,1\}^k} G_{\mathbf{u}^k,\mathbf{v}^k}(\cdot)$$

and deduce by (5)

$$\mathcal{G}(Q_\infty) = Q_\infty. \tag{6}$$

Since $\mathcal{G}$ is generated a finite set of contractive maps, namely by an Iterated Function System, then

$$\begin{array}{c} Q_\infty \text{ is the only compact subset} \\ \text{of } \mathbb{C}^2 \text{ enjoying (6)} \end{array} \tag{U}$$

In order to seek a contradiction, assume now that there exists a compact set $X \subset \mathbb{C}^2$ different from Q_∞ satisfying (5). Then X also satisfies (6): the uniqueness condition (U) provides the required contradiction and concludes the proof. □

Remark 5. *For an estimate of k_ρ, see (Lai et al., 2014).*

4 CHARACTERIZATION OF THE CONVEX HULL OF THE REACHABLE WORKSPACE

Throughtout this section we employ Proposition 4 in order to characterize the $co(R_\infty)$, co denoting the convex hull of a set. We begin by the following general fact.

Lemma 6. *Let $\{F_1,\ldots,F_H\}$ be a finite set of linear maps on a metric space X and assume that there exists and it is unique a compact set Q satisfying*

$$\mathcal{F}(Q) \doteq \bigcup_{h=0}^{H} F_h(Q) = Q.$$

If

$$\mathcal{F}(Y) \subseteq Y \tag{7}$$

for some $Y \subset X$ then

$$Q \subseteq Y \tag{8}$$

Proof. By iterating (7) for one has for every k

$$Y \supseteq \mathcal{F}(Y) \supseteq \mathcal{F}^2(Y) \supseteq \cdots \supseteq \mathcal{F}^k(Y)$$

then as $k \to \infty$, the set sequence $\mathcal{F}^k(Y)$ converges to a set $\bar{Y}$ satisfying

$$\mathcal{F}(\bar{Y}) = \bar{Y} \subseteq Y.$$

By the uniqueness of Q one has $\bar{Y} = Q$ and this completes the proof. □

Theorem 7. *With the notations of previous Section, let $V \subset Q_\infty$ be such that*

$$\mathcal{F}(V) \doteq \bigcup_{u,v \in \{0,1\}} F_{uv}(V) \subseteq co(V). \tag{9}$$

Then

$$co(R_\infty) = co(\pi(V)).$$

Proof. The linearity of F_{uv}'s and (9) imply

$$\mathcal{F}(co(V)) \subseteq co(V). \tag{10}$$

This together with Proposition 4, implies that we may apply Lemma 6 to Q_∞ and $Y = co(V)$ and deduce $Q_\infty \subseteq co(V)$. By assumption we also have $V \subset Q_\infty$, then $co(Q_\infty) = co(V)$. The claim hence follows by the fact that $R_\infty = \pi(Q_\infty)$ and that projection is a convex map.

□

Next result gives a more operative description of $co(R_\infty)$.

Theorem 8. *Let W be a compact subset of Q_∞. If*

$$\pi(\mathcal{F}(W)) \subseteq \pi(co(W)) \tag{11}$$

then

$$co(R_\infty) = co(\pi(W)).$$

Proof. We show the claim by double inclusion. The inclusion $\supseteq$ is trivial, since we assumed $W \subseteq Q_\infty$ and, consequently $\pi(W) \subseteq R_\infty$. Now we show by induction that if (11) holds then for every k

$$\pi(\mathcal{F}^k(W)) \subseteq \pi(co(W)). \tag{12}$$

The case $k = 1$ is given by (11) itself. We then assume as inductive hypothesis

$$\pi(\mathcal{F}^{k-1}(W)) \subseteq \pi(co(W))$$

so that we get for every $(\hat{w}, \sigma(\hat{w})) \in W$

$$\mathcal{F}^k(\hat{w}) = \mathcal{F}(w, \sigma(w)) \quad \text{with } w \in \pi(co(W))$$

In particular, $w = \sum_k \lambda_k w_k$ for some $w_k \in \pi(W)$ and some convex combinators λ_k. Since $W \subseteq Q_\infty$, if $w_k \in \pi(W)$ then $(w_k, \sigma(w_k)) \in W$. Then, by (11)

$$\pi(\mathcal{F}^k(\hat{w})) = \sum_k \lambda_k \pi(\mathcal{F}(w_k, \sigma(w_k))) \subseteq co(\pi(W)). \tag{13}$$

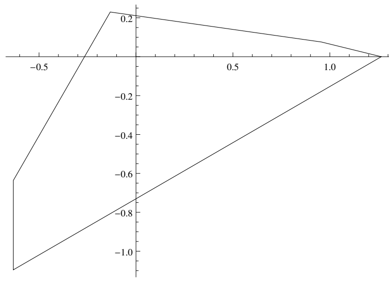

Figure 1: Convex hull of R_∞ with $\rho = \varphi + 1$.

Now, note that $\mathcal{F}^k(W)$ is a non-decreasing sequence of compact sets, consequently as $k \to \infty$ it tends to some compact set $\bar{W}$ satisfying $\mathcal{F}(\bar{W}) = \bar{W}$. By Proposition 4 we get $\bar{W} = Q_\infty$. Consequently

$$R_\infty = \pi(Q_\infty) = \lim_{k\to\infty} \pi(\mathcal{F}^k(W)) \subseteq \pi(co(W)) \quad (14)$$

The claim follows by noting that above inclusion implies $co(R_\infty) \subseteq \pi(co(W))$. □

4.1 Explicit Description of $co(R_\infty)$ in a Particular Case

We now consider the case $\omega = \pi/3$ and we show that $co(R_\infty)$ is a polygon whose vertices are

$\mathbf{v}_1 \doteq \sum_{k=0}^{\infty} \frac{f_k}{\rho^k}$; $\mathbf{v}_2 \doteq e^{-i\omega} \sum_{k=0}^{\infty} \frac{f_k}{\rho^k}$;
$\mathbf{v}_3 \doteq e^{-i\omega} + e^{-i2\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k}$; $\mathbf{v}_4 \doteq e^{-i2\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k}$;
$\mathbf{v}_5 \doteq 1 + e^{-i2\omega} \sum_{k=2}^{\infty} \frac{f_k}{\rho^k}$.

(see Figure 1).

In order to apply Theorem 8, it is useful to introduce the symbols **0** and **1** to denote infinite sequences of 0's and 1's, respectively, and to note that

$\mathbf{v}_1 = x(\mathbf{1}, \mathbf{0})$; $\mathbf{v}_2 = x(\mathbf{1}, 1\mathbf{0})$;
$\mathbf{v}_3 = x(\mathbf{1}, 11\mathbf{0})$; $\mathbf{v}_4 = x(0\mathbf{1}, 11\mathbf{0})$;
$\mathbf{v}_5 = x(10\mathbf{1}, 011\mathbf{0})$;

so that, recalling the definition $\sigma(x(\mathbf{u},\mathbf{v})) = x(\sigma(\mathbf{u}),\sigma(\mathbf{v}))$ where $\sigma(\mathbf{u})$ denotes the unit shift of $\mathbf{u}$, one gets

$\sigma(\mathbf{v}_1) = \mathbf{v}_1$ $\quad \sigma(\mathbf{v}_2) = \mathbf{v}_1$;
$\sigma(\mathbf{v}_3) = \sigma(\mathbf{v}_4) = \mathbf{v}_2$ $\quad \sigma(\mathbf{v}_5) = \mathbf{v}4$.

Let $W = \{(\mathbf{v}_h, \sigma(v_h) \mid h = 1, \ldots, 5\}$. By Theorem 8 one has

$$co(R_\infty) = co(\{\mathbf{v}_h \mid h = 1, \ldots, 5\})$$

if for every $h = 1, \ldots, 5$ and for every $u, v \in \{0, 1\}$

$$\pi(F_{uv}(\mathbf{v}_h, \sigma(\mathbf{v}_h))) \in \pi(co(W)). \quad (15)$$

We will show above inclusion by distinguishing the cases $h = 1, \ldots, 5$, but first we remark that $(0,0) \in co(V)$. Consequently if $\mathbf{z} \in V$ then for every $c \in [0,1]$, $c\mathbf{z} \in co(V)$.

Claim 1.

$$\pi(F_{uv}(\mathbf{v}_1, \sigma(\mathbf{v}_1))) \in co(V). \quad (16)$$

Proof. First notice

$$\pi(F_{uv}(\mathbf{v}_1, \sigma(\mathbf{v}_1))) = x(u\mathbf{1}, v\mathbf{0})$$

Thus

$$x(u\mathbf{1}, v\mathbf{0}) = ue^{-i\omega v} + \sum_{k=1}^{\infty} \frac{f_k}{\rho^k}$$

Case 1.1; $u = v = 0$: one has

$$x(0\mathbf{1}, 0\mathbf{0}) = \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = \mathbf{v}_1 - 1$$

Inclusion (16) hence follows by $\mathbf{v}_1 > 1$.
Case 1.2; $u = 0, v = 1$: one has

$$x(0\mathbf{1}, 1\mathbf{0}) = e^{-i\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = \mathbf{v}_2 - e^{-i\omega} = c\mathbf{v}_2$$

where $c = (1 - 1/\mathbf{v}_1)$ (indeed $\mathbf{v}_2 = e^{-i\omega}\mathbf{v}_1$).
Case 1.3; $u = 1, v = 0$: imediate, indeed

$$x(1\mathbf{1}, 0\mathbf{0}) = 1 + \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = \mathbf{v}_1.$$

Case 1.4; $u = 1, v = 1$: imediate, indeed

$$x(1\mathbf{1}, 1\mathbf{0}) = e^{-i\omega} + e^{-i\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = \mathbf{v}_2.$$

□

Claim 2.

$$\pi(F_{uv}(\mathbf{v}_2, \sigma(\mathbf{v}_2))) \in co(V). \quad (17)$$

Proof. First notice

$$\pi(F_{uv}(\mathbf{v}_2, \sigma(\mathbf{v}_2))) = x(u\mathbf{1}, v1\mathbf{0})$$

Thus

$$x(u\mathbf{1}, v1\mathbf{0}) = ue^{-i\omega v} + e^{-i\omega(v+1)} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k}$$

Case 2.1; $u = v = 0$:

$$x(0\mathbf{1}, 01\mathbf{0}) = e^{-i\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = x(0\mathbf{1}, 1\mathbf{0})$$

Then (17) follows by Claim 1, see Case 1.2.
Case 2.2; $u = 0, v = 1$: imediate, indeed

$$x(0\mathbf{1}, 11\mathbf{0}) = e^{-i2\omega} \sum_{k=1}^{\infty} \frac{f_k}{\rho^k} = \mathbf{v}_4.$$

Case 2.3; $u=1, v=0$:

$$\begin{aligned} x(1\mathbf{1},01\mathbf{0}) &= 1+e^{-i\omega}\sum_{k=1}^{\infty}\frac{f_k}{\rho^k} \\ &= \mathbf{v}_2+1-e^{-i\omega} \\ &= c\mathbf{v}_1+(1-c)\mathbf{v}_2 \end{aligned}$$

with $c=1/\mathbf{v}_1$.

Case 2.4; $u=1, v=1$: imediate, indeed

$$x(1\mathbf{1},11\mathbf{0}) = e^{-i\omega}+e^{-i2\omega}\sum_{k=1}^{\infty}\frac{f_k}{\rho^k}=\mathbf{v}_3.$$

$\square$

Claim 3.

$$\pi(F_{uv}(\mathbf{v}_3,\sigma(\mathbf{v}_3)))\in co(V). \tag{18}$$

Proof. First notice

$$\pi(F_{uv}(\mathbf{v}_3,\sigma(\mathbf{v}_3))) = x(u\mathbf{1},v11\mathbf{0})$$

hence

$$\begin{aligned} x(u\mathbf{1},v11\mathbf{0}) = ue^{-i\omega v} &+ \frac{e^{-i\omega(v+1)}}{\rho} \\ &+e^{-i\omega(v+2)}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k} \end{aligned}$$

Case 3.1; $u=v=0$:

$$\begin{aligned} x(0\mathbf{1},011\mathbf{0}) &= \frac{e^{-i\omega}}{\rho}+e^{-i2\omega}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k} \\ &= c_1\mathbf{v}_2+c_2\mathbf{v}_4 \end{aligned}$$

with $c_1=1/(\rho\mathbf{v}_1)$ and $c_2=1-1/(\rho|\mathbf{v}_4|)$. The claim hence follows by the fact that $c_1+c_2\in[0,1]$ and $0\in co(V)$.

Case 3.2; $u=0, v=1$: recalling $\omega=2\pi/3$

$$\begin{aligned} x(0\mathbf{1},111\mathbf{0}) &= \frac{e^{-i2\omega}}{\rho}+\sum_{k=2}^{\infty}\frac{f_k}{\rho^k}= \\ &= c_1\mathbf{v}_1+c_2\mathbf{v}_4 \end{aligned}$$

where $c_1=1-1/\mathbf{v}_1-1/(\rho\mathbf{v}_1)$ and $c_2=1-1/(\rho|\mathbf{v}_4|)$. Now, notice that $\mathbf{v}_1=\rho^2/(\rho^2-\rho-1)$ and $|\mathbf{v}_4|=\mathbf{v}_1-1$, consequently $c_1+c_2\in[0,1]$. The claim hence follows by $0\in co(V)$.

Case 3.3; $u=1, v=0$:

$$\begin{aligned} x(1\mathbf{1},011\mathbf{0}) &= 1+\frac{e^{-i\omega}}{\rho}+e^{-i2\omega}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k} \\ &= c_1\mathbf{v}_1+c_2\mathbf{v}_2+(1-c_1-c_2)\mathbf{v}_4 \end{aligned}$$

with $c_1=\frac{\rho^3-2\rho-1}{\rho(\rho^2+\rho+1)}$ and $c_2=\frac{\rho^2+1}{\rho(\rho^2+\rho+1)}$. The claim hence follows by $c_1,c_2\in[0,1]$.

Case 3.4; $u=1, v=1$:

$$\begin{aligned} x(1\mathbf{1},111\mathbf{0}) &= e^{-i\omega}+\frac{e^{-i2\omega}}{\rho}+e^{-i3\omega}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k} \\ &= c_1\mathbf{v}_1+c_2\mathbf{v}_2+(1-c_1-c_2)\mathbf{v}_3 \end{aligned}$$

with $c_1=\frac{2\rho+1}{\rho^3}$ and $c_2=\frac{(2\rho+1)(\rho^2-\rho-1)}{\rho^3(\rho+1)}$. The claim hence follows by $c_1,c_2\in[0,1]$.

$\square$

Claim 4.

$$\pi(F_{uv}(\mathbf{v}_4,\sigma(\mathbf{v}_4)))\in co(V). \tag{19}$$

Proof. First notice

$$\pi(F_{uv}(\mathbf{v}_4,\sigma(\mathbf{v}_4))) = x(u0\mathbf{1},v11\mathbf{0})$$

Thus

$$x(u0\mathbf{1},v11\mathbf{0}) = ue^{-i\omega v}+e^{-i\omega(v+2)}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k}$$

Case 4.1; $u=v=0$: the claim follows by $0\in co(V)$, indeed

$$x(00\mathbf{1},111\mathbf{0}) = e^{-i2\omega}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k}=\mathbf{v}_4\left(1-\frac{1}{\rho|v_4|}\right).$$

Case 4.2; $u=0, v=1$: the claim follows by $0\in co(V)$ and $\omega=2\pi/3$, indeed

$$x(00\mathbf{1},111\mathbf{0}) = \sum_{k=2}^{\infty}\frac{f_k}{\rho^k}<\mathbf{v}_1.$$

Case 4.3; $u=1, v=0$:

$$x(10\mathbf{1},011\mathbf{0}) = 1+e^{-i2\omega}\sum_{k=2}^{\infty}\frac{f_k}{\rho^k}=\mathbf{v}_5$$

Case 4.4; $u=1, v=1$: inclusion (19) follows by

$$\begin{aligned} x(10\mathbf{1},111\mathbf{0}) &= e^{-i\omega}+\sum_{k=2}^{\infty}\frac{f_k}{\rho^k} \\ &= \frac{1}{\mathbf{v}_1}\mathbf{v}_2+\left(1-\frac{1}{\mathbf{v}_1}-\frac{1}{\rho\mathbf{v}_1}\right)\mathbf{v}_1. \end{aligned}$$

$\square$

Claim 5.

$$\pi(F_{uv}(\mathbf{v}_5,\sigma(\mathbf{v}_5)))\in co(V). \tag{20}$$

Proof. First notice

$$\pi(F_{uv}(\mathbf{v}_4,\sigma(\mathbf{v}_5))) = x(u10\mathbf{1},v011\mathbf{0})$$

Thus

$$\begin{aligned} x(u10\mathbf{1},v011\mathbf{0}) =& ue^{-i\omega v}+\frac{1}{\rho}e^{-i\omega(v+1)} \\ &+e^{-i\omega(v+2)}\sum_{k=3}^{\infty}\frac{f_k}{\rho^k}. \end{aligned}$$

Case 5.1; $u=v=0$:

$$\begin{aligned} x(010\mathbf{1},0011\mathbf{0}) &= \frac{1}{\rho}+e^{-i2\omega}\sum_{k=3}^{\infty}\frac{f_k}{\rho^k} \\ &= c_1\mathbf{v}_1+c_2\mathbf{v}_4 \end{aligned}$$

with $c_1 = 1/(\rho \mathbf{v}_1)$ and $c_2 = \frac{3\rho+2}{\rho^2(\rho+1)}$. The claim hence follows by $c_1, c_2, 1-c_1-c_2 \in [0,1]$ and $0 \in co(V)$.
Case 5.2; $u = 0, v = 1$:

$$x(010\mathbf{1}, 1011\mathbf{0}) = \frac{1}{\rho} e^{-i\omega} + e^{-i2\omega} \sum_{k=3}^{\infty} \frac{f_k}{\rho^k} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_4$$

with $c_1 = 2/\rho^4 + 3/\rho^3$ and $c_2 = 1/(\rho \mathbf{v}_1)$. The claim hence follows by $c_1, c_2, 1-c_1-c_2 \in [0,1]$ and $0 \in co(V)$. Case 5.3; $u = 1, v = 0$:

$$x(110\mathbf{1}, 0011\mathbf{0}) = 1 + \frac{1}{\rho} + e^{-i2\omega} \sum_{k=3}^{\infty} \frac{f_k}{\rho^k}$$
$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_4 + (1-c_1-c_2)\mathbf{v}_5$$

with $c_1 = 1/\mathbf{v}_1$ and $c_2 = 1/\rho^2$. The claim hence follows by $c_1, c_2 \in [0,1]$.
Case 5.4; $u = 1, v = 1$: recalling $\omega = 2\pi/3$

$$x(110\mathbf{1}, 1011\mathbf{0}) = \left(1 + \frac{1}{\rho}\right) e^{-i\omega} + \sum_{k=3}^{\infty} \frac{f_k}{\rho^k} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$$

with $c_1 = 2/\rho^4 + 3/\rho^3$ and $c_2 = \frac{p^3-2p-1}{p^3}$. The claim hence follows by $c_1, c_2, 1-c_1-c_2 \in [0,1]$ and $0 \in co(V)$. $\square$

5 CONCLUSIONS

We introduced a robot hand model composed by an arbitrarily large number of hyper-redundant binary planar manipulators. The lenght of each link scales according to the Fibonacci sequence. Our assumptions (e.g. binary controls, kinematic redundancy, planar motion...) have twofold motivations. In one hand they facilitate the development of a theory relating fractal geometry and automatic control. On the other hand they appear validated by practical motivations in a wide literature.

We described the kinematics of each finger by giving an explicit formula for the position of the end-effectors (Section 2). We then addressed the investigation of the reachable workspace, by characterizing it as a projection of the attractor of a suitable IFS (Section 3). The relation with iteration function systems also allows to describe the convex hull of the reachable workspace: this technique is finally applied to the explicit characterization in a particular case.

The results in the present paper extend techiques previously developed for the case of links with a constant ratio. The several explicit results obtained also in this more complicated case suggest that the relation with IFSs is a deep connection and a powerful theoretical tool for the investigation of automatic control.

In this paper we studied the purely discrete case in order to give closed formulae and to enlight the relation with IFSs. However we plan to investigate the continuous case in a future work. The issues concerning the practical implementation of our model are beyond the purposes of the present paper; but of course it would be interesting to establish the link between the theoretical approach and its application. Other open problems include a tuning of parameters in order to avoid self-intersecting configurations, grasping algorithms and optimal control strategies.

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