

Complexity of Rule Sets Induced from Incomplete Data with Attribute-concept Values and “Do Not Care” Conditions

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Abstract: In this paper we study the complexity of rule sets induced from incomplete data sets with two interpretations of missing attribute values: attribute-concept values and “do not care” conditions. Experiments are conducted on 176 data sets, using three kinds of probabilistic approximations (lower, middle and upper) and the MLEM2 rule induction system. The goal of our research is to determine the interpretation and approximation that produces the least complex rule sets. In our experiment results, the size of the rule set is smaller for attribute-concept values for 12 combinations of the type of data set and approximation, for one combination the size of the rule sets is smaller for “do not care” conditions and for the remaining 11 combinations the difference in performance is statistically insignificant (5% significance level). The total number of conditions is smaller for attribute-concept values for ten combinations, for two combinations the total number of conditions is smaller for “do not care” conditions, while for the remaining 12 combinations the difference in performance is statistically insignificant. Thus, we may claim that attribute-concept values are better than “do not care” conditions in terms of rule complexity.

1 INTRODUCTION

Rough set theory has been applied to many areas of data mining. Fundamental concepts of rough set theory are standard lower and upper approximations. In this paper we will use probabilistic approximations. A probabilistic approximation, associated with a probability α , is a generalization of the standard approximation. For $\alpha = 1$, the probabilistic approximation is reduced to the lower approximation; for very small positive α , it is reduced to the upper approximation. Research on theoretical properties of probabilistic approximations started from (Pawlak et al., 1988) and then continued in many papers, see, e.g., (Pawlak and Skowron, 2007; Pawlak et al., 1988; Ślęzak and Ziarko, 2005; Yao, 2008; Yao and Wong, 1992; Ziarko, 2008).

Incomplete data sets may be analyzed using global approximations such as singleton, subset and concept (Grzymala-Busse, 2003; Grzymala-Busse, 2004a; Grzymala-Busse, 2004b). Probabilistic approximations for incomplete data sets and based on an arbitrary binary relation were introduced in (Grzymala-

Busse, 2011). The first experimental results using probabilistic approximations were published in (Clark and Grzymala-Busse, 2011).

For our experiments we use 176 incomplete data sets, with two types of missing attribute values: attribute-concept values (Grzymala-Busse, 2004c) and “do not care” conditions (Grzymala-Busse, 1991; Kryszkiewicz, 1995; Stefanowski and Tsoukias, 1999). Additionally, in our experiments we use three types of approximations: lower, middle, and upper. The middle approximation is the most typical probabilistic approximation, with $\alpha = 0.5$.

In (Clark and Grzymala-Busse, 2014), the results indicate that rule set performance, in terms of error rate, for both missing attribute value interpretations is not significantly different. As a result, given two rule sets with the same error rate, the more desirable would be the least complex, both for comprehension and computation performance. Therefore, the main objective of this paper is research on the complexity of rule sets induced from data sets with attribute-concept values and “do not care” conditions. Complexity is defined in terms of the number of rules

and the number of rule conditions, with larger numbers indicating greater complexity. Our main result is that the simpler rule sets are induced from data sets in which missing attribute values are interpreted as attribute-concept values.

Our secondary objective is to identify the approximation (lower, middle or upper) that produces the lowest rule complexity. Our conclusion is that all three kinds of approximations do not differ significantly with respect to the complexity of induced rule sets.

2 INCOMPLETE DATA

We assume that the input data sets are presented in the form of a *decision table*. An example of a decision table is shown in Table 1. Rows of the decision table represent *cases*, while columns are labeled by *variables*. The set of all cases will be denoted by U . In Table 1, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Independent variables are called *attributes* and a dependent variable is called a *decision* and is denoted by d . The set of all attributes will be denoted by A . In Table 1, $A = \{\text{Education, Skills, Experience}\}$. The value for a case x and an attribute a will be denoted by $a(x)$.

In this paper we distinguish between two interpretations of missing attribute values: attribute-concept values and "do not care" conditions. *Attribute-concept values*, denoted by "-", indicate that the missing attribute value may be replaced by any specified attribute value for a given concept. For example, if a patient is sick with flu, and if for other such patients the value of temperature is high or very-high, then we will replace the missing attribute values of temperature by values high and very-high, for details see (Grzymala-Busse, 2004c). "Do not care" conditions, denoted by "*", mean that the original attribute values are irrelevant, so we may replace them by any attribute value, for details see (Grzymala-Busse, 1991; Kryszkiewicz, 1995; Stefanowski and Tsoukias, 1999). Table 1 presents an incomplete data set affected by both attribute-concept values and "do not care" conditions.

One of the most important ideas of rough set theory (Pawlak, 1982) is an indiscernibility relation, defined for complete data sets. Let B be a nonempty subset of A . The indiscernibility relation $R(B)$ is a relation on U defined for $x, y \in U$ as follows:

$$(x, y) \in R(B) \text{ if and only if } \forall a \in B (a(x) = a(y)).$$

The indiscernibility relation $R(B)$ is an equivalence relation. Equivalence classes of $R(B)$ are called *elementary sets* of B and are denoted by $[x]_B$. A subset of

Table 1: A decision table.

Case	Attributes			Decision
	Education	Skills	Experience	
1	higher	high	-	high
2	*	high	low	high
3	secondary	-	high	high
4	higher	*	high	high
5	elementary	high	low	low
6	secondary	-	high	low
7	-	low	high	low
8	elementary	*	-	low

U is called *B-definable* if it is a union of elementary sets of B .

The set X of all cases defined by the same value of the decision d is called a *concept*. For example, a concept associated with the value *low* of the decision *Productivity* is the set $\{1, 2, 3, 4\}$. The largest *B*-definable set contained in X is called the *B-lower approximation* of X , denoted by $\underline{\text{appr}}_B(X)$, and defined as follows

$$\cup\{[x]_B \mid [x]_B \subseteq X\},$$

while the smallest *B*-definable set containing X , denoted by $\overline{\text{appr}}_B(X)$ is called the *B-upper approximation* of X , and is defined as follows

$$\cup\{[x]_B \mid [x]_B \cap X \neq \emptyset\}.$$

For a variable a and its value v , (a, v) is called a variable-value pair. A *block* of (a, v) , denoted by $[(a, v)]$, is the set $\{x \in U \mid a(x) = v\}$ (Grzymala-Busse, 1992).

For incomplete decision tables the definition of a block of an attribute-value pair is modified in the following way.

- If for an attribute a there exists a case x such that $a(x) = -$, then the corresponding case x should be included in blocks $[(a, v)]$ for all specified values $v \in V(x, a)$ of attribute a , where

$$\begin{aligned} V(x, a) &= \\ &= \{a(y) \mid a(y) \text{ is specified, } y \in U, d(y) = d(x)\}, \end{aligned}$$

- If for an attribute a there exists a case x such that $a(x) = *$, then the case x should be included in blocks $[(a, v)]$ for all specified values v of the attribute a .

For the data set from Table 1, $V(1, \text{Experience}) = \{\text{low, high}\}$, $V(3, \text{Skills}) = \{\text{high}\}$,

$V(6, Skills) = \{low, high\}$, $V(7, Education) = \{elementary, secondary\}$ and $V(8, Experience) = \{low, high\}$.

For the data set from Table 1 the blocks of attribute-value pairs are:

$$\begin{aligned} [(Education, elementary)] &= \{2, 5, 7, 8\}, \\ [(Education, secondary)] &= \{2, 3, 6, 7\}, \\ [(Education, higher)] &= \{1, 2, 4\}, \\ [(Skills, low)] &= \{4, 6, 7, 8\}, \\ [(Skills, high)] &= \{1, 2, 3, 4, 5, 6, 8\}, \\ [(Experience, low)] &= \{1, 2, 5, 8\}, \\ [(Experience, high)] &= \{1, 3, 4, 6, 7, 8\}. \end{aligned}$$

For a case $x \in U$ and $B \subseteq A$, the *characteristic set* $K_B(x)$ is defined as the intersection of the sets $K(x, a)$, for all $a \in B$, where the set $K(x, a)$ is defined in the following way:

- If $a(x)$ is specified, then $K(x, a)$ is the block $[(a, a(x))]$ of attribute a and its value $a(x)$,
- If $a(x) = -$, then the corresponding set $K(x, a)$ is equal to the union of all blocks of attribute-value pairs (a, v) , where $v \in V(x, a)$ if $V(x, a)$ is nonempty. If $V(x, a)$ is empty, $K(x, a) = U$,
- If $a(x) = *$ then the set $K(x, a) = U$, where U is the set of all cases.

For Table 1 and $B = A$,

$$\begin{aligned} K_A(1) &= \{1, 2, 4\}, \\ K_A(2) &= \{1, 2, 5, 8\}, \\ K_A(3) &= \{3, 6\}, \\ K_A(4) &= \{1, 4\}, \\ K_A(5) &= \{2, 5, 8\}, \\ K_A(6) &= \{3, 6, 7\}, \\ K_A(7) &= \{6, 7, 8\}, \\ K_A(8) &= \{2, 5, 7, 8\}. \end{aligned}$$

Note that for incomplete data there are a few possible ways to define approximations (Grzymala-Busse, 2003), we used *concept* approximations (Grzymala-Busse, 2011) since our previous experiments indicated that such approximations are most efficient (Grzymala-Busse, 2011). A *B-concept lower approximation* of the concept X is defined as follows:

$$\underline{BX} = \cup\{K_B(x) \mid x \in X, K_B(x) \subseteq X\},$$

while a *B-concept upper approximation* of the concept X is defined by:

$$\begin{aligned} \overline{BX} &= \cup\{K_B(x) \mid x \in X, K_B(x) \cap X \neq \emptyset\} = \\ &= \cup\{K_B(x) \mid x \in X\}. \end{aligned}$$

For Table 1, *A-concept lower* and *A-concept upper* approximations of the concept $\{5, 6, 7, 8\}$ are:

$$\begin{aligned} \underline{A}\{5, 6, 7, 8\} &= \{6, 7, 8\}, \\ \overline{A}\{5, 6, 7, 8\} &= \{2, 3, 5, 6, 7, 8\}. \end{aligned}$$

3 PROBABILISTIC APPROXIMATIONS

For completely specified data sets a *probabilistic approximation* is defined as follows

$$appr_\alpha(X) = \cup\{[x] \mid x \in U, P(X \mid [x]) \geq \alpha\},$$

α is a parameter, $0 < \alpha \leq 1$, see (Grzymala-Busse, 2011; Grzymala-Busse and Ziarko, 2003; Pawlak et al., 1988; Wong and Ziarko, 1986; Yao, 2008; Ziarko, 1993). Additionally, for simplicity, the elementary sets $[x]_A$ are denoted by $[x]$. For discussion on how this definition is related to the variable precision asymmetric rough sets see (Clark and Grzymala-Busse, 2011; Grzymala-Busse, 2011).

Note that if $\alpha = 1$, the probabilistic approximation becomes the standard lower approximation and if α is small, close to 0, in our experiments it is 0.001, the same definition describes the standard upper approximation.

For incomplete data sets, a *B-concept probabilistic approximation* is defined by the following formula (Grzymala-Busse, 2011)

$$\cup\{K_B(x) \mid x \in X, Pr(X \mid K_B(x)) \geq \alpha\}.$$

For simplicity, we will denote $K_A(x)$ by $K(x)$ and the *A-concept probabilistic approximation* will be called a *probabilistic approximation*.

For Table 1 and the concept $X = [(Productivity, low)] = \{5, 6, 7, 8\}$, there exist three distinct three distinct probabilistic approximations:

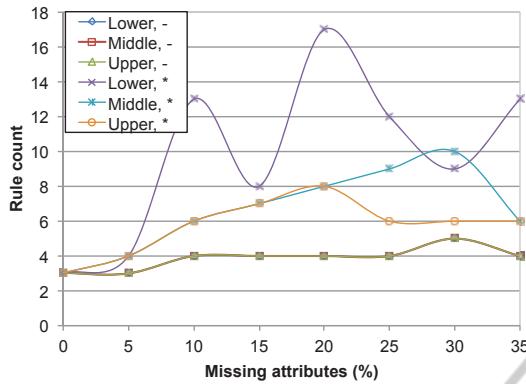
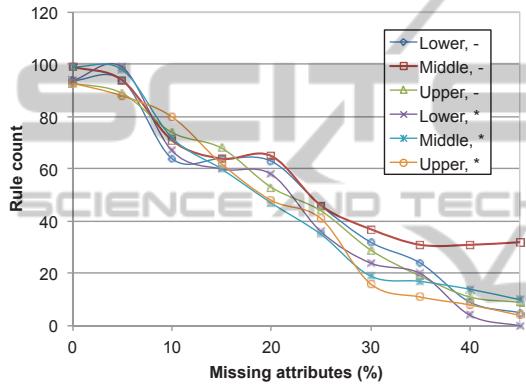
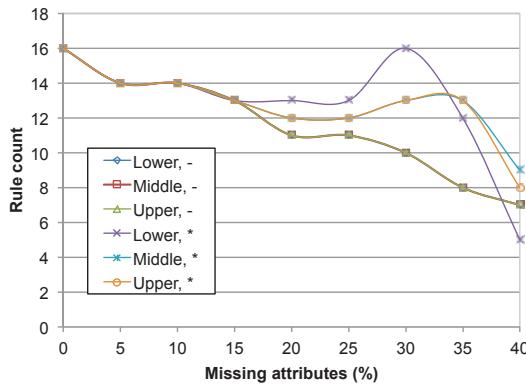
$$\begin{aligned} appr_{1.0}(\{5, 6, 7, 8\}) &= \{6, 7, 8\}, \\ appr_{0.75}(\{5, 6, 7, 8\}) &= \{2, 5, 6, 7, 8\}, \\ \text{and} \\ appr_{0.001}(\{5, 6, 7, 8\}) &= \{2, 3, 5, 6, 7, 8\}. \end{aligned}$$

The special probabilistic approximations with the parameter $\alpha = 0.5$ will be called a *middle approximation*.

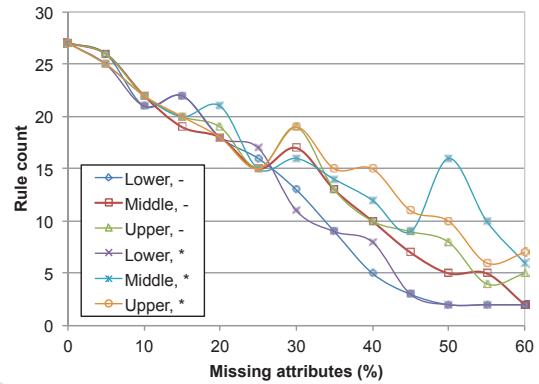
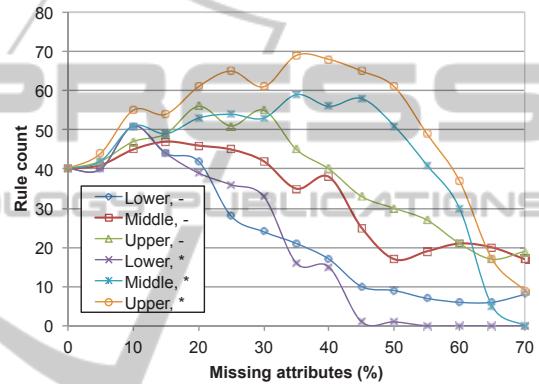
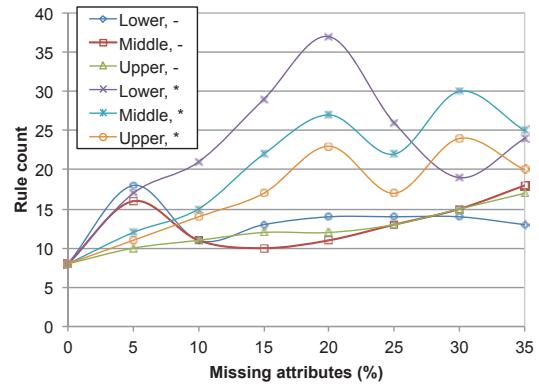
4 EXPERIMENTS

Our experiments are based on eight data sets available from the University of California at Irvine *Machine Learning Repository*, see Table 2.

For every data set a set of templates is created by incrementally replacing a percentage of existing specified attribute values (at a 5% increment) with *attribute-concept values*. Thus, we started each series of experiments with no *attribute-concept values*,

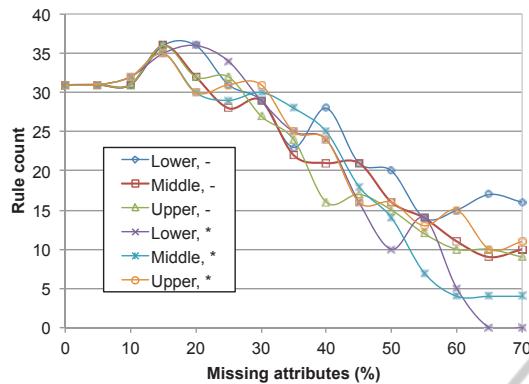
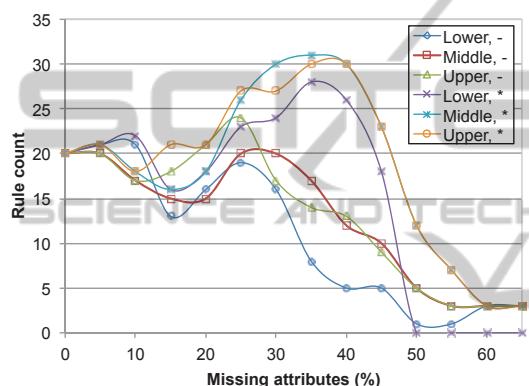
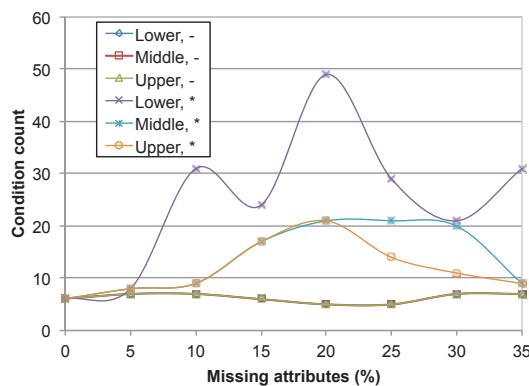

 Figure 1: Size of the rule set for the *Bankruptcy* data set.

 Figure 2: Size of the rule set for the *Breast cancer* data set.

 Figure 3: Size of the rule set for the *Echocardiogram* data set.

then we changed 5% of specified values to *attribute-concept values*, then we changed an additional 5% of specified values to *attribute-concept values*, etc., until at least one entire row of the data set is full of *attribute-concept values*. Then three attempts were made to change the configuration of new *attribute-concept values* and either a new data set with an extra 5% of *attribute-concept* is created or the process is

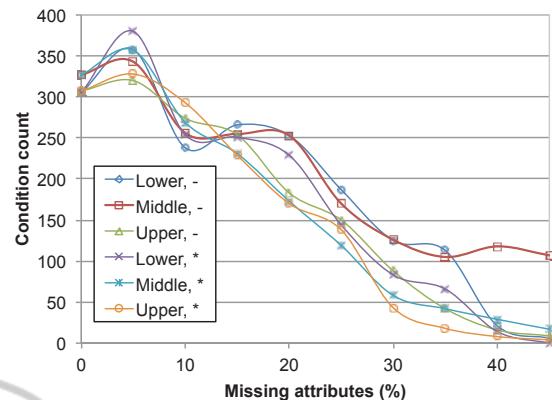
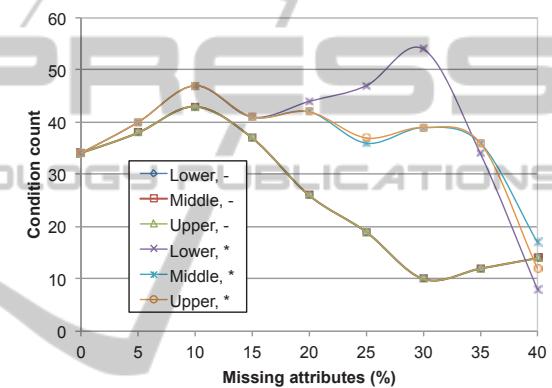
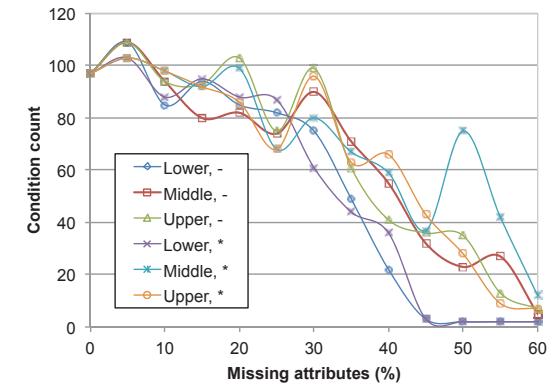

 Figure 4: Size of the rule set for the *Hepatitis* data set.

 Figure 5: Size of the rule set for the *Image segmentation* data set.

 Figure 6: Size of the rule set for the *Iris* data set.

terminated. Additionally, the same formed templates are edited for further experiments by replacing each “-”, representing *attribute-concept values* with “*”, representing “*do not care*” conditions.

For any data set there is some maximum for the percentage of missing attribute values. For example, for the *bankruptcy* data set, it is 35%. Hence, for the *bankruptcy* data set, there exist seven data sets with


 Figure 7: Size of the rule set for the *Lymphography* data set.

 Figure 8: Size of the rule set for the *Wine recognition* data set.

 Figure 9: Number of conditions for the *Bankruptcy* data set.

attribute-concept values and seven data sets with “do not care” conditions, for a total of 15 data sets (the additional data set is complete, with no missing attribute values). In a similar process for the *breast cancer*, *echocardiogram*, *hepatitis*, *image segmentation*, *iris*, *lymphography* and *wine recognition* data sets we created 19, 17, 25, 29, 15, 29, and 27 data sets. Hence the total number of data sets is 176.


 Figure 10: Number of conditions for the *Breast cancer* data set.

 Figure 11: Number of conditions for the *Echocardiogram* data set.

 Figure 12: Number of conditions for the *Hepatitis* data set.

Results of our experiments are presented in Figures 1–16.

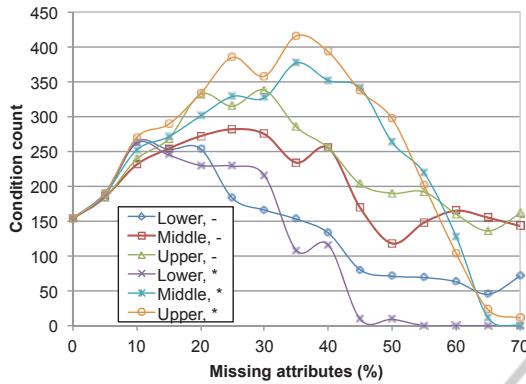


Figure 13: Number of conditions for the *Image segmentation* data set.

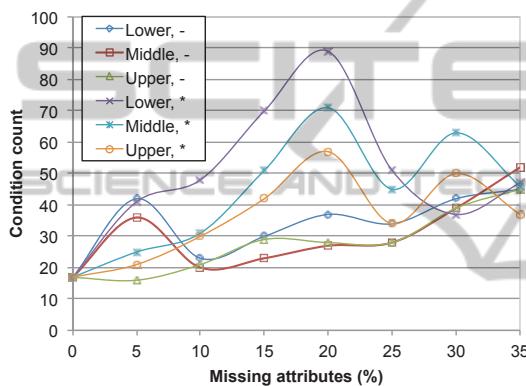


Figure 14: Number of conditions for the *Iris* data set.

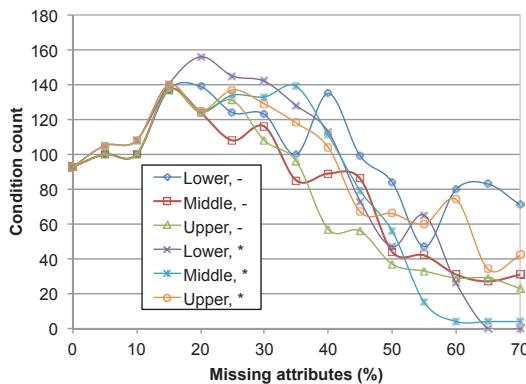


Figure 15: Number of conditions for the *Lymphography* data set.

5 DISCUSSION

First we compare two interpretations of missing attribute values, attribute-concept values and "do not care" conditions with respect to the rule set size. For

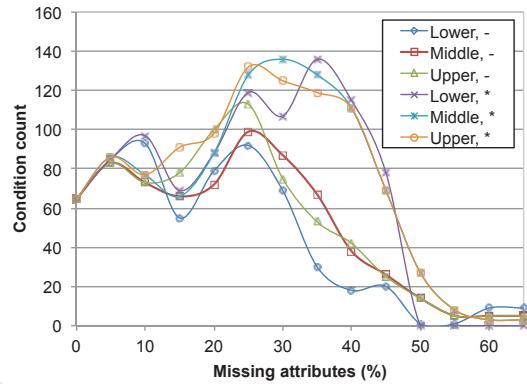


Figure 16: Number of conditions for the *Wine recognition* data set.

Table 2: Data sets used for experiments.

Data set	Number of cases	Number of attributes	Number of concepts
Bankruptcy	66	5	2
Breast cancer	277	9	2
Echocardiogram	74	7	2
Hepatitis	155	19	2
Image segmentation	210	19	7
Iris	150	4	3
Lymphography	148	18	4
Wine recognition	178	13	3

every data set type, separately for lower, middle and upper approximations, the Wilcoxon matched-pairs signed rank test is used with a 5% level of significance two-tailed test. With eight data set types and three approximation types, the total number of combinations is 24.

For 12 combinations the rule set size is smaller for attribute-concept values: *bankruptcy* data set with all three types of approximations, *hepatitis* data set with middle and upper approximations, *image segmentation* data set with middle and upper approximations, *iris* data set with all three types of approximations and *wine recognition* data set with middle and upper approximations. For one combination, *breast cancer* data set with middle approximations, the size of the rule set is smaller for "do not care" conditions. For the remaining 11 combinations, the difference in the rule set size between attribute-concept values and "do not care" conditions is insignificant. Therefore there is strong evidence that attribute-concept values provide for smaller rule set sizes than "do not care" conditions.

Similarly, for the total number of conditions in a rule set, in ten combinations this number is smaller for

attribute-concept values: *bankruptcy* data set with all three types of approximations, *echocardiogram* data set with all three types of approximations, *lymphography* data set with upper approximations and *wine recognition* data set with all three types of approximations. For two combinations the total number of conditions in the rule set is smaller for “do not care” conditions: *breast cancer* data set with middle approximations and *image recognition* data set with lower approximations. For the remaining 12 combinations the difference in the total number of conditions in rule sets between attribute-concept values and “do not care” conditions is insignificant. Thus there is evidence that attribute-concept values provide for a smaller total number of conditions in rule sets than “do not care” conditions.

Next, for a given interpretation of missing attribute values we compare all three types of approximations in terms of the rule set size and the total number of conditions in the rule set. For all eight types of data sets, we compare lower approximations with middle and upper approximations, and middle approximations with upper approximations. This experiment setup results in a total of 24 combinations and the Friedman Rank Sums test, with 5% significance level, is used.

The rule set size is smaller for lower approximations than for upper approximations in four combinations of the type of data set and type of missing attribute value: *hepatitis* data set with attribute-concept values, *image segmentation* data set with both attribute-concept values and “do not care” conditions and *wine recognition* data set with “do not care” conditions. The rule set size is smaller for lower approximations than for middle approximations for three combinations of the data set type and missing attribute value type: *image segmentation* data set with both attribute-concept values and “do not care” conditions and *hepatitis* data set with attribute-concept values. The rule set size is smaller for upper approximations than for lower approximations in three combinations: *bankruptcy* data set with “do not care” conditions, *iris* data set with “do not care” conditions and *lymphography* data set with attribute-concept values. Finally, the rule set size is smaller for upper approximations than for middle approximations in one combination: *breast cancer* data set with attribute-concept values. In the remaining 13 combinations the difference between all three approximations is insignificant. There is weak evidence that lower approximations might be better than the remaining approximations.

The total number of conditions in a rule set is smaller for lower approximations than for upper approximations in three combinations of data set type

and missing attribute value type: *hepatitis* data set with attribute-concept values and *image segmentation* data set with both attribute-concept values and “do not care” conditions. The total number of conditions in a rule set is smaller for lower approximations than for middle approximations in one combination: *image segmentation* data set with “do not care” conditions. The total number of conditions in a rule set is smaller for middle approximations than for lower approximations in one combination: *lymphography* data set with attribute-concept values. The total number of conditions in a rule set is smaller for upper approximations than for lower approximations in three combinations: *bankruptcy* data set with “do not care” conditions, *iris* data set with “do not care” conditions and *lymphography* data set with attribute-concept values. Finally, the total number of conditions in a rule set is smaller for upper approximations than for middle approximations in one combination: *lymphography* data set with the attribute-concept values. In the remaining 15 combinations the difference between all three approximations is insignificant. Practically speaking, we do not have enough evidence to tell which approximation is the best.

6 RELATED WORK

Rough set concepts and the indiscernibility relation are introduced in (Pawlak, 1982) with additional research explaining theoretical concepts of probabilistic approximations in (Pawlak et al., 1988). Further research was conducted in the area of probabilistic types of approximations in other efforts, comparing them to deterministic approaches and studying other extensions in (Pawlak and Skowron, 2007; Pawlak et al., 1988; Ślezak and Ziarko, 2005; Yao, 2008; Yao and Wong, 1992; Ziarko, 2008).

In (Grzymala-Busse, 2003; Grzymala-Busse, 2004a; Grzymala-Busse, 2004b), three global approximations are defined: singleton, subset and concept. The papers also include a study of these approximations with incomplete data. In our work, concept approximations are used in conjunction with probabilistic approximations on incomplete data sets. These concepts were introduced in (Grzymala-Busse, 2011) and include definitions of *B-concept probabilistic approximations* with discussions on how the definition is related to variable precision asymmetric rough sets. In addition, the first experimental results are studied in (Clark and Grzymala-Busse, 2011).

7 CONCLUSIONS

As follows from our experiments, there is evidence that the rule set size is smaller for the attribute-concept interpretation of missing attribute values than for the "do not care" condition interpretation. The total number of conditions in rule sets is also smaller for attribute-concept interpretation of missing attribute values. Thus we may claim attribute-concept values are better than "do not care" conditions as an interpretation of a missing attribute value in terms of rule complexity.

Furthermore, all three kinds of approximations (lower, middle and upper) do not differ significantly with respect to the complexity of induced rule sets.

Future direction this work might take is an investigation of other interpretations of missing attribute values and a comparison of the complexity of the rule sets produced.

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