A Mixed-Integer Linear Program for Routing and Scheduling Trains through a Railway Station

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Abstract: This paper studies a train routing and scheduling problem faced by railway station infrastructure managers to generate a conflict-free timetable which consists of two parts, commercial movements and technical movements. Firstly, we present the problem and propose a discrete-time mixed-integer linear mathematical model formulation. Due to the computational complexity of integer programming methods, we need to improve the calculation performance. On one hand, we consider the problem in continuous-time domain which decrease the computational size. The integrality of the scheduling variables is proved. On the other hand, the redundant constraints are cut off by probing the potential conflicts between trains and movements. The full practical problem is large: 247 trains consisting of 503 movements per day should be considered. The proposed approach can solve an instance made of 60 trains and 121 movements representing 385 minutes of traffic within less than 2 minutes.

1 INTRODUCTION

In most countries, rail network is a busy system with increasing patterns of train services that require accurate scheduling and routing to adapt to the limited infrastructures. The traditional process to generate a timetable for a railway network is divided into several stages (Watson., 2001). First, a draft timetable is generated by train activities managers (national, regional, freight) based on the traffic frequencies, the volume of traffic, the rough layout of the railway network between the railway stations together with the desired lines and their connection requirements (Schrijver and Steenbeek, 1994) (Serafini and Ukovich., 1989). Then, station operators need to check whether the draft timetable is feasible within the railway station while satisfying capacity, safety and customer service (Kroon and Zwaneveld, 1995) (Zwaneveld et al., 1996). At the same time, schedules for the trains through the railway station are generated by including all the required technical operations such as carriage preparation, maintenance, etc. So far, the conflicts of proposed train times, lines and platforms are found and resolved by hand. Most of the studies focus on the problem of railway network with a global point of view (D'Ariano et al., 2007) (D'Ariano, 2008) (Caimi, 2009). Nevertheless the routing and scheduling problem in large, busy, complex train stations is also a complex issue with respect to time and space.

This paper studies a train routing and scheduling problem faced by railway station managers to generate a conflict-free timetable which consists of two sets of circulations. The first set is made of *commercial circulations* given by several administrative levels (national, regional, freight) over a large time horizon (typically one year before the effective realization of the production). The other set corresponds to *technical circulations* added by the railway station managers to prepare or repair the trains. The routing problem is the problem of assigning each of the involved trains to a route through the railway station and to a platform in the station. Thus, routes and platforms in the station are here the critical resources of the system. The scheduling problem is to adjust the timetable of technical circulations to guarantee on-time arrivals and leavings of all the commercial circulations. A conflict-free timetable with acceptable commercial circulations and needed technical circulations is generated. Commercial circulations with unsolvable conflicts will return to their original activity managers. with suggestions for the modification of the arrival and leaving times.

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(Carey, 1994b) proposes a mixed integer program to find the paths of trains in a one-way track system. The numerical example provided in Carey's paper has 10 nodes, 28 links, and 10 train services and requires a significant amount of time to be solved. In another article, (Carey, 1994a) extends the model from one-way to two-way tracks system. The resulting model is also a mixed integer program, which is easier to solve than his earlier model, but this newer study does not provide testing results. (Kroon et al., 1997) consider computational complexity of the problem of routing trains through railway stations. They show that the problem is NP-complete if each train has three or more routing possibilities. (Zwaneveld et al., 2001) describe the routing problem of trains through a railway station with the given arrival and leaving times of trains and the detailed layout of the railway station. The algorithm which consists of preprocessing, valid inequalities and branch-and-cut approach is proposed to find the optimal routing solution. The conflicts of routes are solved by the theory of dynamics referred in (Zwaneveld, 1997). (Carey and Carville, 2003) consider the problem of train planning or scheduling for large, busy, complex train stations. A scheduling heuristics analogous to those successfully adopted by train planners using "manual" methods is developed. Heuristic techniques are designed according to train planners' objectives, and take account of a weighted combination of costs and preference trade-off. But the robustness of the schedule is not considered.

In view of the above, a suitable and effective model is still needed for generating a robust conflict-free timetable in the railway station. In this paper, we propose to extend our earlier study given in (Bai et al., 2013), which defines and formulates the problem as an integer linear program. This paper is structured as follow. Section 2 starts with the definition of the problem and related notions. In section 3, a mathematical model is proposed as a mixed-integer linear program. In section 4, we give practical improvements to our formulation and assess their efficiency by giving numerical experiments. In section 5, we give a conclusion and discuss further development and application of the algorithm.

2 PROBLEM FORMALIZATION

Definition 1 (Railway Station)**.** A railway station $R = (\mathbb{S}, \mathbb{L}, \mathbb{P})$ is defined by a set of lines $\mathbb{L}$ on which trains follow some paths in a set $\mathbb{P}$, defined using switches in the set $\mathbb{S}$.

Switches (s_k)**.** The set $\mathbb{S} = \{s_1, s_2, \ldots, s_{\mathsf{S}}\} = \{s_k\}_{k\in[\![1,\mathsf{S}]\!]}$ designates a set of switches. The cardinal number of $\mathbb{S}$ is denoted as S.

Lines (l)**.** The set of lines is defined by $\mathbb{L} = \{l_1, l_2, \ldots, l_{\mathsf{L}}\} = \{l_f\}_{f\in[\![1,\mathsf{L}]\!]}$. L denotes the cardinal of the set of lines $\mathbb{L}$. We make a distinction between *internal* and *external* lines. Passengers board or get off the train on the platforms in front of internal lines. They are denoted by the set $\mathbb{L}^{\ominus\!\rightarrow}$. External lines are located at the entrance of the railway station. They are denoted by the set $\mathbb{L}^{\rightarrow\!\ominus}$. Internal and external lines can be connected together using the set of switches, through a small railway network inside the railway station. Every line $l \in \mathbb{L}$ is connected to a unique *"entrance"* switch denoted as $\zeta(l) \in \mathbb{S}$, while a switch may be connected to multiple lines.

Paths (p_c)**.** The set of paths is denoted by $\mathbb{P} = \{p_1, p_2, \ldots, p_{\mathsf{P}}\} = \{p_c\}_{c\in[\![1,\mathsf{P}]\!]}$. P denotes the cardinal number of $\mathbb{P}$. A path $p \in \mathbb{P}$ consists of a set of ordered switches $p = [s_1^p, s_2^p, \ldots, s_{\mathsf{S}^p}^p] = \{s_k^p\}_{k\in[\![1,\mathsf{S}^p]\!]}$ with the cardinal number S^p. Switches of a path are always described from railway station to the outside. For each path, we consider two special switches s_1^p (*internal switch*) and $s_{\mathsf{S}^p}^p$ (*external switch*). The set $\mathbb{P}$ reflects the topology of the railway station, and some sequences of switches are not valid paths.
The subset of paths that connect the internal line $l_i \in \mathbb{L}^{\ominus\!\rightarrow}$ and the external line $l_e \in \mathbb{L}^{\rightarrow\!\ominus}$ is denoted by $\mathbb{P}^{(l_i,l_e)} = \{p_c\}_{c\in[\![1,\mathsf{P}^{(l_i,l_e)}]\!]}$. The subset of internal lines l_i reachable from an external line $l_e \in \mathbb{L}^{\rightarrow\!\ominus}$ is denoted by $\mathbb{L}_{l_e}^{\ominus\!\rightarrow}$.

The traffic in the railway station is defined by a set of trains. Each train may be composed of several *"technical"* movements and *"commercial"* movements. A commercial movement denotes a circulation of a train taking passengers onboard. A technical movement denotes a circulation without passengers, corresponding to the locomotive only or to empty wagons.

Definition 2 (Movement)**.** Let $R = (\mathbb{S}, \mathbb{L}, \mathbb{P})$ be a railway station. The set of considered movements is defined as $\mathbb{M} = \{m, m_2, \ldots, m_{\mathsf{M}}\} = \{m_j\}_{j\in[\![1,\mathsf{M}]\!]}$. M is the cardinal of $\mathbb{M}$.

A movement $m \in \mathbb{M}$ is defined by its type, its reference time, its actual time interval, its internal line (generally unknown to be determined), its external line (given) and its path (to be determined).

Lines of a Movement ($l_m^{\ominus\!\rightarrow}$ **and** $l_m^{\rightarrow\!\ominus}$)**.** The internal line (resp. external line) of a movement $m \in \mathbb{M}$ is denoted by $l_m^{\ominus\!\rightarrow} \in \mathbb{L}^{\ominus\!\rightarrow}$ (resp. $l_m^{\rightarrow\!\ominus} \in \mathbb{L}^{\rightarrow\!\ominus}$).
The subset of movements going through an exernal line $l \in \mathbb{L}^{\rightarrow\!\ominus}$ (i.e. for which $l_m^{\rightarrow\!\ominus} = l$) is denoted

by $\mathbb{M}^l$.

Paths of a Movement (p_m). Let $m \in \mathbb{M}$ be a movement. The path of the movement m is denoted by $p_m \in \mathbb{P}$. Since this path should describe a circulation between lines $l_m^{\ominus\rightarrow}$ and $l_m^{\rightarrow\ominus}$, we have obviously:

$$s_1^{p_m} = \zeta(l_m^{\ominus\rightarrow}) \text{ and } s_{\mathsf{S}^{\mathsf{pm}}}^{p_m} = \zeta(l_m^{\rightarrow\ominus}) \qquad (1)$$

which restricts the number of possible paths for the movement m.

Actual Time Interval of a Movement ($[\alpha_m, \beta_m]$). Let $m \in \mathbb{M}$ be a movement. The actual time interval of the movement m is defined by $[\alpha_m, \beta_m]$ with $\alpha_m, \beta_m \in \mathbb{N}$ and $\alpha_m < \beta_m$. In this paper, we consider that the movement occupies all corresponding resources (i.e. the switches) over its actual time interval. In our case study, the length of this time interval is fixed to $S = 5$ minutes, so we have $\beta_m = \alpha_m + S$.

Type of a Movement. Four types of movements are defined depending on their commercial or technical nature , and their direction (entering or leaving the railway station).

In the following paragraphs, the technical movements are denoted by a semi-arrow $\rightharpoonup$; the commercial movements are denoted by a full arrow $\hookrightarrow$; a train leaving the railway station is denoted by $\ominus\!\!\rightarrow$; a train entering the railway station is denoted by $\rightarrow\!\!\ominus$ (the full circle $\bigcirc$ being a mnemotechnic way to denote the railway station side). We divide thus the set of movements $\mathbb{M}$ into four subsets such that: $\mathbb{M} = \mathbb{M}^{\ominus\rightharpoonup} \bigcup \mathbb{M}^{\rightharpoonup\ominus} \bigcup \mathbb{M}^{\ominus\hookrightarrow} \bigcup \mathbb{M}^{\hookrightarrow\ominus}$.

Reference Times (α_m^{ref} and β_m^{ref}). We define *reference times* α_m^{ref} and β_m^{ref} depending on the type of the considered movement. These reference times constrain the possible values for the actual time interval of a movement, allowing to advance or postpone some technical movements in order to free the railway network for other commercial circulations. In this study, we consider that the adjustment should not last more than $L = 10$ minutes.

- A technical movement $m \in \mathbb{M}$ entering the railway station is associated to a reference time β_m^{ref} denoting the latest termination date of this movement such that:

$$\forall m \in \mathbb{M}^{\rightharpoonup\ominus},\ \exists \beta_m^{\text{ref}} \in \mathbb{N} \text{ s.t. } \beta_m^{\text{ref}} - L \leq \beta_m \leq \beta_m^{\text{ref}} \qquad (2)$$

- A technical movement $m \in \mathbb{M}$ leaving the railway station is associated to a reference time α_m^{ref} denoting its earliest starting date such that:

$$\forall m \in \mathbb{M}^{\ominus\rightharpoonup}, \exists \alpha_m^{\text{ref}} \in \mathbb{N} \text{ s.t. } \alpha_m^{\text{ref}} + L \geq \alpha_m \geq \alpha_m^{\text{ref}} \qquad (3)$$

- A commercial movement $m \in \mathbb{M}$ entering the railway station should arrive exactly at the reference time β_m^{ref} such that:

$$\forall m \in \mathbb{M}^{\hookrightarrow\ominus}, \beta_m = \beta_m^{\text{ref}} \qquad (4)$$

- A commercial movement $m \in \mathbb{M}$ leaving the railway station should leave exactly at the reference time α_m^{ref} such that:

$$\forall m \in \mathbb{M}^{\ominus\hookrightarrow}, \alpha_m = \alpha_m^{\text{ref}} \qquad (5)$$

A set of technical and commercial movements can define a train whose properties are inherited from its movements.

Definition 3 (Train)**.** Let $R = (\mathbb{S}, \mathbb{L}, \mathbb{P})$ be a railway station. The set of trains is denoted by $\mathbb{T} = \{t_1, t_2, \ldots, t_{\mathsf{T}}\} = \{t_i\}_{i \in [\![1,\mathsf{T}]\!]}$. The cardinal number of $\mathbb{T}$ is denoted by T. Every train $t \in \mathbb{T}$ consists of a set of movements $\mathbb{M}^t = \{m_1^t, m_2^t, \ldots, m_{\mathsf{M}^t}^t\} \subset \mathbb{M}$. We denote by M^t the cardinal number of $\mathbb{M}^t$.

Internal Lines of a Train. Each movement of a train must be executed on the same internal line, denoted by $\lambda_t \in \mathbb{L}$ such that:

$$\forall t \in \mathbb{T}, \exists \lambda_t \in \mathbb{L} \text{ s.t. } \forall m \in \mathbb{M}^t, l_m^{\ominus\rightarrow} = \lambda_t \qquad (6)$$

Actual Time Interval of a Train ($[A_t, B_t]$). Let $t \in \mathbb{T}$ be a train and $\lambda_t \in \mathbb{L}$ its internal line. The train t occupies the line λ_t during the interval $[A_t, B_t]$, such that:

$$A_t = \min_{m \in \mathbb{M}^t} \alpha_m \qquad (7)$$

$$B_t = \max_{m \in \mathbb{M}^t} \beta_m \qquad (8)$$

Obviously, every movement of the train occurs during this interval of time:

$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, [\alpha_m, \beta_m] \subset [A_t, B_t] \qquad (9)$$

We partition the set of movements of a train t according to the types of movements defined above: $\mathbb{M}^t = \mathbb{M}^{t\ominus\rightharpoonup} \bigcup \mathbb{M}^{t\rightharpoonup\ominus} \bigcup \mathbb{M}^{t\ominus\hookrightarrow} \bigcup \mathbb{M}^{t\hookrightarrow\ominus}$. Obviously, the constraints (2) to (5) must be applied to the movements of a train. Furthermore, we need additional constraints to ensure the safety of trains movements.

Lines Occupation Constraint. A line can not be occupied by two trains at the same time:

$$\forall t, t' \in \mathbb{T} \text{ s.t. } \lambda_t = \lambda_{t'}, [A_t, B_t] \cap [A_{t'}, B_{t'}] = \varnothing \qquad (10)$$

Switches Occupation Constraint. Two movements using paths containing a common switch cannot be sheduled during the same time interval:

$$\forall s \in \mathbb{S}, \forall m, m' \in \mathbb{M} \text{ s.t. } s \in \mathbb{S}^{p_m} \cap \mathbb{S}^{p_{m'}}$$
$$[\alpha_m, \beta_m] \cap [\alpha_{m'}, \beta_{m'}] = \varnothing \quad (11)$$

In the next section, we propose a mixed-integer linear program to solve the allocation problem described above.

3 MIXED-INTEGER LINEAR PROGRAMMING MODEL

Hereafter, the function $\delta(Q)$ is an indicator such that $\delta(Q) = 1$ if the condition Q is valid, otherwise 0.

3.1 Parameters

- C is a sufficiently large constant.
- L is the adjustable time interval of the technical movements.
- α_m^{ref} is the reference starting time of the movement m.
- β_m^{ref} is the reference ending time of the movement m.
- S is the time allocated to a movement. In our context, $S = 5$ minutes.
- $Y_{m,t}$ identifies the movements belonging to trains, $Y_{m,t} = \delta(m \in \mathbb{M}^t)$.
- $Y_{s,p}^{SP}$ identifies the switches composing a path p. $Y_{s,p}^{SP} = \delta(s \in p)$.
- $Y_{l,m}^{L \hookrightarrow\!\!\ominus M}$ identifies the external line of the movement. $Y_{l,m}^{L \hookrightarrow\!\!\ominus M} = \delta(l_m^{\hookrightarrow\!\!\ominus} = l)$.
- $Y_{p,p'}^{P}$ identifies the conflict of switches between two paths. $Y_{p,p'}^{P} = \delta(p \cap p' \neq \varnothing)$.

3.2 Variables

In the practical situation, the arrival and leaving times of trains are measured in minutes. The scheduling decision variables are thus defined as integers with units of minutes, characterizing a discrete-time sheduling problem.

- α_m is actual starting time of the movement m.
- β_m is actual ending time of the movement m, $\alpha_m + S = \beta_m$.
- A_t is the starting time of occupation of the railway station by the train t.
- B_t is the ending time of occupation of the railway station by the train t.

All the scheduling decision variables have values that fit the length of one day (1440 minutes). The routing decision variables are defined as binary variables.

- $X_{t,m}^{A}$ identifies the first movement of trains. $X_{t,m}^{A} = \delta(A_t = \alpha_m)$.
- $X_{t,m}^{B}$ identifies the last movement of trains. $X_{t,m}^{B} = \delta(B_t = \beta_m)$.
- $X_{l,t}^{L \ominus\!\!\to T}$ identifies the internal lines of trains. $X_{l,t}^{L \ominus\!\!\to T} = \delta(\lambda_t = l)$.
- $X_{l,m}^{L \ominus\!\!\to M}$ identifies the internal lines of movements. $X_{l,m}^{L \ominus\!\!\to M} = \delta(l = l_m^{\ominus\!\!\to})$.
- $X_{p,m}^{PM}$ identifies the path of movements. $X_{p,m}^{PM} = \delta(p = p_m)$.
- $X_{t,t'}^{OrderT}$ identifies the time order of two trains using the same line. $X_{t,t'}^{OrderT} = \delta(t$ circulates before $t')$.
- $X_{m,m'}^{OrderM}$ identifies the time order of two movements using two paths with the same switch(es). $X_{m,m'}^{OrderM} = \delta(m$ circulates before $m')$.

3.3 Constraints

The constraints (1) to (11) are expressed as linear constraints with the parameters and variables defined above.

Time Interval of a Train. According to equations (7)(8) , the time interval of a train covers all the movements of the train, which can be formulated in a classical way as below:

$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, \quad A_t \leq \alpha_m \quad (12)$$
$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, \quad \alpha_m \leq A_t + C \cdot (1 - X_{t,m}^{A}) \quad (13)$$
$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, \quad B_t \geq \beta_m \quad (14)$$
$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, \quad B_t \leq \beta_m + C \cdot (1 - X_{t,m}^{B}) \quad (15)$$

Time Constraints. The constraints (2) to (5) are expressed as follows:

$$\forall m \in \mathbb{M}^{\dashv\!\ominus}, \quad \beta_m^{ref} - L \leq \beta_m \leq \beta_m^{ref} \quad (16)$$
$$\forall m \in \mathbb{M}^{\ominus\!\!\rightharpoonup}, \quad \alpha_m^{ref} + L \geq \alpha_m \geq \alpha_m^{ref} \quad (17)$$
$$\forall m \in \mathbb{M}^{\hookrightarrow\!\!\ominus}, \quad \beta_m = \beta_m^{ref} \quad (18)$$
$$\forall m \in \mathbb{M}^{\ominus\!\!\to}, \quad \alpha_m = \alpha_m^{ref} \quad (19)$$

Allocation of Lines. For the movements passing on a given external line l_e, we allocate an internal line $l_i \in \mathbb{L}_{l_e}^{\ominus\!\rightarrow}$ which is reachable from the line l_e. This property and equation (6) can be expressed as follows:

$$\forall t \in \mathbb{T}, \forall m \in \mathbb{M}^t, \forall l_i \in \mathbb{L}^{\ominus\!\rightarrow}, \qquad X_{l_i,m}^{L^{\ominus\!\rightarrow}M} = X_{l_i,t}^{L^{\ominus\!\rightarrow}T} \quad (20)$$

$$\forall l_e \in \mathbb{L}^{\rightarrow\!\oplus}, \forall m \in \mathbb{M}^{l_e}, \qquad \sum_{l_i \in \mathbb{L}_{l_e}^{\ominus\!\rightarrow}} X_{l_i,m}^{L^{\ominus\!\rightarrow}M} = 1 \quad (21)$$

Allocation of Paths. According to equation (1), the choice of paths for a movement can be expressed as below:

$$\forall l_e \in \mathbb{L}^{\rightarrow\!\oplus}, \forall m \in \mathbb{M}^{l_e^{\rightarrow\!\oplus}}, \forall l_i \in \mathbb{L}_{l_e}^{\ominus\!\rightarrow}$$
$$\sum_{p \in \mathbb{P}^{(l_i,l_e)}} X_{p,m}^{PM} = X_{l_i,m}^{L^{\ominus\!\rightarrow}M} \quad (22)$$

Compatibility of Lines. The constraints of occupation of lines (10) indicate that two trains cannot occupy a same line at the same time. This rule is expressed as follows:

$$\forall t, t' \in \mathbb{T}, t \neq t', \forall l \in \mathbb{L}^{\ominus\!\rightarrow}, \quad (23)$$
$$B_t \leq A_{t'} + C \cdot (3 - X_{l,t}^{L^{\ominus\!\rightarrow}T} - X_{l,t'}^{L^{\ominus\!\rightarrow}T} - X_{t,t'}^{OrderT})$$
$$\forall t, t' \in \mathbb{T}, t \neq t', \qquad X_{t,t'}^{OrderT} + X_{t',t}^{OrderT} = 1 \quad (24)$$

The constraint (23) indicates that if two trains t and t' are allocated to the same line l in the railway station and if the train t circulates before t', then the term $3 - X_{l,t}^{L^{\ominus\!\rightarrow}T} - X_{l,t'}^{L^{\ominus\!\rightarrow}T} - X_{t,t'}^{OrderT} = 0$. We have then $B_t \leq A_{t'}$. Otherwise this term is larger than zero, and the constraint (23) is relaxed.

Compatibility of Switches. The constraint of occupation of switches (11) indicates that two movements m and m' cannot pass the same switches at the same time. Such constraint is enforced as above.

$$\forall m, m' \in \mathbb{M}, m \neq m', \forall p, p' \in \mathbb{P}, p \neq p' \quad (25)$$
$$\beta_m \leq \alpha_{m'} + C \cdot (4 - X_{p,m}^{PM} - X_{p',m'}^{PM} - X_{m,m'}^{OrderM} - Y_{p,p'}^{P})$$
$$\forall m, m' \in \mathbb{M}, m \neq m',$$
$$X_{m,m'}^{OrderM} + X_{m',m}^{OrderM} = 1 \quad (26)$$

Objective Function. The objective we focus on is to minimize the lines' occupancy rate, which can be expressed as follows:

$$\min \sum_{t \in \mathbb{T}} (B_t - A_t) \quad (27)$$

4 IMPROVEMENT OF THE MATHEMATICAL MODEL

4.1 Continuous-time Model

The first major issue for scheduling problems concerns the time representation. All existing scheduling formulations can be classified into two main categories: discrete-time models and continuous-time models (Floudas and Lin, 2004).

The time horizon (24 hours) of discrete-time scheduling formulations is divided into 1440 minutes as the train time table showed to passengers. The division of the long time horizon into small time interval length leads to very large combinatorial problems of intractable size.

In continuous-time models, events can be potentially associated with any point in the continuous domain of time. Because of the possibility of eliminating a major fraction of the inactive event-time interval assignments with the continuous-time approach, the mathematical programming problems are usually of much smaller size and require less computational efforts for their solution.

Based on the studies of the mathematical model in section 3, we can prove that all the time variables are integer-valued. We divide the whole problem into two separate parts: routing problem and scheduling problem. We focus on the scheduling problem and suppose the routing issue is known. The scheduling corresponding equations (12)-(15) (23)(25) are reformulated as the following equation (28), and equations(16)-(19) can be rewritten in the form of equation(29):

$$A.x \leq b \quad (28)$$
$$c \leq x \leq d \quad (29)$$

where A is an $m \times n$ matrix of $\{0, 1, -1\}$, and b, c and d are positive integer m-vector. In our problem, x represents a vector including all the scheduling variables α_m, β_m, A_t and B_t, $x = [\alpha_1 \ldots \alpha_M, \beta_1 \ldots \beta_M, A_1 \ldots A_T, B_1 \ldots B_T]$. In certain rows of A, there is one variable weighted 1, another weighted -1 and all others weighted 0. Other rows contain only 0. So each column of A^T contains zero or two non-zeroes. The set N of row indices of A^T can be partitioned into $N_1 \cup N_2$, N_1 is the set of rows including only 0, and N_2 is the set of rows with 1 and -1. Recall the results of (Heller and Tompkins, 1956) and (Hoffman and Kruskal, 1956):

Proposition 1. (Heller and Tompkins, 1956) A matrix A is totally unimodular if

- each entry is 0,1 or -1;

- each column contains at most two non-zeroes;
- the set N of row indices of A can be partitioned into $N_1 \cup N_2$, so that in each column j with two non-zeroes we have $\sum_{i\in\mathbb{N}_1} a_{i,j} = \sum_{i\in\mathbb{N}_2} a_{i,j}$.

For the matrix A^T, we have $\sum_{i\in\mathbb{N}_1} a_{i,j} = \sum_{i\in\mathbb{N}_2} a_{i,j} = 0$. According to the proposition above, A^T is a totally unimodular matrix.

Proposition 2. If A is TU then A^T also TU.

So the matrix A in equation (28) is totally unimodular.

Theorem 1. (Hoffman and Kruskal, 1956) Let A be an integral $m \times n$ matrix, the polyhedron $P(A,b) = \{x : x \geq 0, A.x \leq b\}$ is integral for all integral vectors $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.

Based on Hoffman and Kruskal's theorem, every vertex solution, the n-vector x, is integral. The integrality of the scheduling decision variables is guaranteed. So the scheduling decision variables α_m, β_m, A_t and B_t are defined in the continuous-time domain.

4.2 Reduction of Model

To improve the calculation performance, we seek to reduce the number of constraints. We design an indicator as the probe of potential conflicts between movements $C^{refM}_{m,m'}$ and between trains $C^{refT}_{t,t'}$. In this way, the constraints are created only for the movements and trains with potential conflicts. The undesired constraints are cut off. Four additional parameters need to be created as below.

${\alpha_m}^{Early}$ is the earliest departure time of the movement m.

${\beta_m}^{Late}$ is the latest arrival time of the movement m.

${A_t}^{Early} = \min_{m\in\mathbb{M}^t} {\alpha_m}^{Early}$

${B_t}^{Late} = \max_{m\in\mathbb{M}^t} {\beta_m}^{Late}$

The possible time interval of technical movements is $[{\alpha_m}^{Early}, {\beta_m}^{Late}]$. The possible time interval of trains is $[{A_t}^{Early}, {B_t}^{Late}]$. These parameters can be precalculated using the given problem instance. In this case, for all $m \in \mathbb{M}^{t\oplus}$, we have $[{\alpha_m}^{Early}, {\beta_m}^{Late}] = [{\alpha_m}^{ref}, {\beta_m}^{ref} + L]$. For all $m \in \mathbb{M}^{t\ominus}$, we have $[{\alpha_m}^{Early}, {\beta_m}^{Late}] = [{\alpha_m}^{ref} - L, {\beta_m}^{ref}]$.

$C^{refM}_{m,m'} = \delta([{\alpha_m}^{Early}, {\beta_m}^{Late}] \cap [{\alpha_{m'}}^{Early}, {\beta_{m'}}^{Late}] \neq \varnothing)$ indicate the potential time conflict of two movements. $C^{refT}_{t,t'} = \delta([{A_t}^{Early}, {B_t}^{Late}] \cap [{A_{t'}}^{Early}, {B_{t'}}^{Late}] \neq \varnothing)$ indicate the potential time conflict of two trains. So $C^{refT}_{t,t'} = 1$ is added as a condition in the equation (23) and (24), $C^{refM}_{m,m'} = 1$ is added as a condition in the equation (25) and (26).

The numerical experiments in section 4.3 show that the number of constraints decreases considerably.

4.3 Numerical Experiments

The computational study is conducted using AMPL and CPLEX version 12.5. The hardware architecture is x86-64, with Intel i5-2520M CPU at 2.5GHz and 8GB memory RAM.

We compare the original model and the improved model using a real railway station with 18 switches, 15 internal lines and 10 external lines. There are 247 trains 504 movements per day. In the rush hours, there are up to 3 trains running at the same time and up to 10 trains staying at the platforms.

Once the variables and constraints are sent to the solver, the problem will be adjusted by CPLEX presolve which eliminates the redundant constraints and variables. The whole problem is divided into small size problems in chronological order. So we have 50 groups of 5 trains, 24 groups of 10 trains, 16 groups of 15 trains, 12 groups of 20 trains, 9 groups of 25 trains and 8 groups of 30 trains. The draft timetable that defines the problem instances includes the parameters reference times of commercial movements and technical movements without any feasibility checking.

We try to solve the problems with three different models that are described in Section 3 and Section 4: discrete-time model (DT), continuous-time model (CT) and reduced continuous-time model (RCT). The results are separately presented in Table 1, Table 2 and Table 3. In each group, the complexity of the problems is different. The problem instance solved in the minimum or the maximum solve time is presented in the tables(rows labeled `Min` and `Max` respectively). The problem with the average solve time is to be constructed using the solve information of the whole groups.

Compared with the discrete-time model in Table 1, the continuous-time model has the same amount of variables and constraints, but the solve time decreases by 17.5% on average. The discrete-time model has 9 problems unsolved, and the continuous-time model has 5 problems unsolved. The solutions are all integral as we have proven in the section 4.1. So the improvements of continuous-time model are qualified.

Compared with the complete model, the reduced continuous-time model drops 22.1% variables and 66.2% constraints on average. The solve time decreases by 45.7% compared with the discrete-time model and decreases by 30.6% compared with the continuous-time model.

The infeasibility case is caused by the conflicts between the technical movements and the commercial movements. The adjustable time interval for technical movements $L = 10$ minutes in equations (16) and (17) is too tight to ensure the existence of solution.

Table 1: Discrete-time model.

	Trains	Movements		Time	Before presolve		After presolve		First solution		Number of groups		
	per group	tech.	comm.	interval	Variables	Constraints	Variables	Constraints	Solution	Solve time	solved	infeasible	no result
Min	5	2	8	77	2140	4624	318	1913	156	0,02			
Average	5	3	7	77.6	2175	15199	391	7772	155	0.11	50	0	0
Max	5	5	5	51	2140	24684	455	14178	118	1.15			
Min	10	3	17	122	4530	28666	754	13457	365	0.08			
Average	10	7	13	105.7	4598	59962	929	32104	312	2.07	24	0	0
Max	10	9	12	64	4764	89769	1030	49525	254	14.98			
Min	15	7	23	94	7170	85283	1429	45439	524	1.15			
Average	15	10	20	140.3	7268	129144	1604	68617	485	9.80	13	3	0
Max	15	12	18	137	7170	194985	1748	113906	475	64.88			
Min	20	13	31	214	11168	150755	2318	74511	815	2.03			
Average	20	13	27	176.8	10233	215515	2391	112708	650	10.52	8	3	1
Max	20	13	27	140	10060	184949	2300	101232	646	23.59			
Min	25	17	33	213	13200	362041	3015	183220	769	9.02			
Average	25	17	33	220.4	13377	345295	3300	178056	811	19.93	5	1	3
Max	25	21	31	243	13790	265689	3015	142350	902	29.58			
Min	30	18	42	188	16590	475438	4348	240245	980	12.36			
Average	30	23	38	269.0	16800	533594	4490	272449	978	50.97	3	0	5
Max	30	23	39	243	17220	357778	3962	193601	1090	73.88			

Table 2: The continuous-time model.

	Trains	Movements		Time	Before presolve		After presolve		First solution		Number of groups		
	per group	tech.	comm.	interval	Variables	Constraints	Variables	Constraints	Solution	Solve time	solved	infeasible	no result
Min	5	2	8	77	2140	4624	318	1913	156	0.02			
Average	5	3	7	77.6	2175	15199	391	7772	154	0.07	50	0	0
Max	5	5	5	51	2140	24684	455	14178	118	0.50			
Min	10	3	17	122	4530	28666	754	13457	300	0.09			
Average	10	7	13	105.7	4598	59962	929	32104	313	1.03	24	0	0
Max	10	9	12	64	4764	89769	1030	49525	254	5.30			
Min	15	7	23	94	7170	85283	1429	45439	528	1.05			
Average	15	10	20	140.3	7268	129144	1604	68617	487	7.63	13	3	0
Max	15	12	18	137	7170	194985	1748	113906	475	63.45			
Min	20	13	31	214	11168	150755	2318	74511	827	1.83			
Average	20	8	33	176.8	10233	215515	2391	112708	648	7.03	8	4	0
Max	20	13	27	137	10334	151907	2368	79990	617	11.79			
Min	25	14	36	149	13494	220090	3184	116923	844	8.92			
Average	25	14	38	220.4	13377	345295	3300	178056	823	10.61	5	3	1
Max	25	21	31	163	13200	298832	3110	156363	889	15.16			
Min	30	18	42	188	16590	475438	4348	240245	992	11.67			
Average	30	23	38	269.0	16800	533594	4490	272449	973	24.94	3	1	4
Max	30	23	39	376	16590	767565	5160	383502	858	35.24			

Table 3: The reduced continuous-time model.

	Trains	Movements		Time	Before presolve		After presolve		First solution		Number of groups		
	per group	tech.	comm.	interval	Variables	Constraints	Variables	Constraints	Solution	Solve time	solved	infeasible	no result
Min	5	2	8	77	2130	2812	304	1013	156	0.00			
Average	5	3	4	77.6	2165	5651	350	3265	154	0.04	50	0	0
Max	5	5	5	45	2130	18610	422	13545	140	0.20			
Min	10	9	11	102	4510	9048	771	6472	256	0.03			
Average	10	7	13	105.7	4578	18413	753	12649	314	1.71	24	0	0
Max	10	9	12	112	4743	31484	845	23404	254	26.96			
Min	15	7	23	94	7140	19883	1100	13133	543	0.94			
Average	15	10	20	140.3	7238	31139	1164	22499	489	15.13	13	3	0
Max	15	12	18	137	7140	73169	1348	57475	473	166.27			
Min	20	13	31	214	11124	30407	1490	18974	841	1.93			
Average	20	13	27	176.8	10192	42426	1573	30343	657	4.38	8	4	0
Max	20	9	31	163	10020	41532	1560	29595	775	6.30			
Min	25	21	31	243	13738	45417	1905	33419	914	4.93			
Average	25	17	33	220.4	52886	13326	1995	38258	818	10.33	5	1	3
Max	25	11	40	148	13443	44199	2041	30294	832	24.84			
Min	30	18	42	188	16530	73634	2386	53136	994	10.20			
Average	30	23	38	269.0	16739	74726	2407	55610	998	11.43	3	2	3
Max	30	27	33	376	16530	88703	2503	66487	877	12.76			

When the value of L is increased, one can find an optimal solution but the solving time can also be greatly increased. For example, setting $L = 30$ helps to find solutions to three cases previously labeled infeasible at the expense of solve time 330 seconds instead of the average time of 12 seconds. Further experimentations are necessary so that the value of L is adjusted in order to get the best tradeoff between the solution feasibility and the solving time.

5 CONCLUSIONS

This paper describes a mixed-integer linear program for routing and scheduling trains through a railway station to find a conflict-free schedule, given the detailed information of commercial movements. Considering the time representation, we compare the continuous-time and discrete-time models. The continuous-time mathematical model satisfies our computational requirement and decreases the problem size. Furthermore, to speed up the calculation, we try to cut off the redundant constraints and concentrate on the potential conflicts. Based on the numerical experiments, the improvements of the reduced continuous-time model are qualified. For the moment, we can solve example up to 60 trains, 121 movements during 385 minutes. The solve time of the first feasible solution is 97.8438 seconds. The solve time depends on the testing example.

To solve problems of larger size, we propose to use the decomposition methods (Benders, 1962) (Binato et al., 2001) (Cordeau et al., 1975). All trains are divided into groups in chronological sequence. The group solved is considered as the valid constraints of shared resources for the succedent groups. The adjacent groups have common trains as a buffer, i.e. the group size is 40 and the buffer group size is 20. The partitioning procedures are followed until the end of the problem. This method can be used to solve the real-time train routing and scheduling problem.

REFERENCES

Bai, L., Bourdeaud'huy, T., Castelain, E., and Rabenasolo, B. (2013). Recherche de plans d'occupation de voies dans les réseaux ferroviaires. In *14e conférence ROADEF de la société Française de Recherche Opérationnelle et Aide à la Décision*.

Benders, J. (1962). Partitioning procedures for solving mixed-variables programming problems. In *Numerische Mathematik, 4: 238-252*.

Binato, S., Pereira, M., and Granville, S. (2001). A new benders decomposition approach to solve power transmission network design problems. In *Numerische Mathematik, 16(2): 235-240*.

Caimi, G. (2009). Algorithmic decision support for train scheduling in a large railway network. In *PhD thesis*. ETH Zurich.

Carey, M. (1994a). Extending a train pathing model from one-way to two-way track. In *Transportation Research B 28 (5): 395-400*.

Carey, M. (1994b). A model and strategy for train pathing with choice of lines, platforms and routes. In *Transportation Research B 28 (5): 333-353*.

Carey, M. and Carville, S. (2003). Scheduling and platforming trains at busy complex stations. In *Transportation Research A 37 (3): 195-224*.

Cordeau, J., Soumis, F., and Desrosiers, J. (1975). A benders decomposition approach for the locomotive and car assignment problem. In *Transportation Science, 34(2): 133-149*.

D'Ariano, A. (2008). Improving real-time train dispatching: model, algorithms and applications. In *PhD thesis*. TU Delft.

D'Ariano, A., Pacciarelli, D., and Pranzo, M. (2007). A branch and bound algorithm for scheduling trains in railway network. In *European Journal of Operational Research 183: 643-657*.

Floudas, C. and Lin, X. (2004). Continuous-time versus discrete-time approaches for scheduling of chemical processes: A review. In *Computers and Chemical Engineering 28: 2109-2129*.

Heller, I. and Tompkins, C. (1956). An extension of a theorem of Dantzig's. In *Linear inequalities and related systems: 247-254*. Princeton Univ. Press, Princeton, NJ.

Hoffman, A. and Kruskal, J. (1956). Integral boundary points of convex polyhedra. In *Linear inequalities and related systems: 223-246*. Princeton Univ. Press, Princeton, NJ.

Kroon, L., Romeijn, H., and Zwaneveld, P. (1997). Routing trains through railway stations: Complexity issues. In *European Journal of Operational Research 98: 485-498*.

Kroon, L. and Zwaneveld, P. (1995). Stations: Final report of phase 1, ERASMUS management report series no.201. In *Rotterdam School of Manangement*. Erasmus University Rotterdam. Rotterdam. Netherlands.

Schrijver, A. and Steenbeek, A. (1994). Dienstregeling ontwikkeling voor Railned (timetable development for railway station). In *Report Cadans 1.0*. C.W.I. Amsterdam, Netherlands.

Serafini, P. and Ukovich., W. (1989). Mathematical model for periodic scheduling problems. In *SIAM Journal on Discrete Mathematics 2: 550-581*.

Watson., R. (2001). The effects of railway privatisation on train planning. In *Transport Reviews 21(2): 181-193*.

Zwaneveld, P. (1997). Railway planning-routing of trains and allocation of passenger lines. In *PhD thesis*. Erasmus University Rotterdam. Rotterdam. Netherlands.

Zwaneveld, P., Kroon, L., and Ambergen, H. (1996). A decision support system for routing trains through railway stations. In *Proceedings of Comp Rail 96:217-226*. Computational Mechanics Publications, Southampton, Berlin.

Zwaneveld, P., Kroon, L., and van Hoesel, S. (2001). Routing trains through a railway station based on a node packing model. In *European Journal of Operational Research 128: 14-33*.