A First Algorithm to Calculate Force Histograms in the Case of 3D Vector Objects

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Abstract: In daily conversation, people use spatial prepositions to denote spatial relationships and describe relative positions. Various quantitative relative position descriptors can be found in the literature. However, they all have been designed with 2D objects in mind, most of them cannot be extended to handle 3D objects in vector form, and there is currently no implementation able to process such objects. In this paper, we build on a descriptor called the histogram of forces, and we present the first algorithm for quantitative relative position descriptor calculation in the case of 3D vector objects. Experiments validate the approach.

1 INTRODUCTION

In daily conversation, people use spatial prepositions to denote spatial relationships and describe relative positions (e.g., the apple *in* the bowl, the bowl *near* the vase, the vase in *front* of the window). Most research on relative position descriptors and models of spatial relationships has focused so far on qualitative approaches and 2D objects (or 2D perspectives of 3D objects), often with the assumption that the objects were far enough from each other and could be approximated by their centres or minimum bounding rectangles. Unacceptable processing times, human cognitive limitations, a strong inhibitor factor (our long history with 2D research), the ubiquity of 2D data and the increased complexity of 3D modelling have channelled the researchers' attention away from quantitative approaches, 3D objects and intricate configurations. In the past few years, however, computer processing speed as well as storage and memory capacity have kept improving at exponential rates, technical limitations to the handling of 3D spatial data have been decreasing, and there has been a surge of wide-ranging interest in 3D contents.

In this paper, we present what we believe is the first algorithm for quantitative relative position descriptor calculation in the case of 3D objects in vector form. Various descriptors can be found in the literature (Miyajima and Ralescu, 1994); (Wang and Makedon, 2003); (Kwasnicka and Paradowski, 2005); (Zhang et al., 2010). As far as we know, however, they all have been designed with 2D objects in mind (mainly objects in raster form), most of them cannot be extended to handle 3D vector objects, and there is currently no implementation able to process such objects. After a thorough comparative analysis, we have chosen to build on a descriptor called the histogram of forces (Matsakis et al., 2011). Its mathematical definition holds in any Euclidean space, and theory endows it with remarkable properties. It is able to handle a variety of objects (e.g., connected or disconnected, with or without holes, disjoint or intersecting). Its behaviour towards affine transformations is known. It can easily be normalized to achieve invariance under translations, rotations, reflections and scalings. It lends itself to the design of quantitative models of spatial relationships that also satisfy remarkable properties. From a practical point of view, in the case of 2D objects, it has shown to be robust to noise, its discriminative power is high, the existing algorithms are highly parallelizable and include subalgorithms often implemented in the firmware or hardware of graphics cards. As a result, force histograms have been used to interpret human-to-robot commands and generate robot-to-human feedback (Skubic et al., 2004), for scene matching (Sjahputera and Keller, 2007), in a geospatial information retrieval and indexing system (Shyu et al., 2007), in a land cover classification system (Vaduva et al., 2010), etc.

The concept of the histogram of forces is described in Section 2. The new algorithm for the handling of 3D vector objects is introduced in Sec-

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tion 3. Experimental results follow in Section 4, and Section 5 concludes the paper.

2 BACKGROUND

Section 2.1 gives an informal definition of a force histogram, while Section 2.2 briefly describes the existing algorithm for the handling of 2D vector objects. Finally, Section 2.3 explains why the handling of 3D objects is related to the problem of finding an even distribution of points on the unit sphere.

2.1 Force Histograms

The mathematical definition of the histogram of forces presented in (Matsakis et al., 2011) is quite general, and it holds in any Euclidean space. We give here a narrower and less formal definition. Consider two distinct points p and q. They are seen as infinitesimal particles of mass 1. According to Newton's law of gravity, p exerts on q the force

$$qp \,/\, |qp|^3 \tag{1}$$

where qp is the vector from q to p and $|qp|$ its length. This force tends to move q towards p, and its magnitude is $1 \,/\, |qp|^2$. Now, consider two subsets A and B of the Euclidean space. Assume each one is an *object*, i.e., a nonempty bounded set of points, equal to the closure of its interior, and with a finite number of connected components. In dimension 2, each component is seen as a homogeneous plate with a density (mass per unit area) of 1. In dimension 3, it is seen as a homogeneous solid with a density (mass per volume) of 1. Every point p of the object A exerts on every $q \neq p$ of B an infinitesimal gravitational force. The vector sum of all these forces, i.e., the resultant force exerted by A on B, can be found using integral calculus. Instead, however, consider a real number r and a unit vector θ, replace (1) with (2), and calculate the magnitude $h_r^{AB}(\theta)$ of the vector sum of all the infinitesimal forces in direction θ (Fig. 1). The function h_r^{AB} so defined is called a *force histogram*. It is one possible representation of the position of A relative to B.

$$qp \,/\, |qp|^{r+1} \tag{2}$$

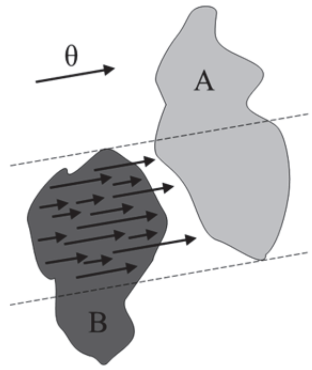


Figure 1: Every point of A exerts on every point of B an infinitesimal force. Using integral calculus, find the vector sum of the forces in direction θ. Its magnitude is $h_r^{AB}(\theta)$.

2.2 The Case of 2D Vector Objects

An algorithm for calculating force histograms in the case of 2D vector objects is presented in (Recoskie et al., 2012). The objects considered are fuzzy subsets of the Euclidean plane. It is assumed that the number of distinct α-cuts of an object is finite and that each α-cut can be expressed—using the union and difference set operations—in terms of a finite number of simple polygons. No other assumptions are made. An α-cut may therefore be convex or concave, connected or disconnected, and may have holes in it. Moreover, pairs of overlapping objects can be handled. Let us briefly describe the case of a pair of crisp objects A and B with non-intersecting interiors. Here is how to calculate $h_r^{AB}(\theta)$. The straight lines in direction θ that pass through the objects' vertices divide the objects into trapezoidal pieces A_1, A_2, etc., and B_1, B_2, etc. (Fig. 2*a*). We have:

$$h_r^{AB}(\theta) = \Sigma_i \, \Sigma_j \, h_r^{A_iB_j}(\theta) \tag{3}$$

$h_r^{A_iB_j}(\theta) = 0$ unless the pieces A_i and B_j are between two consecutive lines. If they are, $h_r^{A_iB_j}(\theta)$ can be expressed in terms of r, θ, the edge lengths and the distances between the edges of the two pieces. There are nine possible expressions, depending on the configuration (Fig. 2*bcd*) and the value for r. These expressions are relatively complex closed-form expressions that result from the symbolic calculation of definite triple integrals. Note that $h_r^{AB}(\theta)$ is computed in $O(\eta \log \eta)$ time, where η is the total number of object vertices.

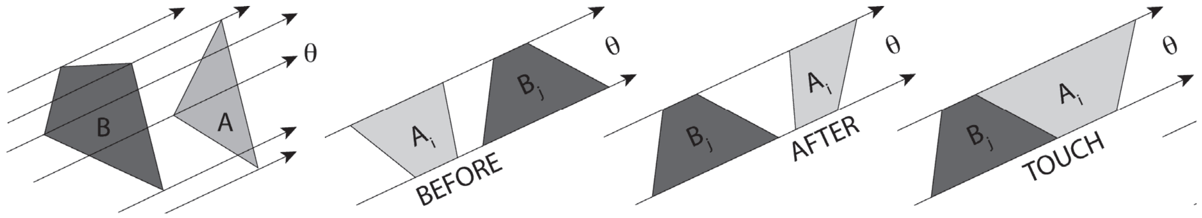


Figure 2: *(a)* The objects *A* and *B* are broken into trapezoidal (or triangular) pieces *A*1, *A*2, etc. and *B*1, *B*2, etc. Note that *A* and *B* are not necessarily convex, and there may be more than two pieces between two consecutive lines. Two pieces *Ai* and *Bj* between the same two consecutive lines can be arranged in three possible ways: *(b)*, *(c)* or *(d)*. Note that in *(b)*, the trapezoids may share one vertex or one edge; in *(c)* they may only share one vertex.

2.3 Reference Directions

Practically, of course, only a finite number of directions can be considered when calculating a force histogram. An important issue is the choice of these *reference directions*. The higher the number of reference directions, the more complete the collected histogram data, but the longer the processing time.

In the 2D case, the reference directions are usually chosen so that they are evenly distributed in space (Matsakis et al., 2011). Since a direction θ can be represented by a point p on the unit circle centred at the origin ω (choose p such that ωp=θ), the problem comes down to finding an even distribution of points on the circle. The set of reference directions therefore corresponds to the set of vertices of a regular convex polygon.

In the 3D case, the unit circle becomes the unit sphere and regular convex polygons become regular convex polyhedra. There are only five such polyhedra (known as the Platonic solids). One must thus reflect on what an even distribution of an arbitrary number of points on a sphere is. The topic has attracted the attention of a wide variety of researchers (Saff and Kuijlaars, 1997) (Darvas, 2007), and many different criteria for point distribution can be found in the literature. The general idea is to optimize some function of the positions of the points on the sphere. As an example, one may want to see the points as electrons that repel each other with a force given by Coulomb's law and determine the minimum energy configuration. This is the Thomson problem (Thomson, 1904). Practically, points are first randomly generated on the sphere, and then an iterative process allows a stable configuration to be found. For example, Bourke uses hill climbing (Bourke, 1996), while Semechko uses a more efficient adaptive Gauss-Seidel update scheme (Semechko, 2012).

3 ALGORITHM

The calculation of a force histogram that represents the relative position of two 3D connected objects in vector form is described, in pseudocode, on the next page. It relies on a simple numerical integration technique called the *composite midpoint rule*. The integral of a function f over an interval $[a,b]$ is calculated as follows: $[a,b]$ is divided into subintervals of equal length; the integral of f over each subinterval $[a_i, a_{i+1}]$ is approximated by $(a_{i+1}-a_i)$ $f((a_i+a_{i+1})/2)$; the integral of f over $[a,b]$ is obtained by adding up all the results. In our algorithm, each force histogram value h(θ) is approximated using this technique (hence the **for** loop; line 13). First, the direction θ is rotated together with the objects so that it lies in the xy-plane (line 14; Fig. 3). The domain of integration $[a,b]$ can then be easily determined by sorting the vertices along the z-axis (line 15; Fig. 3).

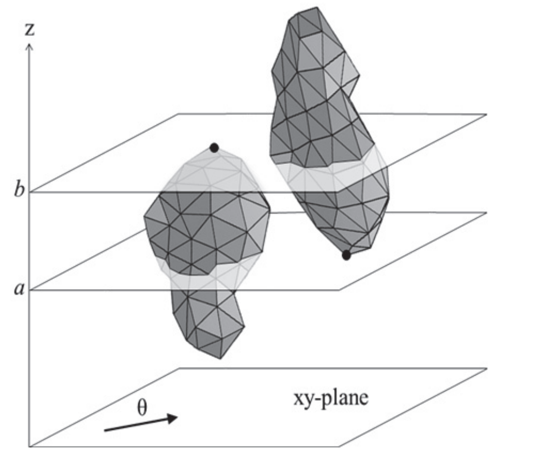


Figure 3: Each plane parallel to the xy-plane and with z-coordinates between *a* and *b* slices the 3D objects into 2D objects. The force in direction θ between the 3D objects is the integral over $[a,b]$ of the force in direction θ between the 2D objects.

```
1  function forceHistogramCalculation
2
3    input two objects A and B
4          a set of reference directions
5          a relative error tolerance relErrTol
6    output the force histogram h that represents the position of A relative to B
7
8      absErrTol ← 0
9      c_A ← centroid of object A
10     c_B ← centroid of object B
11     θ_0 ← reference direction that maximizes the dot product θ · c_Bc_A
12
13     for each reference direction θ, starting with θ_0
14          rotate θ together with the objects so that θ lies in the xy-plane
15               determine the domain of integration [a,b]   % these are z-values
16               if a >= b then h(θ) ← 0
17                         continue
18               n ← INITIAL_NUMBER_SUBINTERVALS
19               approx1 ← integral approximation using n subintervals
20               do
21                      n ← n*2
22                      approx2 ← integral approximation using n subintervals
23                      absErr ← |approx1−approx2|
24                      if approx2 = 0 then  % it seems that the histogram value is 0
25                                           % (we have tried enough slices):
26                                           relErr ← 0
27                      else relErr ← absErr / approx2
28                      approx1 ← approx2
29               while relErr > relErrTol and absErr > absErrTol
30               if θ = θ_0 then absErrTol ← approx2 * relErrTol
31               h(θ) ← approx2
```

Each plane parallel to the xy-plane and with z-coordinates between a and b slices the two 3D vector objects into two 2D vector objects (which may have multiple connected components). $h(\theta)$ is the integral over $[a,b]$ of the force in direction θ between these 2D objects (a force calculated as in Section 2.2). The number of subintervals considered is first set to *INITIAL_NUMBER_ SUBINTERVALS* (line 18). This number is then repeatedly doubled (line 21) until the approximation of the integral is found satisfactory (line 29). The accuracy of numerical integration is controlled by the relative and absolute error tolerances *relErrTol* and *absErrTol* (line 29). The absolute difference between two consecutive approximations is used as an estimate of the absolute error *absErr* (line 23). An estimate of the relative error *relErr* follows (line 27). The reference direction θ_0 closest to the direction defined by the centroids of the objects (lines 9-11) is likely to give one of the highest force histogram values. It is therefore considered first (line 13) and used to determine *absErrTol* (line 30). The combined use of *relErrTol* and *absErrTol* (line 29) stems from the following: assume the relative error tolerance in input (line 5) is 1% and the true force histogram values in some directions θ_1 and θ_2 are 100 and 10; if we accept 99 as an approximation of the first value (relative error 1%, absolute error 1), we should accept 9 as an approximation of the second value (relative error 10%, absolute error 1).

4 EXPERIMENTS

The experimental setup is described in Section 4.1 and the results are given and discussed in Section 4.2.

4.1 Setup

The algorithm for force histogram calculation in the case of 3D vector objects was implemented in C. The experiments were conducted on a machine running the Linux 3.11.1 kernel with the AMD Phenom II X6 1055T processor, 2.8GHz, 8 GB. They involve the five objects A, B, C, D and T shown in Fig. 4. The scene is relatively simple, but it is a familiar scene, with common real-world objects, and approximating each object by its centre or minimum bounding box would be doomed to failure.

Three sets of reference directions are used for force histogram calculation: one with 102 directions (neighbour directions are about 20° apart), one with 414 directions (10° apart), and one with 1646 directions (5° apart). Each set of reference directions is a set of evenly distributed directions that includes the set {above, below, left, right, front, behind} of cardinal directions. See Fig. 5.

Assume h_1 and h_2 are two force histograms calculated using the same set of reference directions. How to compare these histograms? In (Matsakis et al., 2004), over twenty similarity measures are examined for the comparison of 2D force histograms, and two are retained: the *Tversky index*

$$\frac{\sum_{\theta} \min(h_1(\theta), h_2(\theta))}{\sum_{\theta} \max(h_1(\theta), h_2(\theta))} \quad (4)$$

and the *Pappis' measure*

$$1 - \frac{\sum_{\theta} |h_1(\theta) - h_2(\theta)|}{\sum_{\theta} |h_1(\theta) + h_2(\theta)|} \quad (5)$$

Both can be applied to 3D histograms as well, and they are used in Section 4.2.

A force histogram associated with two 2D objects *A* and *B* allows various spatial relationships between these objects to be assessed. In particular, the histogram can be used to calculate the truth value of a proposition such as "*A* is in direction θ of *B*" (e.g., "*A* is to the right of *B*", "*A* is above *B*"). Different methods can actually be applied (Matsakis et al., 2011). Those considered in Section 4.2 are the *aggregation* and *effective force* methods, as they can easily be extended to the handling of 3D histograms.

Finally, note that the relative error tolerance *relErrTol* (Section 3) may take three different values in Section 4.2: 0.1 (10%), 0.01 (1%), or 0.001 (0.1%). Moreover, the constant *INITIAL_NUMBER_SUBINTERVALS* is set to 2.

4.2 Results

A force histogram h associated with a pair of 3D objects can be graphically represented by the surface

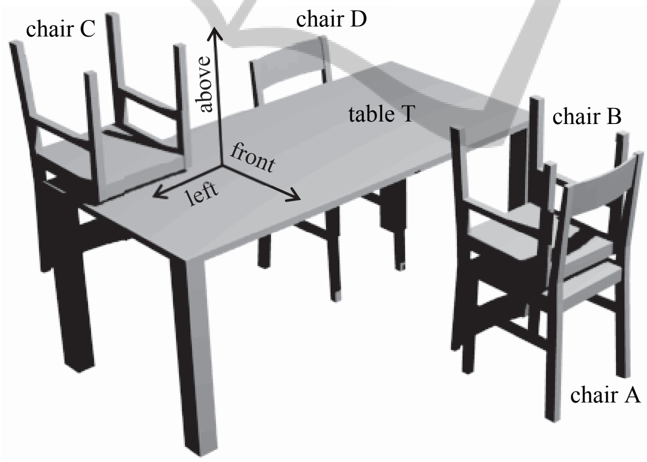


Figure 4: The objects: 1 table (48 vertices) and 4 chairs (128 vertices each).

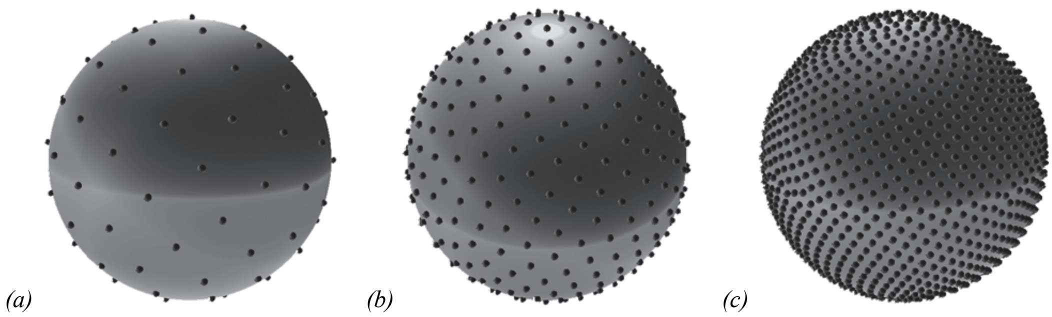


Figure 5: *(a)* The set of 102 reference directions. *(b)* 414 reference directions. *(c)* 1646 reference directions.

(h(θ)+R)θ, where the constant R is a positive real number and the variable θ is a direction (i.e., a unit vector). This surface is wrapped around the sphere of radius R, and bumps on it indicate the presence of forces. See Fig. 6.

Table 1 reports the truth values in the cardinal directions for each object pair. We believe the reader will find these values consistent with their own perception of the scene. 0 means "totally false" while 1 means "totally true". According to the aggregation method, an object may be, to some extent, simultaneously above and below—or to the left and to the right, front and behind—another one. The effective force method disagrees with this point of view, and has more clear-cut opinions. These are the main differences between the two methods.

Not surprisingly, the processing times increase when the relative error tolerance decreases (Table 2*a*). On average, rotating the objects and determining a domain of integration are procedures that take about 0.0001 second each (Table 2*b*), i.e., only 2% (resp. 1%, 0.2%) of the time needed to calculate a force histogram value when *relErrTol* is 10% (resp. 1%, 0.1%). See the numbers in bold in Table 2*ab* (0.0001/0.0054≈2%, 0.0001/0.0137≈1%, etc.). Calculating a 2D force is as fast, but the procedure is repeated many times. In the end, it represents about 11% (resp. 15%, 15%) of the time needed to calculate a force histogram value when *relErrTol* is 10% (resp. 1%, 0.1%). See the numbers in bold in Table 2*ac* (6×0.0001/0.0054≈11%, 20×0.0001/0.0137≈15%, etc.). Slicing the objects is by far the most time consuming. Overall, it represents about 55% (resp. 73%, 75%) of the time needed to calculate a force histogram value when *relErrTol* is 10% (resp. 1%, 0.1%). See the numbers in bold in Table 2*ac* (6× 0.0005/0.0054≈56%, 20×0.0005/0.0137≈73%, etc.).

The relative error tolerance has a visible impact on the force histogram. See Fig. 7. However, in applications where histograms are calculated only to be compared with each other, setting *relErrTol* to 0.01 seems to be a good choice. Indeed, a histogram calculated with *relErrTol* = 0.01 is obtained much faster than and is very similar (a 98% to 99% similarity, as shown in Table 3) to the histogram calculated with *relErrTol* = 0.001. In applications where truth values are extracted from force histograms, *relErrTol* = 0.1 might be enough when using the aggregation method, since *relErrTol* = 0.001 only gives a 1% absolute difference in truth value (at worst). See Table 4. When using the effective force method, *relErrTol* = 0.1 (resp. 0.01) vs. *relErrTol* = 0.001 gives a less than 1% absolute difference in truth value, on average, but that difference may reach 17% (resp. 9%) in the worst case. The number of reference directions seems to have even a bigger impact on truth values than the relative error tolerance. When using the aggregation method (resp. effective force method), 414 vs. 1646 directions gives a less than 1% absolute difference in truth value, on average, but that difference may reach 5% (resp. 13%) in the worst case. See Table 5.

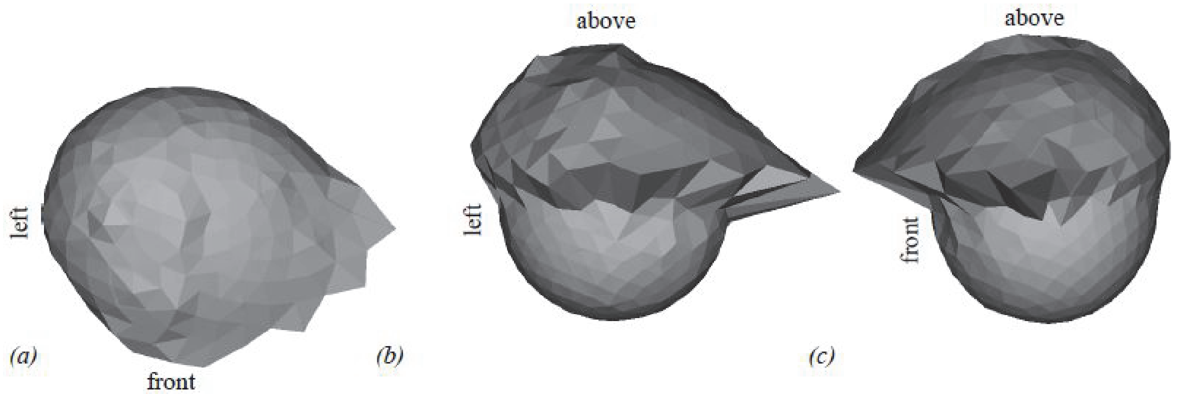


Figure 6: Three different views of the same force histogram: *(a)* from above, *(b)* from front, *(c)* from right. The histogram represents the position of the chair C relative to the table T. It has been calculated using 414 reference directions and a relative error tolerance of 1%.

Table 1: The truth values in the cardinal directions for each object pair in the scene. These values are extracted from the force histograms (414 directions, 1% relative error tolerance) using the *agg*regation and *eff*ective force methods.

B / A	*agg*	*eff*	**C / B**	*agg*	*eff*	**A / T**	*agg*	*eff*	**C / T**	*agg*	*eff*
above	**0.44**	**0.86**	above	0.13	0.46	above	0.03	0	above	**0.37**	**0.84**
below	0.02	0	below	0.00	0	below	0.19	0.40	below	0.01	0
left	0.14	0	left	**0.48**	**0.75**	left	0.15	0	left	0.21	0.06
right	0.16	0.06	right	0	0	right	0.17	0.02	right	0.15	0
front	0.12	0	front	0	0	front	**0.57**	**0.84**	front	0.10	0
behind	0.22	0.17	behind	0.48	0.72	behind	0	0	behind	0.26	0.24

C / A	*agg*	*eff*	**D / B**	*agg*	*eff*	**B / T**	*agg*	*eff*	**D / T**	*agg*	*eff*
above	0.17	0.47	above	0.02	0	above	0.05	0	above	0.04	0
below	0.00	0	below	0.08	0.38	below	0.15	0.30	below	**0.37**	**0.57**
left	0.40	0.65	left	0.08	0.23	left	0.16	0	left	0.18	0
right	0	0	right	0.03	0	right	0.17	0.06	right	0.21	0.05
front	0	0	front	0	0	front	**0.56**	**0.85**	front	0.05	0
behind	**0.55**	**0.74**	behind	**0.83**	**0.92**	behind	0	0	behind	0.24	0.36

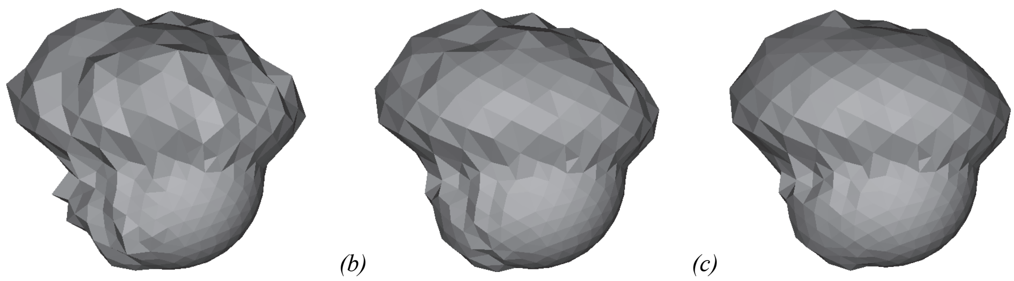
D / A	*agg*	*eff*	**D / C**	*agg*	*eff*
above	0.02	0	front	0.00	0
	0.02	0		0.24	0.51
	0.02	0		0	0
	0.02	0		**0.72**	**0.85**
	0	0		0.05	0.17
	0.94	**1.00**		0.04	0

Table 2: Processing times (in seconds). The force histograms are calculated for every pair of objects in the scene, using 414 reference directions.

(a)	Processing time								
	relErrTol = 0.1			*relErrTol* = 0.01			*relErrTol* = 0.001		
procedure	min	ave	max	min	ave	max	min	ave	max
calculating a force histogram	1s	2s	4s	1s	6s	14s	2s	24s	86s
calculating a force histogram value	0.0007	**0.0054**	0.0935	0.0007	**0.0137**	0.7628	0.0007	**0.0574**	3.5135

(b)	processing time		
procedure	min	ave	max
rotating the objects	0.0001	**0.0001**	0.0004
determining a domain of integration	0.0001	**0.0001**	0.0001

(c)	processing time			number of times the procedure is applied (per direction)								
				relErrTol = 0.1			*relErrTol* = 0.01			*relErrTol* = 0.001		
procedure	min	ave	max	min	ave	max	min	ave	max	min	ave	max
slicing the objects	0.0003	**0.0005**	0.0009									
calculating a 2D force	0.0000	**0.0001**	0.0014	0	**6**	126	0	**20**	1022	0	**86**	4094



(a) *(b)* *(c)*

Figure 7: These force histograms, which are shown from the same point of view, represent the position of the chair A relative to the chair B. They have been calculated using 414 reference directions and a relative error tolerance of *(a)* 10% *(b)* 1% *(c)* 0.1%.

Table 3: Comparing force histograms. For every pair of objects in the scene, compare the force histograms calculated using 414 reference directions and two different relative error tolerances.

	similarity					
	Tversky index			Pappis' measure		
relative error tolerance	min	ave	max	min	ave	max
0.1 vs. 0.01	0.84	0.92	0.97	0.91	0.96	0.98
0.1 vs. 0.001	0.84	0.91	0.96	0.91	0.95	0.98
0.01 vs. 0.001	0.97	**0.98**	0.99	0.99	**0.99**	1.00

Table 4: Comparing truth values (I). For every pair of objects in the scene, calculate the force histograms using 414 reference directions and two different relative error tolerances; then, compare the truth values obtained in every reference direction.

	absolute difference					
	aggregation method			effective force method		
relative error tolerance	min	ave	max	min	ave	max
0.1 vs. 0.01	0	0.002	0.013	0	0.005	0.169
0.1 vs. 0.001	0	0.002	**0.012**	0	0.005	**0.169**
0.01 vs. 0.001	0	0.001	0.004	0	0.002	**0.086**

Table 5: Comparing truth values (II). For every pair of objects in the scene, calculate the force histograms using a relative error tolerance of 0.01 and two different sets of reference directions; then, compare the truth values obtained in every cardinal direction (above, below, left, right, front, behind).

	absolute difference					
	aggregation method			effective force method		
number of directions	min	ave	max	min	ave	max
102 vs. 414	0	0.012	0.063	0	0.025	0.195
102 vs. 1646	0	0.014	0.093	0	0.029	0.280
414 vs. 1646	0	**0.005**	**0.054**	0	**0.009**	**0.134**

5 CONCLUSIONS

We have presented in this paper the first algorithm for quantitative relative position descriptor calculation in the case of 3D objects in vector form. We have built on the histogram of forces because its mathematical definition holds in any Euclidean space and theory endows it with remarkable properties. A force histogram associated with 2D objects allows various spatial relationships between these objects to be assessed through the calculation of truth values; we have shown that the same holds for a force histogram associated with 3D objects, and we have shown that the assessments are consistent with human perception. The new algorithm is an approximation algorithm with two parameters: the set of reference directions and the relative error tolerance. The higher the number of reference directions, the more complete the collected histogram data; the lower the relative error tolerance, the more accurate the collected data; but the longer the processing time. We have provided some insight on how the two parameters impact the processing times, the force histograms, and the truth values that can be extracted from the histograms. In future work, we will show that the processing times can be drastically reduced. In particular, we will use a much more sophisticated numerical integration technique, and we will calculate special sets of reference directions that will allow directions to be grouped and batch processed.

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