Image Analysis through Shifted Orthogonal Polynomial Moments

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Abstract: Image analysis is significant from the standpoint of image description. A well described image has merits in different research areas, e.g., image compression, machine learning, computer vision etc. This paper is an attempt to analyze graylevel images through shifted orthogonal polynomial moments, computed on a discrete disc. This removes the difficulty of computing the moments on an analytic disc. Excellent rotational invariance as well as illumination invariance is observed.

1 INTRODUCTION

Image analysis through moments has recently gained a good amount of attention during the last two decades in the community of image processing, computer vision and pattern recognition, though its initiation was made in 1962 when Hu (Hu, 1962) did his pioneering work on moment invariants. Afterwards, various works based on both non-orthogonal and orthogonal moments were carried out. Among the non-orthogonal moments, some of the reported works can be found in (Prokop and Reeves, 1992), (Reddi, 1981), (Abu-Mostafa and Psaltis, 1984). Similarly, works based on orthogonal moments can be found in (Teague, 1980), (Teh and Chin, 1988), (Z.L. Ping and Sheng, 2002), (H. Ren and Sheng, 2003) and (T. Xia and Luo, 2007). Attempts on discrete orthogonal moments using Chebyshev moments were made by (P.T. Yap and Ong, 2003) and (R. Mukundan and Lee, 2001), while Zhu et al. (H.Q. Zhu and Coatrieux, 2007) introduced a kind of orthogonal polynomials defined on non-uniform lattice, known as Racah polynomials. A good survey of works on moments can be found in the article of Shu et al. (H. Shu and Coatrieux, 2007) and (Jan Flusser and Zitova, 2009). In the present paper, moment-based rotational invariance using orthogonal shifted polynomials on discrete disc (Biswas and Chaudhuri, 1985) is proposed. Shifting function bijectively maps the interval $[0,1]$ to the interval $[-1,1]$. Shifted polynomials are, therefore, orthogonal on $[0,1]$, i.e.,on the unit disc. It should be noted that it is difficult to use analytic disc because of the pixel mapping problem on the analytic disc. On the other hand, using discrete disc has many advantages. The mapping is unique and straightforward because of the mathematical description of the discrete disc. This makes the algorithms straightforward. Results show excellent behavior of invariance under rotation and different conditions of illumination. This facilitates significant image description through orthogonal shifted polynomial image moments.

Below in Section 2, we briefly discuss discrete circles, rings and discs to help readers understand the mapping on discrete disc. Section 3 describes the proposed three different methods, while Section 4 demonstrates results and discussion. Finally, in Section 5 we present our conclusion.

2 DISCRETE CIRCLE, RING AND DISC

Consider a 2-dimensional discrete array space of $m \times n$ points or pixels so that any point or pixel (x, y), $0 \leq x \leq m-1$, $0 \leq y \leq n-1$. $x,y,m,n \in I$ (set of integers) can be mapped to the continuous real plane by a unit square about the center point (x$\pm\frac{1}{2}$, y$\pm\frac{1}{2}$). Also, for simplicity and convenience, let the radii of the discrete circle, ring and disc be integer valued with center of the unit squares.

Discrete Circle (dc)
A dc is a discrete space approximation to the circle defined in Euclidean geometry. In the present scheme of generation, a dc is defined as follows

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Definition 1. A dc with radius r and center (α, β) is a set S_r of 8-connected pixels so that each pixel (x, y) satisfies the inequality

$$r-\frac{1}{2}<|\sqrt{(x-\alpha)^2+(y-\beta)^2}|<r+\frac{1}{2} \quad (1)$$

Uniqueness of a dc under the above definition may be easily established.

Rings and Discs
Since a pixel covers a square area in real space, a dc has some width in real space. A ring or a disc can therefore be generated by the union of circles of radii $r, r+1, \cdots\cdots, r+m$. However it is interesting to observe the following properties in connection with the generation of a ring and a disc by the present method.

Definition 2. A discrete ring (dr) with integer radius r_1 and r_2, $r_2 > r_1$ and integer center (α, β) is given by

$$R(r_1,\ r_2,\ \alpha,\ \beta)=\bigcup_{r=r_1}^{r_2} S_r \quad (2)$$

if $r_1 = 0$ a discrete disc (dd) is generated. Here S_0 is assumed to be the center pixel itself. It is easy to show that there exist no hole or gap in the dr or dd generated according to the definition 2. Thus, a discrete disc (dd) with integer radius r and integer center (α,β) is given by

$$D(r,\alpha,\beta)=\bigcup_{r=0}^{r} S_r \quad (3)$$

It should be noted that when an image is mapped on a discrete disc, the circumference of different circles constituting the disc has different pixels of the image mapped onto it. As this map is unique, positions of pixels on each circle are also unique.

3 PROPOSED METHODS

We now examine three different shifted orthogonal polynomials. Proposed polynomials include the shifted Legendre and shifted Chebyshev polynomials of both the first and second kind. Note that shifted orthogonal polynomials have certain advantages over their non-shifted versions. The advantages are centered about the orthogonality on the unit interval $[0,1]$. Below we discuss these polynomials.

3.1 Shifted Legendre Polynomial

The shifted Legendre polynomial is given by

$$P_n^*(x)=P_n(2x-1) \quad (4)$$

Here, the shifting function shifts $x \longrightarrow 2x-1$. This shifting function is an affine transformation (i.e., it preserves straight lines which means all points lying on a straight line will lie on a line after the transformation. Ratios of distances between points lying on a straight line will remain unaffected but it does not necessarily preserve angles or lengths. Sets of parallel lines will remain parallel to each other). The shifting function is chosen to bijectively map the interval $[0,1]$ to the interval $[-1,1]$. As a result, the shifted Legendre polynomials are orthogonal on $[0,1]$. Since,

$$P_n(x)=\frac{1}{2^n}\sum_{t=0}^{[n/2]}(-1)^t\frac{n!}{(n-t)!t!}\frac{(2n-2t)!}{n!(n-2t)!}x^{n-2t}, \quad (5)$$

where $[n/2]$ is the maximum integer in $n/2$, we get

$$P_n^*(x)=\frac{1}{2^n}\sum_{t=0}^{[n/2]}(-1)^t\frac{(2n-2t)!}{t!(n-t)!(n-2t)!}(2x-1)^{n-2t}. \quad (6)$$

The orthogonality condition for this polynomial can be written as

$$\int_0^1 P_n^*(x)P_m^*(x)dx=\frac{1}{2n+1}\delta_{nm} \quad (7)$$

This orthogonality condition can be suitably changed to

$$\int_0^1 \sqrt{(2n+1)}P_n^*(x)\sqrt{(2m+1)}P_m^*(x)dx=\delta_{nm}. \quad (8)$$

To show the rotational invariance behavior of a polynomial, it must be converted to polar form, i.e., it must be expressed as a function of radius r and polar angle θ. Thus, it should have the form of $V(r,\theta)$ which in turn, for invariant representation under rotation of axes about the origin, can be explicitly written into its radial part $R_n(r)$ and polar part $e^{im\theta}$ (Bhatia and Wolf, 1954) as

$$V(r,\theta)=g_1(r)g_2(\theta)=R_n(r)e^{im\theta}. \quad (9)$$

Hence, we must express the shifted Legendre polynomial as the product of two functions. Now, one can observe the orthogonality of the radial part as

$$\int_0^1 R_n(r)R_m(r)rdr=\delta_{mn} \quad (10)$$

To compute the radial part for the shifted Legendre polynomial, we equate the integrands from equation (8) and equation (10), i.e.,

$$\sqrt{(2n+1)}P_n^*(x)\sqrt{(2m+1)}P_m^*(x)=R_n(r)R_m(r)r, \quad (11)$$

or,

$$\begin{aligned} R_n(r) &= \sqrt{(2n+1)}P_n^*(r)r^{-1/2}, \\ &= \sqrt{(2n+1)}r^{-1/2}\tfrac{1}{2^n}\sum_{t=0}^{[n/2]}(-1)^t \\ &\quad \tfrac{(2n-2t)!}{t!(n-t)!(n-2t)!}\times(2r-1)^{n-2t}. \end{aligned} \quad (12)$$

Therefore, in polar co-ordinates we finally get

$$V_{nm}(r,\theta) = R_n(r)e^{im\theta},$$

and since $V_{nm}(r,\theta$ is orthogonal on the unit disc, we write

$$\int_0^{2\pi}\int_0^1 V_{nm}(r,\theta)V_{pk}(r)rdrd\theta = \delta_{npmk}. \quad (13)$$

If we assume $f(x,y)$ is the digital graylevel image, then it should be suitably mapped to a discrete disc to get $f(r,\theta)$. The Legendre moment of the image $f(x,y)$ can be computed by

$$A^P_{nm} = \int_0^{2\pi}\int_0^1 f(r,\theta)R_n(r)e^{-im\theta}rdrd\theta. \quad (14)$$

Writing $r/r_{max} = \rho$, we get $\rho = 0$ when $r = 0$, and $\rho = 1$ when $r = r_{max}$. Hence, A^P_{nm} on the discrete unit disc can be written as

$$A^P_{nm} = \sum_{\rho=0}^{\rho=1}\rho R_n(\rho)(\sum_{\theta=0}^{2\pi} f(\rho,\theta)e^{-im\theta}). \quad (15)$$

3.2 Normalized Shifted Legendre Moments

To consider normalized Legendre moments A^L_{nm}, we must normalize $f(\rho,\theta)$, $R_n(r)$ and $e^{-im\theta}$. We normalize $f(\rho,\theta)$ by dividing it by the square root of the sum of its elements, i.e., the normalized value $\tilde{f}(\rho,\theta)$ is

$$\tilde{f}(\rho,\theta) = \frac{f(\rho,\theta)}{\sqrt{\sum_{\rho=0}^{\rho=1}\sum_{\theta=0}^{2\pi}[f(\rho,\theta)]^2}}, \quad (16)$$

so that

$$\sum_{\rho=0}^{\rho=1}\sum_{\theta=0}^{2\pi}[\tilde{f}(\rho,\theta)]^2 = 1 \quad (17)$$

Similarly, we normalize $R_n(\rho)$ by dividing it by the square root of the product of the maximum value of n, i.e., r_{max} and the sum of the squired values of $R_n(\rho)$.

$$\tilde{R}_n(\rho) = \frac{R_n(\rho)}{\sqrt{r_{max}\sum_{\rho\in D}[R_n(\rho)]^2}}, \quad (18)$$

where D is the discrete unit disc. Finally, $|e^{-im\theta}| = 1$. Hence, the normalized shifted Legendre moments is given by

$$\tilde{A}^{P^*}_{nm} = \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{f}(\rho,\theta)e^{-im\theta}). \quad (19)$$

3.3 Invariance and Illumination

It should be noted that equation (19) holds good, in general, for all shifted orthogonal polynomials.

3.3.1 Invariance

When a graylevel image $f(\rho,\theta)$ rotates about a point by an angle α, it becomes noisy and blurred to some extent. Normalized moments are capable of handling this situation. To observe this, we consider

$$g(\rho,\theta+\alpha) = f(\rho,\theta+\alpha) + n(\rho,\theta+\alpha) \quad (20)$$

Therefore,

$$\begin{aligned}\tilde{A}^{L(g)}_{nm} &= \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{g}(\rho,\theta+\alpha)e^{-im(\theta+\alpha)}),\\ &= \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{f}(\rho,\theta+\alpha)e^{-im(\theta+\alpha)})\\ &+\\ &\sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{n}(\rho,\theta+\alpha)e^{-im(\theta+\alpha)})\\ &\approx \tilde{A}^{L(f(\theta+\alpha))}_{nm}.\end{aligned} \quad (21)$$

L in equation (21), is used for P^* to indicate the validity of the equation for all shifted polynomials. since, the orthogonal moments $\tilde{A}_{nm}$ can be viewed as the correlation between the image and the moment kernel (Yap and Raveendran, 2004), the second term is zero because the correlation of the low spatial frequency moment kernel and the high spatial frequency of the noise is small. In other words, when n and m in $\tilde{A}_{nm}$ are low,

$$\sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{n}(\rho,\theta)e^{-im\theta}) \approx 0 \quad (22)$$

Therefore,

$$\begin{aligned}\tilde{A}^{L(f(\theta+\alpha))}_{nm} &= \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{f}(\rho,\theta+\alpha)\times\\ &e^{-im(\theta+\alpha)})\\ &= Tr(\alpha)\sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{f}(\rho,\theta)\times\\ &e^{-im(\theta+\alpha)})\\ &= \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}Tr(\alpha)(\tilde{f})(\rho,\theta)\times\\ &e^{-im(\theta+\alpha)})\\ &= \tilde{A}^{L(Tr(\alpha)f(\theta))}_{nm}e^{-im\alpha}\end{aligned} \quad (23)$$

where $Tr(\alpha)$ is the rotational transformation on the image in the discrete domain. Thus,

$$|\tilde{A}^{L(f(\theta+\alpha))}_{nm}| \approx |\tilde{A}^{L(f(\theta))}_{nm}| \quad (24)$$

3.4 Illumination

When the illumination changes by a factor of c, the new image $g(x,y) = cf(x,y)$. In polar co-ordinates $g(\rho,\theta) = cf(\rho,\theta)$. Now,

$$\tilde{g}(\rho,\theta) = \frac{c\tilde{f}(\rho,\theta)}{\sqrt{\sum_{\rho=0}^{\rho=1}\sum_{\theta=0}^{2\pi}|cf(\rho,\theta)|^2}} = \tilde{f}(\rho,\theta) \qquad (25)$$

Thus, under illumination change

$$\begin{aligned}\tilde{A}_{nm}^{L(g)} &= \sum_{\rho=0}^{\rho=1}\rho\tilde{R}_n(\rho)(\sum_{\theta=0}^{2\pi}\tilde{g}(\rho,\theta)\tilde{e}^{-im\theta}), \\ &= \tilde{A}_{nm}^{L(f)}.\end{aligned} \qquad (26)$$

It is already established that moments can serve well as features. From the computational point of view, low order normalized moments can be computed very quickly and hence the high order normalized moments can be easily obtained. We consider the ratio of high to low order moments as our feature. Low order moment is chosen as $M_l = \sqrt{(\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}|\tilde{A}_{nm}|^2}$, while the high order moment is $M_h = 1 - M_l$. M_l and M_h correspond to the low spatial frequency (low-pass) and high spatial frequency (high-pass) components of the image. The ratio of these two moments, $R_M = M_h/M_l$ can be taken as an effective feature.

To consider this ratio feature of moments for other orthogonal polynomials, we simply consider the underlying polynomials and their radial form. Computation of moments for them is self explanatory.

3.5 Shifted Chebyshev Polynomial of First Kind

Shifted Chebyshev polynomial of the first kind is defined by $T_n^*(x) = T_n(2x-1)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind and is described by

$$T_n(x) = \frac{n}{2}\sum_{t=0}^{[n/2]}(-1)^t\frac{(n-t-1)!}{t!(n-2t)!}(2x)^{n-2t}. \qquad (27)$$

Hence,

$$\begin{aligned}T_n^*(x) &= T_n(2x-1), \\ &= \tfrac{n}{2}\sum_{t=0}^{[n/2]+1}(-1)^t\frac{(n-t-1)!}{t!(n-2t)!} \\ &\quad\times[2(2x-1)]^{n-2t}.\end{aligned} \qquad (28)$$

The orthogonality condition is

$$\begin{aligned}\int_0^1(x-x^2)^{-1/2}T_n^*(x)T_m^*(x)dx &= \tfrac{\pi}{2}\delta_{nm}, n\neq 0 \\ &= \pi\delta_{nm}, n=0\end{aligned} \qquad (29)$$

Therefore, in the polar form, the radial polynomial can be written as

$$\begin{aligned}R_n(r) = \sqrt{\pi}(r-r^2)^{-\frac{1}{4}}r^{-\frac{1}{2}}\tfrac{n}{2}\sum_{t=0}^{[n/2]+1}(-1)^t\times \\ \tfrac{(n-t-1)!}{t!(n-2t)!}[2(2r-1)]^{n-2t},\ for\ n=0.\end{aligned} \qquad (30)$$

It should be noted that the shifted Chebyshev polynomial of the first kind is orthogonal on the interval $[0,1]$ with respect to the weight function $w(x) = (x-x^2)^{-\frac{1}{2}}$. In the polar form this becomes $w(\rho) = (\rho-\rho^2)^{-\frac{1}{2}}$. Obviously, when $\rho = 1$, $w(\rho)$ becomes infinite. It is therefore, clear that on the unit discrete disc, the shifted Chebyshev polynomial moments of first kind on the circumference are undefined and hence cannot be used for recognition features. However, one can easily get over this problem by computing the moments on a discrete unit ball because a ball does not consider its circumference. To enhance accuracy, one can consider the value of $\rho = r_{max}/(r_{max}+1)$ when $r = r/r_{max} = 1$.

3.6 Shifted Chebyshev Polynomial of Second Kind

Shifted Chebyshev polynomial of second kind, U_n^* is free from any kind of computational problem because it is orthogonal on the interval $[0,1]$ with respect to the weight function $w(x) = (x-x^2)^{\frac{1}{2}}$. And,

$$\begin{aligned}U_n^*(x) &= U_n(2x-1), \\ &= \sum_{t=0}^{[n/2]+1}(-1)^t\frac{(n-t)!}{t!(n-2t)!} \\ &\quad\times[2(2x-1)]^{n-2t}.\end{aligned} \qquad (31)$$

$U_n(x)$ is the shifted Chebyshev polynomial of second kind. The corresponding radial polynomial can be computed as

$$\begin{aligned}R_n(r) &= U_n^*(r)r^{-1/2} \\ &= \sqrt{8/\pi}(\tfrac{1-r}{r})^{\frac{1}{4}}\sum_{t=0}^{[n/2]+1}(-1)^t\frac{(n-t)!}{t!(n-2t)!} \\ &\quad\times[2(2r-1)]^{n-2t},\end{aligned} \qquad (32)$$

4 EVALUATION OF THE MERIT OF THE WORK

Almost all the authors have computed moments of Chebyshev polynomials on a rectangle. Ping et al. (Z.L. Ping and Sheng, 2002) introduced shifted

Figure 1: (a) Lena image, $c = 1.0$ (b) Map on the discrete disc (c)Lena image, $c = 0.6$ (d)Lena image, $c = 0.4$.

Chebyshev polynomial moment of the second kind on an analytic disc. As the discrete points or pixels are defined on a rectangular grid, this needs a special mapping technique to find the pixel position on a disc. One of the solutions is to use a polar raster. The problems with the polar raster and the errors of the reconstruction are analyzed in detail in (Mikolajczyk and Schmid, 2004). Our method uses a discrete disc for unique mapping of pixels. As a result, computation in our case becomes easy and straightforward, and removes the difficulty of mapping pixels on an analytic disc. Besides, in addition to examining shifted Chebyshev polynomial of the second kind, we have also examined the feasibility of using the shifted Chebyshev polynomial of first kind and shifted Legendre polynomial. Result shows all of them are equally efficient.

5 RESULTS AND DISCUSSION

Fig. 1 shows the Lena image and its map on a discrete disc, while Fig. 2 shows the straw image at different illuminations. Table 1 describes the result of rotational invariance with $\theta = 0$ and varying α in equation (24), while Table 2 describes the result of illumination invariance with varying c in equation (25).

6 CONCLUSIONS

Image description through moments of orthogonal shifted polynomials has been proposed. This description is rotationally invariant as well as illumination invariant. Therefore, it can be used in many applications, such as, compression, computer vision and recognition purposes. We have investigated the in-

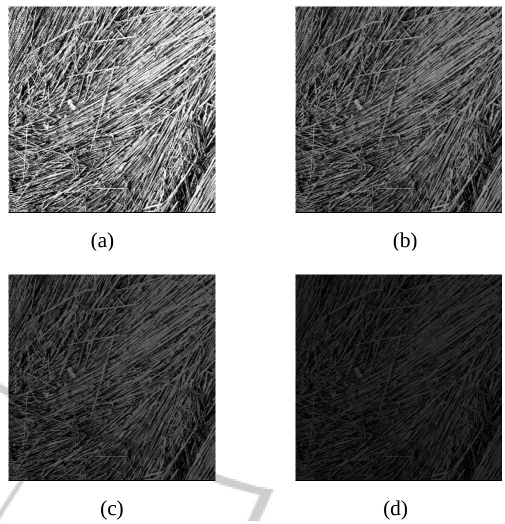

Figure 2: (a)Straw image, $c = 1$ (b)Straw image, $c = 0.6$ (c)Straw image, $c = 0.4$ (d)Straw image, $c = 0.2$.

Table 1: Rotational Invariance for Lena Image, $N = M = 6$.

Polynomial	Image	Value of α in eqn.(24)	R_M
Shifted Legendre	Lena	0^o	1.556585e+2
		30^o	1.555999e+2
		60^o	1.552167e+2
		90^o	1.556327e+2
		120^o	1.556154e+2
		150^o	1.552846e+2
		180^o	1.556736e+2
		210^o	1.553731e+2
		240^o	1.554150e+2
		270^o	1.555708e+2
Shifted Chebysheb First Kind	Lena	0^o	6.023204e+1
		30^o	6.025693e+1
		60^o	6.004949e+1
		90^o	6.033060e+1
		120^o	6.035521e+1
		150^o	6.011921e+1
		180^o	6.030006e+1
		210^o	6.007316e+1
		240^o	6.006964e+1
		270^o	6.016374e+1
Shifted Chebysheb Second Kind	Lena	0^o	1.981176e+1
		30^o	1.976791e+1
		60^o	1.974387e+1
		90^o	1.974037e+1
		120^o	1.972534e+1
		150^o	1.972488e+1
		180^o	1.977120e+1
		210^o	1.978165e+1
		240^o	1.980989e+1
		270^o	1.981933e+1

Table 2: Illumination Invariance for textured images, $N = M = 6$.

Polynomial	Image	Value of c	R_M	Image	Value of c in eqn.(25)	R_M
Shifted Legendre	Straw	1.0	1.819310e+2	Lena	1.0	1.556327e+2
		0.8	1.811447e+2		0.8	1.551542e+2
		0.6	1.799795e+2		0.6	1.544699e+2
		0.4	1.782080e+2		0.4	1.536190e+2
		0.2	1.765312e+2		0.2	1.547770e+2
Chebysheb First Kind	Straw	1.0	7.114174e+1	Lena	1.0	6.023204e+1
		0.8	7.058486e+1		0.8	5.985514e+1
		0.6	6.972407e+1		0.6	5.929079e+1
		0.4	6.825735e+1		0.4	5.837455e+1
		0.2	6.548399e+1		0.2	5.715533e+1
Shifted Chebysheb Second Kind	Straw	1.0	1.974037e+1	Lena	1.0	1.981176e+1
		0.8	1.972534e+1		0.8	1.986103e+1
		0.6	1.972488e+1		0.6	1.996325e+1
		0.4	1.981933e+1		0.4	2.023949e+1
		0.2	1.981933e+1		0.2	2.160718e+1

variance through computation of global moments of images. For invariance, we have computed the ratio features of moments. It is found that the computation of invariance through ratio of moments over local subimages is more powerful than that computed computed over the entire image. Such features can be used in correspondence problem. Our future work is based on such local invariance of patches in images.

REFERENCES

Abu-Mostafa, Y. and Psaltis, D. (1984). Recognitive aspects of moment invariants. *IEEE Trans Pattern Anal. Machine Intell.*, 6:698–706.

Bhatia, A. B. and Wolf, E. (1954). On the circular polynomials of zernike and related orthogonal sets. In *Proc. Cambridge Philosos. Soc. 50*.

Biswas, S. N. and Chaudhuri, B. B. (1985). On the generation of discrete circular objects and their properties. *Comput. Vision, Graphics Image Process.*, 32:158–170.

H. Ren, Z. Ping, W. B. W. W. and Sheng, Y. (2003). Multi-distortion invariant image recognition with radial harmonic fourier moments. *J. Opt. Soc. Am. A*, 20:631–637.

H. Shu, L. L. and Coatrieux, J. L. (2007). Moment-based approaches in image. In *IEEE Engineering in Medicine and Biology Magazine*.

H. Q. Zhu, H. Z. Shu, J. L. L. L. and Coatrieux, J. (2007). Image analysis by discrete orthogonal racah moments. *Signal Processing*, 87:687–708.

Hu, M. (1962). Visual pattern recognition by moment invariants. *IRE Trans. Inform. Theory IT*, 8:179–187.

Jan Flusser, T. S. and Zitova, B. (2009). *Moments and Moment Invariants in Pattern Recognition*. John Wiley and Sons, UK.

Mikolajczyk, K. and Schmid, C. (2004). Scale and affine invariant interest point detectors. *International Journal of Computer Vision*, 60:63–86.

Prokop, R. and Reeves, A. (1992). A survey of moment-based techniques for unoccluded object representation and recognition. *CVGIP: Graphical Models and Image Process.*, 54:438–460.

P. T. Yap, R. P. and Ong, S. (2003). Image analysis by krawtchouk moments. *IEEE Trans. Image Process.*, 12:1367–1377.

R. Mukundan, S. H. O. and Lee, P. A. (2001). Image analysis by tchebichef moments. *IEEE Trans. Image Process.*, 10:1357–1364.

Reddi, S. (1981). Radial and angular moment invariants for image identification. *IEEE Trans Pattern Anal. Machine Intell.*, 3:240–242.

T. Xia, H.Q. Zhu, H. S. P. H. and Luo, L. (2007). Image description with generalized pseudo-zernike moments. *J. Opt. Soc. Am. A*, 24:50–59.

Teague, M. (1980). Image analysis via the general theory of moments. *J. Opt. Soc. Am.*, 70:920–930.

Teh, C. and Chin, R. (1988). On image analysis by the methods of moments. *IEEE Trans. Pattern Anal. Mach. Intell.*, 10:496–513.

Yap, P. T. and Raveendran, P. (2004). Image focus measure based on chebyshev moments. In *IEE Proc. -Vis. Image Signal Process.*

Z. L. Ping, R. W. and Sheng, Y. (2002). Image description with chebyshev-fourier moments. *J. Opt. Soc. Am. A*, 19:1748–1754.