

A Methodology for the Design of Fuzzy Fractional PID Controllers

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Abstract: This paper proposes two novel fuzzy fractional PID structures. The tuning of the fuzzy fractional controllers is based on the prior knowledge of fractional-order control tuning rules. The digital implementation of these controllers is also investigated. The effectiveness and robustness of the proposed tuning methodology is illustrated through its application on a fractional-order plant. The simulation results show that the control system performance is better than that of conventional fractional PID control.

1 INTRODUCTION

In recent years, the fractional-order PID (FO-PID) controllers have been a fruitful field of research (Podlubny, 1999a; Podlubny, 1999b). However, no effective and simple tuning rules still exist for these controllers as those given for the integer PID controllers (Astrom and Hagglund, 1995). It is well known that the FO-PID extends the capabilities of the classical counterpart and, thus, have a wider domain of application, such as in suspension systems, robotics, signal processing, control and diffusion (Oldham and Spanier, 1974; Podlubny, 1999a; Podlubny, 1999b). On the other hand, the fuzzy logic controllers (FLC) have also been successfully applied in the control of many physical systems, particularly those with uncertainty, unmodelled, disturbed and/or nonlinear dynamics (Lee, 1990; Li and Gatland, 1996; Carvajal et al., 2000).

In this paper, we combine the features of fuzzy controllers with those of fractional controllers of PID-type. The resulting fuzzy fractional PID (FF-PID) controller is investigated in terms of its digital implementation and robustness. The combined advantages of the two controllers results in a better controller with superior robustness and wider domain of application. The tuning methodology of these controllers is based on the prior knowledge of fractional-order control. First, a fractional-order controller is built and tuned (or used one already implemented). Then, we replace it with a linear fuzzy fractional controller displaying exactly the same step response. After, we make the

controller nonlinear and fine tune it in order to get better control of the system. The fuzzy fractional controller will give, at least, the same performance of its linear counterpart.

The paper is organized as follows. Section 2 presents the basic ideas of continuous and discrete fractional PID controllers. Section 3 outlines a procedure for the design of FF-PID controllers. In section 4, we test the proposed fuzzy fractional controllers and assess their applicability and robustness on a fractional-order plant. Finally, section 5 draws the main conclusions and addresses perspectives to future developments.

2 FRACTIONAL PID CONTROLLERS

The fractional-order controller of PID-type, usually named $PI^\lambda D^\mu$ controller, may be given as (Podlubny, 1999b; Barbosa et al., 2010):

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s^\lambda} + K_d s^\mu \quad (1)$$

where K_p , K_i and K_d are the proportional, integral and derivative gains, and usually the fractional orders $(\lambda, \mu) \in [0, 1]$. Clearly, taking $(\lambda, \mu) \equiv \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ we get the classical {PID, PI, PD, P}-controllers, respectively. The $PI^\lambda D^\mu$ -controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of a control system (Podlubny, 1999b).

The time domain equation of the $PI^\lambda D^\mu$ controller is:

$$u(t) = K_p e(t) + K_i D^{-\lambda} e(t) + K_d D^\mu e(t) \quad (2)$$

where $D^{(*)}$ ($\equiv {}_0 D_t^\alpha$) denotes the differential operator of integration and differentiation (differintegral) to a fractional-order $\alpha = \{-\lambda, \mu\} \in \mathfrak{R}$.

The two most commonly used definitions for the differintegral are the Riemann-Liouville definition and the Grünwald-Letnikov definition. For our purpose we use the Grünwald-Letnikov definition, which can be written as ($\alpha \in \mathfrak{R}$):

$$D^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor t/h \rfloor} (-1)^j \binom{\alpha}{j} f(t - jh) \quad (3a)$$

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1) \Gamma(\alpha - j + 1)} \quad (3b)$$

where $f(t)$ is the applied function, $\Gamma(\cdot)$ is the Gamma function, h is the time increment, and $\lfloor \cdot \rfloor$ means the integer part.

From a control and signal processing perspective, approach (3) seems to be the most useful and intuitive, particularly for a discrete-time implementation (Barbosa et al., 2006; Machado, 1997). In fact, using (3), a discrete fractional $PI^\lambda D^\mu$ control equation can be obtained from (2) as ($h \approx T$, T is the sampling period):

$$u(k) = K_p e(k) + K_i D^{-\lambda} e(k) + K_d D^\mu e(k) \quad (4)$$

with

$$D^\alpha e(k) \approx \frac{1}{T^\alpha} \sum_{j=0}^k (-1)^j \binom{\alpha}{j} e(k - j) \quad (5)$$

The difference control equation (4) is then given by:

$$u(k) = K_p e(k) + \frac{K_i}{T^{-\lambda}} \sum_{j=0}^k (-1)^j \binom{-\lambda}{j} e(k - j) + \frac{K_d}{T^\mu} \sum_{j=0}^k (-1)^j \binom{\mu}{j} e(k - j) \quad (6)$$

Eq. (6) shows that the current value of control signal $u(k)$ depends on all previous values of error $e(k)$, making the computation too heavy as time increases and so unsuitable for a practical implementation of these algorithms. This fact illustrates the global character (i.e., unlimited memory) of the fractional-order operators. For practical implementation of fractional integral and derivative (5) we often apply the short

memory principle (Podlubny, 1999a), resulting in expression:

$$u(k) = K_p e(k) + \frac{K_i}{T^{-\lambda}} \sum_{j=v}^k c_j^{(-\lambda)} e(k - j) + \frac{K_d}{T^\mu} \sum_{j=v}^k c_j^{(\mu)} e(k - j) \quad (7)$$

where $v = 0$ for $k < L/T$ or $v = k - L/T$ for $k > L/T$; L is the memory length and $c_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$ are the binomial coefficients which may be calculated recursively as:

$$c_0^{(\alpha)} = 1; \quad c_j^{(\alpha)} = \left(1 - \frac{1 + \alpha}{j}\right) c_{j-1}^{(\alpha)}, \quad j = 1, 2, \dots \quad (8)$$

Note that (7) is given in the form of a FIR filter. Other discrete-time approximations in the form of IIR filters are also possible (Vinagre et al., 2003; Barbosa et al., 2006; Chen et al., 2004).

3 DESIGN OF FUZZY FRACTIONAL PID CONTROLLERS

Despite of variety of possible fuzzy controller structures, the controller is usually arranged in cascade with the system being controlled. This type of arrangement is shown in Fig. 1 and will be used in this study.

The main idea here is to explore the fact that the FLC, under certain conditions, is equivalent to a PID controller (Mizumoto, 1995; Li and Gatland, 1996; Jantzen, 2007). In a certain sense, the fuzzy PID controllers are a special case of the more general FF-PID controllers, in which are involved two extra tuning parameters: the fractional orders (λ, μ) of controller equation (4).

The basic form of a fuzzy controller is illustrated in Fig. 2 (Passino and Yurkovich, 1998). In general, the mapping between the inputs and the outputs

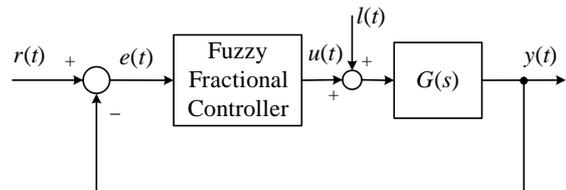


Figure 1: Fuzzy fractional PID controlled system.

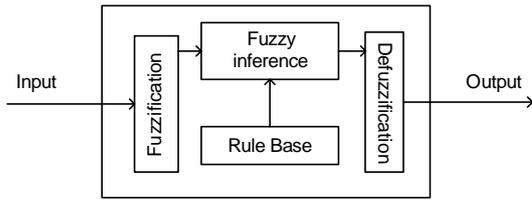


Figure 2: Structure of a fuzzy controller.

of a fuzzy system is nonlinear (Galichet and Foulloy, 1995; Jantzen, 2007). However, it is possible to construct a rule base with a linear input-output mapping (Mizumoto, 1995; Jantzen, 2007). For that, the following conditions must be fulfilled:

- Use triangular input sets that cross at the membership value 0.5;
- The rule base must be complete AND combination (cartesian product) of all input families;
- Use the algebraic product (*) for the AND connective;
- Use output singletons, positioned at the sum of the peak positions of the input sets;
- Use sum-accumulation and centre of gravity for singletons (COGS) defuzzification.

It seems reasonable to start with the design of a conventional integer/fractional PID controller and from there to proceed to a fuzzy control design. In this way, the linear fuzzy controller may be used in a design procedure based on integer/fractional PID control, as follows (Jantzen, 2007; Barbosa et al., 2010; Barbosa, 2010):

1. Build and tune an integer/fractional PID controller;
2. Replace it with an equivalent linear fuzzy controller;
3. Make the fuzzy controller nonlinear;
4. Fine-tune it.

With the above procedure, the design of fuzzy fractional PID controllers will be greatly simplified, particularly if the controller was already implemented and it is desirable to enhance its performance. Moreover, this new type of controllers extends the potentialities of both fuzzy and fractional controllers and performs, at least, as well as its linear fractional counterpart (Jantzen, 2007; Barbosa et al., 2010; Barbosa, 2010).

3.1 Fuzzy Fractional PD Controller

The time-domain equation of a fractional PD^μ-controller is given by ($K_i = 0$ in (2)):

$$u(t) = K_p e(t) + K_d D^\mu e(t) \quad (9)$$

The corresponding discrete-time fractional PD^μ-controller is:

$$u(k) = K_p e(k) + K_d D^\mu e(k) \quad (10)$$

Fig. 3 illustrates the block diagram of the fuzzy fractional PD^μ (FF-PD^μ) controller. As can be seen, the controller acts on the error, $E = K_e e(k)$, and on the fractional change of error, $FE = K_{fe} D^\mu e(k)$. The control signal is $U = K_u u$. The controller has three tuning gains, K_e and K_{fe} , corresponding to the inputs and K_u to the output.

The control signal U is generally a nonlinear function of E and FE :

$$U = f(E, FE) K_u = f(K_e e, K_{fe} D^\mu e) K_u \quad (11)$$

With a proper choice of design, a linear approximation can be obtained as:

$$f(K_e e(k), K_{fe} D^\mu e(k)) \approx K_e e(k) + K_{fe} D^\mu e(k) \quad (12)$$

and

$$\begin{aligned} U(k) &= (K_e e(k) + K_{fe} D^\mu e(k)) K_u \\ &= K_e K_u e(k) + K_{fe} K_u D^\mu e(k) \end{aligned} \quad (13)$$

Comparing (13) with (10), it yields the relation for the gains of the two controllers:

$$\begin{aligned} K_e K_u &= K_p \\ K_{fe} K_u &= K_d \end{aligned} \quad (14)$$

The linear FF-PD^μ-controller provides all the advantages of the conventional fractional PD^μ-controller.

For an equivalent linear FF-PD^μ-controller, the conclusion universe should be the sum of the premise universes and the input-output mapping should be linear. Table 1 lists a linear rule base for the FF-PD^μ controller composed of four rules. There are only two fuzzy labels (Negative and Positive) used for the fuzzy input variables and three fuzzy labels (Negative, Zero and Positive) for the fuzzy output variable. This rule base should satisfy conditions mentioned above in order to provide a linear mapping.

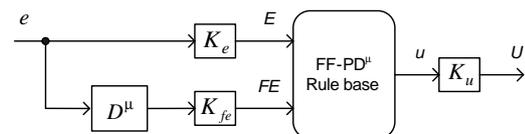

 Figure 3: Fuzzy fractional PD^μ-controller.

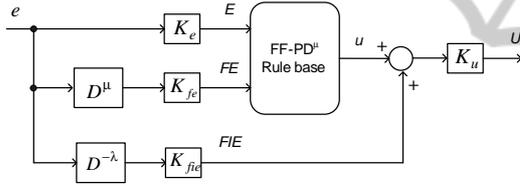
Table 1: Rule base for the FF-PD $^\mu$ controller.

Rule 1	If E is N and FE is N then u is N
Rule 2	If E is N and FE is P then u is Z
Rule 3	If E is P and FE is N then u is Z
Rule 4	If E is P and FE is P then u is P

Scaling the input gains may be necessary to preserve the linearity of the fuzzy controller. However, that should be made without affecting the tuning (Barbosa et al., 2010; Barbosa, 2010). This scaling has some advantages, as it will avoid saturation and will provide a simpler design, since the universes ranges of inputs and outputs are normalized to a prescribed interval, say percentage of full scale $[-100, 100]$.

3.2 Fuzzy Fractional PID Controller

The inclusion of an integral action is necessary whenever the closed-loop system exhibits a steady-state error. The fuzzy fractional PD $^\mu$ +I $^\lambda$ (FF-PD $^\mu$ +I $^\lambda$) controller combine the fractional-order integral action with a fuzzy PD $^\mu$ -controller, as illustrated in Fig. 4.


Figure 4: Fuzzy fractional PD $^\mu$ +I $^\lambda$ controller.

The control signal U is generally a nonlinear function of error E , fractional change of error FE , and fractional integral of error FIE :

$$U = (f(E, FE) + FIE) K_u$$

$$= \left(f(K_e e(k) + K_{fe} D^\mu e(k)) + K_{fie} D^{-\lambda} e(k) \right) K_u \quad (15)$$

Adopting the linear approximation (12) yields the control action:

$$U(k) \approx \left(K_e e(k) + K_{fe} D^\mu e(k) + K_{fie} D^{-\lambda} e(k) \right) K_u$$

$$= K_u K_e e(k) + K_u K_{fe} D^\mu e(k) + K_u K_{fie} D^{-\lambda} e(k) \quad (16)$$

Comparing (16) with the discrete fractional PI $^\lambda$ D $^\mu$ -controller (4), it yields the relation for the gains of the two controllers:

$$\begin{aligned} K_e K_u &= K_p \\ K_{fie} K_u &= K_i \\ K_{fe} K_u &= K_d \end{aligned} \quad (17)$$

The linear FF-PD $^\mu$ +I $^\lambda$ controller provides all the advantages of the conventional fractional PI $^\lambda$ D $^\mu$ -controller.

4 ILLUSTRATIVE EXAMPLE

Many real dynamical processes are modeled by fractional-order transfer functions (Podlubny, 1999a; Oldham and Spanier, 1974). Here we consider the fractional-order plant model given in (Podlubny, 1999b):

$$G(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} \quad (18)$$

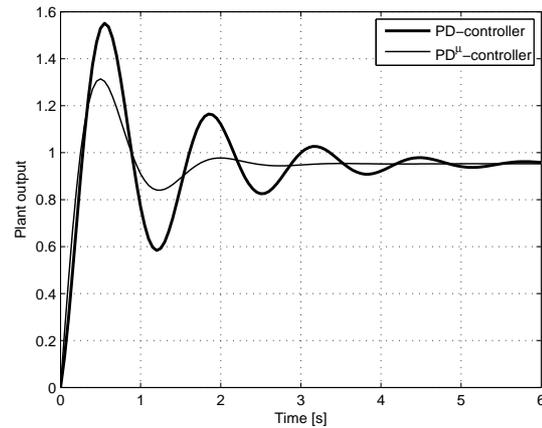
An integer-order PD controller and a fractional-order PD $^\mu$ -controller were designed in (Podlubny, 1999b):

$$C_{PD}(s) = 20.5 + 2.7343s \quad (19)$$

$$C_{PD^\mu}(s) = 20.5 + 3.7343s^{1.15} \quad (20)$$

Fig. 5 shows the unit-step response of the closed-loop fractional-order system with the conventional PD-controller and with the PD $^\mu$ -controller. The comparison shows that for satisfactory feedback control of the fractional-order system is better to use a fractional-order controller. Note, however, that the control system presents a steady-state error, since no integral action is employed.

Let us now design an equivalent linear FF-PD $^\mu$ controller. By configuring the fuzzy inference system (FIS) and selecting three scaling factors, we obtain a FF-PD $^\mu$ -controller that reproduces the exact control performance as the fractional PD $^\mu$ -controller. We first fix $K_e = 100$, since the error universe is chosen to be


Figure 5: Unit-step responses of the fractional-order control system with the PD and PD $^\mu$ -controllers.

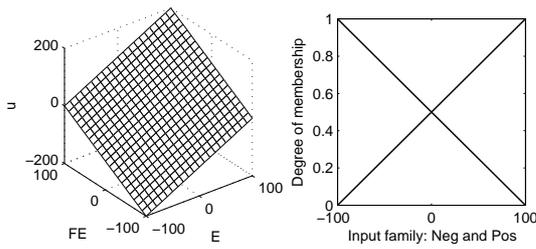


Figure 6: Linear surface with the corresponding input families.

percentage of full scale $[-100, 100]$, and the maximum error to a unit step is 1. The values of K_{fe} and K_u are obtained using expressions (14). Fig. 6 shows the input families and the linear control surface obtained by using the rule base of Table 1 while satisfying conditions outlined in section 3. Note that this result represents the step 2 – replace the conventional controller with an equivalent linear fuzzy controller – of the design procedure. In order to enhance the performance of the control system we proceed to step 3 and 4 of the design – make the fuzzy controller nonlinear and fine-tune it.

Thus, after verifying that the linear FF-PD $^\mu$ -controller is properly designed, we may adjust the FIS settings such as its style, membership functions and rule base to obtain a desired nonlinear control surface. In our example, we choose Gaussian membership functions for the inputs, as illustrated in Fig. 7 with the corresponding nonlinear control surface.

In Fig. 8, the comparison of the unit-step response of the closed-loop system with plant model (18) controlled by the linear PD and FF-PD $^\mu$ -controllers, and with the nonlinear FF-PD $^\mu$ -controller is given. The simulation parameters are: absolute memory computation of approximation (5), fractional-order $\mu=1.15$, scale factor $M = 0.4$ and $T = 0.05$ s. As can be seen, making the controller nonlinear improved the control system performance, namely the overshoot, rise time, settling time, and steady-state error, when compared with the linear fuzzy controller. The fuzzy fractional controller provides greater flexibility than the fractional controller and can be used to better adjust the dynamical properties of a control system.

Now, let us consider the FF-PD $^\mu+I^\lambda$ -controller. In order to test the robustness of the fuzzy controller, we introduce a load disturbance of amplitude $l = 2$ after 7 seconds in system of Fig. 1. We use the same (K_p , K_d) parameters of the linear FF-PD $^\mu$ -controller and tuned the (K_i , λ) for a satisfactory control response. The final tuned parameters are (K_i , λ) = (10, -0.8). With $K_e = 100$, and using (17) we obtain K_{fe} , K_u , and K_{fie} of the fuzzy controller.

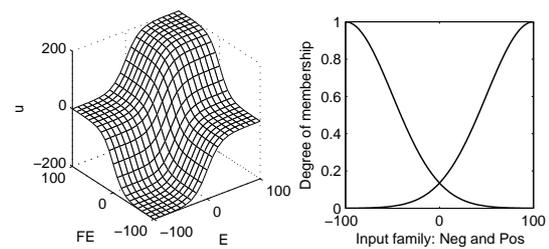


Figure 7: Nonlinear control surface with the corresponding input families.

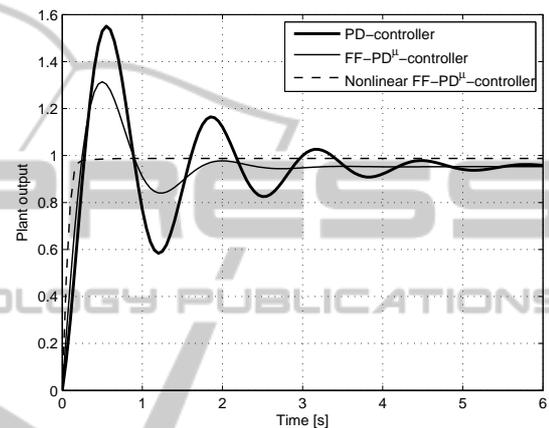


Figure 8: Unit-step responses of the fractional control system with the linear PD and FF-PD $^\mu$ -controllers, and with the nonlinear FF-PD $^\mu$ -controller.

In this experiment, the simulation parameters are: absolute memory computation of approximation (5), scale factor $M = 0.1$ and $T = 0.05$ s. Fig. 9 shows the step and load responses of closed-loop system with FF-PD $^\mu+I^\lambda$ controller, (μ , λ)=(1.15, -0.8), for the linear and nonlinear control surfaces. We observe the better response of the fuzzy controller to the reference and disturbance inputs with the nonlinear rule base compared to their linear counterpart. Once more, we demonstrate the robustness and effectiveness of this type of controller.

5 CONCLUSIONS

This paper introduced two novel fuzzy fractional PID structures: the FF-PD $^\mu$ and FF-PD $^\mu+I^\lambda$ controllers. It was demonstrated that these controllers are equivalent to the conventional fractional PD and PID controllers by using a linear input-output mapping of the rule base of the fuzzy fractional controller. Moreover, by making the controller nonlinear, the performance of the control system proves to be, in most systems, better than its linear counterpart. A methodology for

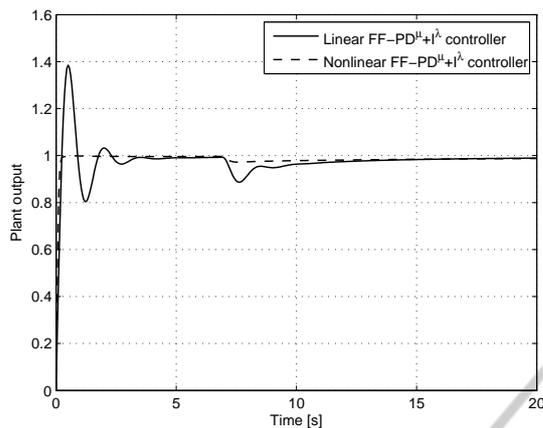


Figure 9: Unit-step and load responses of the fractional control system with the linear and nonlinear FF-PD $^{\mu}$ +I $^{\lambda}$ controllers.

tuning the nonlinear fuzzy fractional PID controllers is also presented. This methodology is simple and effective and can be used to replace an existent fractional/integer PID controller in order to get better performance of the control system. In this perspective, future research on this topic includes the application of the proposed fuzzy fractional PID controllers and tuning methodology in other types of linear and nonlinear plants of integer and/or fractional-order. We expect that the incorporation of fuzzy reasoning into fractional-order controllers will increase the applicability of these controllers.

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REFERENCES

- Astrom, K. J. and Haggund, T. (1995). *PID Controllers: Theory, Design, and Tuning*. Instrument Society of America, USA.
- Barbosa, R. S., Jesus, I. S., and Silva, M. F. (2010). Fuzzy reasoning in fractional-order PD controllers. In *Proceedings of AIC'10, 10th WSEAS International Conference on Applied Informatics and Communications*, pages 252–257, August 20–22, Taipei, Taiwan.
- Barbosa, R. S. (2010). On linear fuzzy fractional pd and pd+i controllers. In *Proceedings of FDA'10, 4th IFAC Workshop Fractional Differentiation and its Applications*, pages 1–6, October 18–20, Badajoz, Spain.
- Barbosa, R. S., Machado, J. A. T., and Silva, M. F. (2006). Time domain design of fractional differintegrators using least-squares. *Signal Processing*, 86:2567–2581.
- Carvajal, J., Chen, G., and Ogmen, H. (2000). Fuzzy PID controller: Design, performance evaluation, and stability analysis. *Journal of Information Science*, 123:249–270.
- Chen, Y. Q., Vinagre, B. M., and Podlubny, I. (2004). Continued fraction expansion approaches to discretizing fractional order derivatives—an expository review. *Nonlinear Dynamics*, 38:155–170.
- Galichet, S. and Foulloy, L. (1995). Fuzzy controllers: Synthesis and equivalences. *IEEE Transactions on Fuzzy Systems*, 3(2):140–148.
- Jantzen, J. (2007). *Foundations of Fuzzy Control*. Wiley and Sons, Chichester, England.
- Lee, C. C. (1990). Fuzzy logic in control systems: fuzzy logic controller-Part I and II. *IEEE Transactions on System Man, and Cybernetics-Part B: Cybernetics*, 20(2):404–435.
- Li, H.-H. and Gatland, H. B. (1996). Conventional fuzzy control and its enhancement. *IEEE Transactions on System Man, and Cybernetics-Part B: Cybernetics*, 26(5):791–797.
- Machado, J. A. T. (1997). Analysis and design of fractional-order digital control systems. *SAMS Journal of Systems Analysis, Modelling and Simulation*, 27:107–122.
- Mizumoto, M. (1995). Realization of PID control by fuzzy control methods. *Fuzzy Sets and Systems*, 70:171–182.
- Oldham, K. B. and Spanier, J. (1974). *The Fractional Calculus*. Academic Press, New York.
- Passino, K. M. and Yurkovich, S. (1998). *Fuzzy Control*. Addison-Wesley, Menlo Park, California.
- Podlubny, I. (1999a). *Fractional Differential Equations*. Academic Press, San Diego.
- Podlubny, I. (1999b). Fractional-order systems and $PI^{\lambda}D^{\mu}$ -controllers. *IEEE Transactions on Automatic Control*, 44(1):208–214.
- Vinagre, B. M., Chen, Y. Q., and Petras, I. (2003). Two direct tustin discretization methods for fractional-order differentiator/integrator. *Journal of the Franklin Institute*, 340:349–362.