The Optimal Quaternion Equilibrium Point
Using an Energy Function to Choose the Optimal Quaternion Equilibrium Point

Margrete Djupaa and Rune Schlanbusch
Department of Technology, Narvik University College, Narvik, Norway

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Abstract: By parameterizing the attitude of a rotating rigid body in a closed-loop system with unit quaternions, the existence of dual equilibria leads to new challenges. In order to optimize the energy consumption due to control effort, the choice of the cheapest equilibria, that is, the one which requires least energy to reach, is essential. A new predicting solution of choosing the optimal equilibrium point for rotational maneuvers of a rigid body is presented in this article. This new solution consists of an energy function which base its prediction on the initial attitude on the rotational sphere, taking account for both potential and kinetic energy of the rigid body. The equilibrium energy function is developed through a previously presented statistical analysis for the system behaviour of a rigid body in closed loop attitude control.

1 INTRODUCTION

For controlling and steering an object there is a need for a controller. New controllers have been evolved during several years, with different focus. There are different needs for different situations, and many of the demands are in contrast to each other. Needs like minimum energy consumption, robust stability, faster and more accurate settling giving enhanced performance and minimizing the control effort, can give the developers some interesting tradeoffs. The development of new techniques for the controller to give more optimal control of a rigid body is needed, especially in spacecrafts and UAV's where resources are sparse.

To control the attitude of a rotational rigid body when the attitude is parameterized by unit quaternions, can be a challenging task. The existence of dual equilibria and possibilities of the unwinding phenomenon (Bhat and Bernstein, 2000) are the main issues. (Kristiansen, 2008) presented a technique to overcome the issue with two equilibria, where the shortest rotational path was chosen initially and kept throughout the maneuver. The technique uses two state-feedback controllers in the development work, one for each equilibrium point, the positive and the negative one. These two controllers steer the rigid body to each of the equilibrium points, and they both give an uniformly asymptotically stable equilibrium point for the chosen equilibrium, while the not chosen equilibrium is rendered unstable. This control strategy is called the initial choice of equilibrium, and it bases it's choice only on the initial attitude scalar value called $\eta(t_0)$. With this method the shortest rotational length towards equilibrium will always be the chosen one, which will function well if the rigid body is not rotating at initial time. This method is further developed from the attitude error function presented by (Fjellstad, 1994), where a signum function is continuously choosing the closest equilibrium. Later it was proved that the signum function is not robust against measurement noise, and that the function can lead the closed-loop system stuck for infinite time, never reaching it's desired position (Mayhew et al., 2009).

The disadvantage of using these mentioned methods of choosing the closest equilibrium is that it is not taking account of rotational motion by the rigid body at the initial time, thus it will only consider the shortest distance. From a resource-consuming point of view of the controller this is not appropriate, and better methods are sought.

The goal is to predict the optimal equilibrium point based on the rigid body's attitude at the initial time, in order to save energy consumed by the control solution. By taking account of both potential and kinetic energy of the rotational sphere in the initial phase, the need of control power can be reduced to a minimum. This is done with a prediction of the most optimal equilibrium based on the initial attitude and angular velocity of the rigid body. The prediction

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method is based on an equilibrium energy function, developed on the basis of the behaviours of the dynamical system seen by the statistical analysis done by (Schlanbusch, 2012).

In Matlab a dynamical closed-loop model is developed of a rigid body having constant inertia matrix. The rigid body could be anything, but its motion can be viewed as a unit sphere. In real life no object is perfectly rigid, but with this approximation the study of attitude maneuvers is less complicated. The control law used is based on the PD+ controller introduced by (Paden and Panja, 1988). This controller is a state-feedback controller which controls the rigid body to a desired position. The only acting torque in the closed-loop system in simulations are the controller, no disturbance torques are accounted for in the system model. The system model and simulations are conducted in Matlab. The simulation results with the use of a PD+-controller and randomized initial values shows that the presented equilibrium energy function is able to predict the optimal equilibrium point in about 97% of the maneuvers. This prediction result is a further improvement compared to previously presented results.

2 MATHEMATICAL BACKGROUND

2.1 Quaternions

The attitude of a rigid object can according to (Fjellstad, 1994) be described by a rotation matrix $\mathbf{R} \in SO(3)$, where $SO(3)$ is the special orthogonal group of order three, which is the set of all rotation matrices with the determinant equal to $+1$, defined as

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \ : \ \mathbf{R}^\top \mathbf{R} = \mathbf{I}, \ \det(\mathbf{R}) = 1\} \quad (1)$$

where $\mathbf{I}$ is the 3×3 identity matrix. The manifold $SO(3)$, which is the group of rotation matrices, is closed and bounded, while the manifold called the Euclidean space $\mathbb{R}^{3\times 3}$, is open and unbounded. This difference in manifolds leads to the fact that there is no homeomorphism between these two spaces.

The quaternion parameterization of $SO(3)$ can be done according to (Fjellstad, 1994) by four parameters $\mathbf{q} = [\eta, \varepsilon^\top]^\top$; the three-dimensional vector $\varepsilon \in \mathbb{R}^3$ and the scalar component $\eta \in \mathbb{R}$. The definition of an unit quaternion is $\mathbf{q} \in S^3 = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x}^\top \mathbf{x} = 1\}$. The rotation matrix can be rewritten as

$$\mathbf{R}_{\eta,\varepsilon} = \mathbf{I} + 2\eta \mathbf{S}(\varepsilon) + 2\mathbf{S}^2(\varepsilon) \quad (2)$$

where $\mathbf{S}(\varepsilon)$ is the skew matrix of the ε vector, such that $\mathbf{S}(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

With unit norm the quaternion's conjugate $\bar{\mathbf{q}} = [\eta, -\varepsilon^\top]^\top$ and the inverse is the same.

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{||\mathbf{q}||^2}. \quad (3)$$

The manifold S^3 forms a group of quaternion multiplication, which is distributive and associative, but not commutative. The quaternion product between two quaternion vectors is defined by (Egeland and Gravdahl, 2002) as

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} \eta_1\eta_2 - \varepsilon_1^\top \varepsilon_2 \\ \eta_1\varepsilon_2 + \eta_2\varepsilon_1 + \mathbf{S}(\varepsilon_1)\varepsilon_2 \end{bmatrix}. \quad (4)$$

S^3 is the set of unit quaternions and is the covering manifold for $SO(3)$. According to (Bhat and Bernstein, 2000) this situation with the covering map provides a globally nonsingular parametrization of $SO(3)$. The covering map is onto and a locally diffeomorphism everywhere, but globally many to one. Mentioned by (Fjellstad, 1994) that due to the redundant fourth parameter there is a double covering of the space $SO(3)$. This gives the existence of the unwanted phenomenon of two equilibria.

2.2 Kinematics and Dynamics

The angular velocity vector gives the rate of change of the attitude of the rigid body with respect to time. Given by (Egeland and Gravdahl, 2002) the definition of the angular velocity vector represents the time derivative of the rotation matrix.

$$\mathbf{S}(\omega_{a,b}^a) = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^\top \quad (5)$$

where $\omega_{a,b}^a \in \mathbb{R}^3$ is the angular velocity of $\mathcal{F}^b$ relative to $\mathcal{F}^a$ represented in $\mathcal{F}^b$ where $\mathcal{F}^{(\cdot)}$ denotes frame $(\cdot)$. Furthermore, (5) can be expressed as (Egeland and Gravdahl, 2002)

$$\dot{\mathbf{R}}_b^a = \mathbf{S}(\omega_{a,b}^a)\mathbf{R}_b^a = \mathbf{R}_b^a \mathbf{S}(\omega_{a,b}^b). \quad (6)$$

The angular momentum vector $\mathbf{h}$ can be expressed by the angular velocity ω and the rigid body's inertia matrix $\mathbf{J} \in \mathbb{R}^{3\times 3} = diag\{J_x, J_y, J_z\}$ about its center of mass, as (Sidi, 1997)

$$\mathbf{h}^b = \mathbf{J}\omega_{i,b}^b. \quad (7)$$

The rotational kinematic of a rigid body which rotates from $\mathcal{F}^b$ to $\mathcal{F}^a$, can according to (Egeland and Gravdahl, 2002) be described by the kinetic differential equations

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\eta} \\ \dot{\varepsilon} \end{bmatrix} = \mathbf{T}(\mathbf{q})\omega \ , \qquad \mathbf{T}(\mathbf{q}) = \frac{1}{2} \begin{bmatrix} -\varepsilon^\top \\ \eta \mathbf{I} + \mathbf{S}(\varepsilon) \end{bmatrix} \in \mathbb{R}^{4\times 3}. \quad (8)$$

The dynamical model of the rotational rigid body can be described by a differential equation for angular velocity, deduced from the Euler's moment equation as described by (Sidi, 1997)

$$\tau = \dot{\mathbf{h}}^i = \dot{\mathbf{h}}^b + \mathbf{S}(\omega^b_{i,b})\mathbf{h}^b \tag{9}$$

where $\mathcal{F}^i$ is the inertial frame and $\mathcal{F}^b$ is the body frame. From (9) it can be seen how the applied torque τ is affecting the rotational motion of the rigid body.

Through the angular momentum (7) and the assumption that the inertia matrix $\mathbf{J}$ of the rigid body is constant, the derivative of the angular momentum is

$$\dot{\mathbf{h}}^b = \mathbf{J}\dot{\omega}^b_{i,b}. \tag{10}$$

By inserting the equations for the angular momentum (7) and (10) within the Euler's moment (9), we obtain

$$\tau = \mathbf{J}\dot{\omega}^b_{i,b} + \mathbf{S}(\omega^b_{i,b})\mathbf{J}\omega^b_{i,b}, \tag{11}$$

and with some rearranging of the terms, the dynamical model of the system is found to be

$$\mathbf{J}\dot{\omega}^b_{i,b} = -\mathbf{S}(\omega^b_{i,b})\mathbf{J}\omega^b_{i,b} + \tau \tag{12}$$

where the external torque τ is the sum of all forces acting on the rigid body. For simplicity through this article, the only force working in the closed-loop dynamical model is the controller τ_a, there are no disturbance torques τ_d.

$$\tau = \tau_a + \tau_d \quad \text{where} \quad \tau_d = 0 \Rightarrow \tau = \tau_a. \tag{13}$$

3 CONTROLLER DESIGN

The controller is designed to control a rigid body from it's state $\mathbf{q}(t)$ towards a reference state $\mathbf{q}_d(t)$ satisfying the kinematic equation

$$\dot{\mathbf{q}}_d = \mathbf{T}(\mathbf{q}_d)\omega_d \tag{14}$$

where ω_d is the desired angular velocity.

Shown by (Schlanbusch, 2012) the quaternion error can be defined by using the quaternion product

$$\tilde{\mathbf{q}} = \bar{\mathbf{q}}_d \otimes \mathbf{q} = \begin{bmatrix} \eta\eta_d - \varepsilon^\top\varepsilon_d \\ \eta_d\varepsilon + \eta\varepsilon_d + \mathbf{S}(\varepsilon)\varepsilon_d \end{bmatrix} \tag{15}$$

where $\tilde{\mathbf{q}} = [\tilde{\eta}, \tilde{\varepsilon}^\top]^\top$. The angular velocity error can be defined as

$$\tilde{\omega} = \omega - \omega_d, \tag{16}$$

and from (8) the error kinematics can be expressed as

$$\dot{\tilde{\mathbf{q}}} = \mathbf{T}(\tilde{\mathbf{q}})\tilde{\omega} \ , \ \ \mathbf{T}(\tilde{\mathbf{q}}) = \frac{1}{2}\begin{bmatrix} -\tilde{\varepsilon}^\top \\ \tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\varepsilon}) \end{bmatrix}. \tag{17}$$

The two equilibria represents the same physical orientation, there is a need to choose which of the equilibrium point that is going to stabilized. For stability reasons, the error functions are introduced to make the equilibrium points from $(\tilde{\mathbf{q}}, \tilde{\omega}) = ([\pm 1, 0, 0, 0]^T, [0, 0, 0]^T)$ for the closed-loop system to become $(\mathbf{e}_{q\pm}, \mathbf{e}_\omega) = ([0, 0, 0, 0]^T, [0, 0, 0]^T)$. The error quaternion $\tilde{\mathbf{q}}$ is earlier found in (15), and the equilibria can be moved to the origin according to

$$\mathbf{e}_{q\pm} = [1 \mp \tilde{\eta}, \tilde{\varepsilon}] \tag{18}$$

and the definition of the angular velocity error is found from (16) as

$$\mathbf{e}_\omega = \omega - \omega_d. \tag{19}$$

The controllers goal is to steer the rigid body's attitude and angular velocity towards the origin, $[\mathbf{e}_{q\pm}, \mathbf{e}_\omega]^\top \to \mathbf{0}$ as $t \to \infty$.

We start by analyzing the positive equilibrium, thus define $\mathbf{e}_q = \mathbf{e}_{q+}$. The control law of the continuous dynamical system can then be expressed as

$$\tau_a = \mathbf{J}\dot{\omega}_d - \mathbf{S}(\mathbf{J}\omega)\omega_d - k_q\mathbf{T}_e^\top\mathbf{e}_q - k_\omega\mathbf{e}_\omega. \tag{20}$$

The closed-loop rotational dynamics is then obtained by inserting (20) into (12) as

$$\mathbf{J}\dot{\mathbf{e}}_\omega - \mathbf{S}(\mathbf{J}\omega)\mathbf{e}_\omega + \kappa_q\mathbf{T}_e^\top\mathbf{e}_q + \kappa_\omega\mathbf{e}_\omega = \mathbf{0}. \tag{21}$$

With the definition of a Lyapunov function candidate which is radially unbounded and positive definitive, the candidate is

$$V(\mathbf{e}_{q+}, \mathbf{e}_\omega) = \frac{1}{2}\mathbf{e}_\omega^\top\mathbf{J}\mathbf{e}_\omega + \frac{1}{2}\mathbf{e}_q^\top k_p\mathbf{e}_q \tag{22}$$

satisfying $V > 0 \ \forall \ \mathbf{e}_\omega \neq \mathbf{0}, \mathbf{e}_q \neq \mathbf{0}$. By differentiation along closed loop trajectories we obtain

$$\dot{V} = \mathbf{e}_q^\top k_q\mathbf{T}_e\mathbf{e}_\omega + \mathbf{e}_\omega^\top\mathbf{S}(\mathbf{J}\omega)\mathbf{e}_\omega - \mathbf{e}_\omega^\top k_\omega\mathbf{e}_\omega - \mathbf{e}_\omega^\top k_q\mathbf{T}_e^\top\mathbf{e}_q \tag{23}$$

where it can be seen that the first term is canceled by the last term and the second term is zero because $\mathbf{S}(\cdot)$ is skew-symmetric, thus yielding

$$\dot{V} = -\mathbf{e}_\omega^\top k_\omega\mathbf{e}_\omega \leq 0. \tag{24}$$

By (Khalil, 2002) we can conclude that the equilibrium point $(\mathbf{e}_{q+}, \mathbf{e}_\omega) = (\mathbf{0}, \mathbf{0})$ is uniformly stable.

To check the convergence properties of the trajectories we make us of Matrosov's theorem (Hahn, 1967) by defining an auxiliary function

$$W(\mathbf{e}_{q+}, \mathbf{e}_\omega) = \mathbf{e}_q^\top\mathbf{T}_e k_q\mathbf{J}\mathbf{e}_\omega. \tag{25}$$

which is continuous and bounded for any states in a closed set. In the set $E : \{\dot{V} = 0\} = \{\mathbf{e}_\omega = \mathbf{0}\}$ the time derivative of (25) is non-zero definite and is negative definite in the attitude error

$$\mathbf{e}_\omega = \mathbf{0} \Rightarrow \dot{W} = -\mathbf{e}_q^\top\mathbf{T}_e k_q\mathbf{T}_e^\top\mathbf{e}_q, \tag{26}$$

which concludes the proof of (Hahn, 1967), thus rendering the equilibrium point of the closed-loop system as uniformly asymptotically stable. The stability proof for the negative equilibrium point $\mathbf{e}_{q-} = [1 + \tilde{\eta}, \tilde{\varepsilon}^\top]^\top$ can be concluded by utilizing the same structure of the candidates V and W, thus the stability result holds for both of the equilibria $(\mathbf{e}_{q\pm}, \mathbf{e}_\omega) = (\mathbf{0}, \mathbf{0})$. It is not possible to achieve a global result due to dual equilibria and also because of the fact that V is not radially unbounded in the attitude.

4 STATISTICAL CHOICE OF EQUILIBRIUM

Due to the constant need of reducing the energy consumption by the attitude controller, the idea of using the rotational energy in the initial maneuver phase took place. A prediction method was developed for finding the optimal equilibrium based on statistical analysis (Schlanbusch, 2012). Multiple simulations with different initial angular velocities and initial attitudes of the rigid body were conducted, and the system behaviour of the attitude control was visualized in plots. Through statistical analysis of the patterns in these plots (Schlanbusch, 2012) found three different motion behaviours, and from these behaviours there were developed three prediction formulas based on the initial attitude and angular velocity. With these statistical based formulas the prediction of which of the two equilibria is cheapest to reach based on energy consumption, were correct in nearly 90% of the maneuvers.

Through a minimum criteria of the controller's torque used to reach each of the equilibrium point, the statistical solution was evolved. The criteria was described as

$$J = \min\{J_{p+}, J_{p-}\} \tag{27}$$

is satisfied where $J_{p\pm}$ called a *performance functional* is described with the integral of the applied control torque

$$J_{p\pm} = \int_{t_0}^{t_f} \tau_a^\top(\mathbf{e}_{q\pm})\tau_a(\mathbf{e}_{q\pm})dt \tag{28}$$

where $\mathbf{e}_{q\pm}$ is defined in (18) and t_0 and t_f denotes the start and end time, respectively. Thus, J denotes whether either the positive or the negative equilibrium is chosen based on the energy criteria.

If the rigid body is standing still in the initial phase, it is obvious that the shortest rotation length is the least resource-demanding. The situation is more complex if the object is moving in the initial phase, both the attitude and the angular velocity can affect the need of the controller's torque. If an object is near the positive equilibrium, but the rigid body rotates in a direction towards the negative equilibrium, the least energy demanding can be to control the body towards the negative equilibrium.

4.1 Simulations for the Statistical Analysis

With use of the performance functional (28) and the control law for the closed-loop system (20), the energy consumption for the controller to maneuver a rigid body to one of the two equilibria from an initial position can be calculated. In our setup we have plotted each result in two different plots. Each dot indicates the initial values for a given simulation, and the distribution of dots between the plots is such that one plot collects all initial values which gives cheapest rotation for the positive equilibrium, and vice versa for the negative equilibrium.

There are three simulation cases with respectively three different standard deviation values for the initial angular velocity. Each of the simulation cases are simulated 1000 times. The initial four element attitude vector $\mathbf{q}(t_0)$ is uniform random generated with values from -1 to 1. The angular velocity is random generated from a normal distribution with a mean value as 0 rad/s, and the standard deviation of 0.01, 0.1 and 1 rad/s, respectively.

With the torque τ_a, the controller is able to maneuver the rigid body either to the positive or the negative equilibrium. In the following simulation results, all other forces than the controller's is set to zero, so no external or noise induced disturbances are accumulated within the performance functional. The desired states are the origin, which gives the desired attitude $\mathbf{q}_d = [\pm 1, \mathbf{0}^\top]^\top$ and the desired angular velocity and acceleration $\omega_d = \dot{\omega}_d = \mathbf{0}$.

4.1.1 Simulation Case 1, Standard Deviation 0.01 rad/s

The plot of the simulations are presented in the Figure 1 and in Figure 2. They show the closed-loop system behaviour for small initial angular velocities, given standard deviation of 0.01 rad/s in the random initial angular velocity vector.

Figure 1 shows scatter in the maneuvers where the positive equilibrium is the cheapest rotational direction based on the given initial values. Figure 2 shows the same for the negative equilibrium. As seen in the scatter plots of the simulations runs, the initial scalar part of the quaternion is the dominated part. The error value $\tilde{\eta}(t_0)$ gives the best solution for which of the

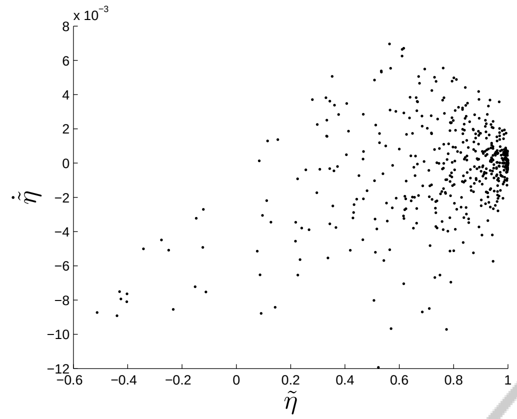

Figure 1: Simulation results of the positive equilibrium from random initial values with 0.01 rad/s standard deviation for angular velocity.

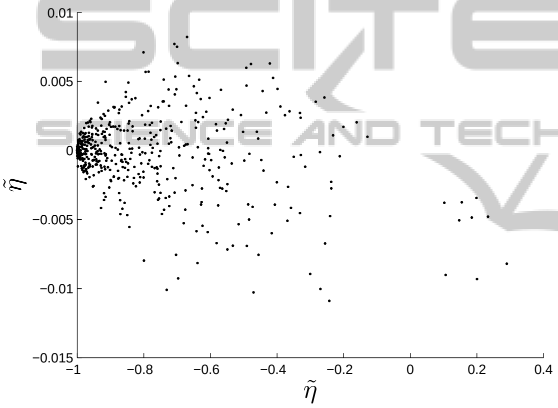

Figure 2: Simulation results of the negative equilibrium from random initial values with 0.01 rad/s standard deviation for angular velocity.

equilibrium to choose. If $\tilde{\eta}(t_0)$ is negative, then maneuvering to the negative equilibrium is least energy consuming. A positive $\tilde{\eta}(t_0)$ indicates that the positive equilibrium is the optimal one. This result can also be concluded by the fact that the kinetic energy is nearly zero. The cheapest rotation path for rigid bodies nearly standing still is the nearest equilibrium, as expected.

The first rule is made for cases with small initial angular velocities and is set up as

$$\mathbf{e}_q = \begin{cases} \mathbf{e}_{q+} & \text{if} \quad k_{\tilde{\eta}}\tilde{\eta}(t_0) + k_{\dot{\tilde{\eta}}}\dot{\tilde{\eta}}(t_0) \geq 0 \\ \mathbf{e}_{q-} & \text{if} \quad k_{\tilde{\eta}}\tilde{\eta}(t_0) + k_{\dot{\tilde{\eta}}}\dot{\tilde{\eta}}(t_0) < 0 \end{cases} \tag{29}$$

with the initial $\tilde{\eta}(t_0)$ values, and $k_{\tilde{\eta}}$ and $k_{\dot{\tilde{\eta}}}$ which are scaling constant. By tuning the *k*-constants through trial/error or statistical methods, this first rule will give a good estimation of which equilibrium point is least energy demanding for the controller.

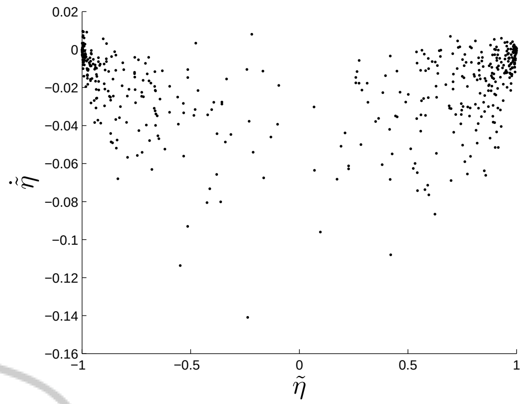

Figure 3: Simulation results for the positive equilibrium from random initial values with 0.1 rad/s standard deviation for angular velocity.

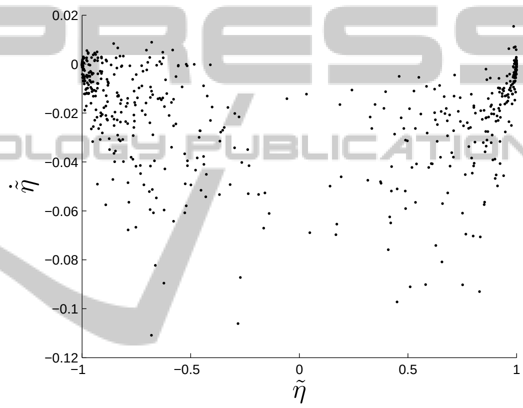

Figure 4: Simulation results for the negative equilibrium from random initial values with 0.1 rad/s standard deviation for angular velocity.

4.1.2 Simulation Case 2, Standard Deviation 0.1 rad/s

The second simulation case is performed with a standard deviation of 0.1 rad/s in the random angular velocity vector, which gives larger initial velocity values compared to the first simulation case.

The hit scatter for the positive equilibrium is located in the Figure 3, while the results for the negative equilibrium is located in Figure 4. From the plots it can now be seen that $\dot{\tilde{\eta}}(t_0)$ is the dominant parameter. The second rule found from this analysis can then be expressed as

$$\mathbf{e}_q = \begin{cases} \mathbf{e}_{q+} & \text{if} \quad \dot{\tilde{\eta}}(t_0) > 0 \\ \mathbf{e}_{q-} & \text{if} \quad \dot{\tilde{\eta}}(t_0) < 0 \end{cases} \quad \forall\ \dot{\tilde{\eta}}(t_0) \neq 0 \tag{30}$$

for cases where $\dot{\tilde{\eta}}(t_0) = 0$ then

$$\mathbf{e}_q = \begin{cases} \mathbf{e}_{q+} & \text{if} \quad \tilde{\eta}(t_0) \geq 0 \\ \mathbf{e}_{q-} & \text{if} \quad \tilde{\eta}(t_0) < 0 \end{cases} \quad . \tag{31}$$

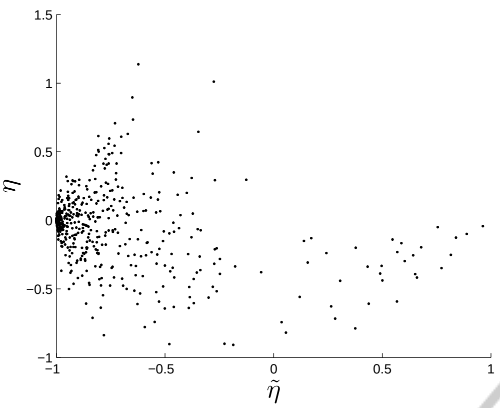

Figure 5: Simulation results for positive equilibrium from random initial values with 1 rad/s standard deviation for angular velocity.

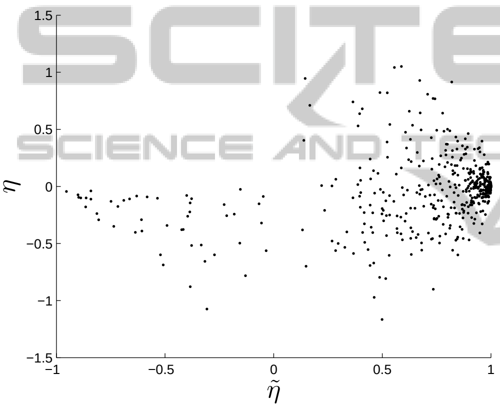

Figure 6: Simulation results for negative equilibrium from random initial values with 1 rad/s standard deviation for angular velocity.

4.1.3 Simulation Case 3, Standard Deviation 1 rad/s

For the third simulation case the random values for angular velocity have a standard deviation of 1 rad/s, which can be considered as large initial angular velocities.In Figure 5 we see the initial values which are preferred for the positive equilibrium, and for the negative equilibrium in Figure 6. The system is now preferring the opposite equilibrium, than observed in simulation case one, for small initial angular velocities ($\omega(t_0) \approx 0.01$ rad/s). The system does not care much for the sign of $\dot{\tilde{\eta}}(t_0$ as in simulation case two, shown in Figures 3 - 4. The behavior can be described by a third rule as

$$\mathbf{e_q} = \begin{cases} \mathbf{e_{q+}} & \text{if} \quad \tilde{\eta}(t_0) < 0 \\ \mathbf{e_{q-}} & \text{if} \quad \tilde{\eta}(t_0) > 0 \end{cases} \quad \forall\ \tilde{\eta}(t_0) \neq 0 \tag{32}$$

for $\tilde{\eta} = 0$ following is set

$$\mathbf{e_q} = \begin{cases} \mathbf{e_{q+}} & \text{if} \quad \dot{\tilde{\eta}}(t_0) \geq 0 \\ \mathbf{e_{q-}} & \text{if} \quad \dot{\tilde{\eta}}(t_0) < 0 \end{cases} . \tag{33}$$

With large initial angular velocity for a rigid body, it seems that it can be more efficient not to stop at the nearest equilibrium, thus demand fast and "hard" maneuvers, but use the initial velocity to rotate towards the opposite point, reducing the angular velocity more slowly. Also, stopping and reverse rotation due to overshoot seems to be more energy consuming, than instead continue in the initial rotational direction. Stated by (Schlanbusch, 2012) it can be seen that the opposite equilibrium compared to case one is preferred by the system, even if the parameter $\dot{\tilde{\eta}} \approx 0$. This is due to the relation $\dot{\tilde{\eta}} = -\frac{1}{2}\tilde{\varepsilon}^\top \mathbf{e}_\omega$. If the vector $\varepsilon(t_0) \approx 0$, that is, the rigid body is initially oriented close to one of the equilibria, large angular velocities $\mathbf{e}_\omega$ will not affect the resultant parameter $\dot{\tilde{\eta}}$.

5 EQUILIBRIUM ENERGY FUNCTION

For the behavioural analysis presented in Section 4.1 three different behaviours were observed, leading to three rules for choosing the cheapest equilibrium based on size of the initial values. By taking these three statistical rules and the initial angular velocity value, it is possible to make a better prediction function which works for all maneuver situations. The goal is to improve the accuracy in predicting the equilibrium point which requires least energy to reach, based on the initial value of the attitude and the angular velocity of the rigid body. If both the kinetic and potential energy in the initial states are accounted for, the energy usage for bringing the states of the dynamical system to the equilibrium can be made as small as possible.

If the rigid body is standing still in the initial phase, it is obvious that the shortest rotational path will be the least resource-demanding one. In situations with a rotational motion in the initial phase, then an energy function predicting the equilibrium based on the initial kinetic energy level is needed to predict which is optimal equilibrium point to reach.

From the statistical analysis presented in Section 4, three behaviours were found for the dynamical system. For small initial angular velocities $|\tilde{\omega}(t_0)| \approx 0$ it can then be seen that the behaviour is dependant of the initial value of $\tilde{\eta}(t_0)$. For large initial angular velocities over 1 rad/s, it can be seen the behaviour of the dynamical system is dependant on the same parameter, although with opposite behaviour, that is, the preferable equilibrium point is the one which is most far away from the initial attitude. For initial angular velocities between 0 rad/s to 1 rad/s, the system's behaviour is dependant of the $\dot{\tilde{\eta}}(t_0)$.

Following element of the equilibrium energy function[1] can be found for the first case - note that for all energy functions it is implicit that for $E \geq 0$ we choose $\mathbf{e}_{q+}$ and for $E < 0$ we choose $\mathbf{e}_{q-}$ - where there are small initial angular velocities

$$E_{\pm} = k_1 \tilde{\eta}(t_0), \tag{34}$$

thus the sign of $\eta(t_0)$ in the error quaternion decides the prefered equilibrium. This can be related to an object standing still, where it is "cheaper" to move to the nearest equilibrium.

The energy function can be expanded with an element from the second case

$$E_{\pm} = k_1 \tilde{\eta}(t_0) + k_2 \dot{\tilde{\eta}}(t_0) \tag{35}$$

for angular velocities up to 1 rad/s a comparison of direction and speed of the rotation, where it can be seen that the initial orientation is less significant, than in the first case. As stated in Section 4.1.3 we have that

$$\dot{\tilde{\eta}} = -\frac{1}{2} \tilde{\varepsilon}^{\top} \mathbf{e}_{\omega}. \tag{36}$$

where the initial kinetic energy disappears from the energy function when the initial orientation is close to either equilibria. To counter this effect we add a new cross coupling term between initial potential and kinetic energy according to

$$E_{\pm} = k_1 \tilde{\eta}(t_0) + k_2 \dot{\tilde{\eta}}(t_0) - k_3 \tilde{\eta}(t_0) \tilde{\omega}^{\top}(t_0) \tilde{\omega}(t_0), \tag{37}$$

where k_1, k_2 and k_3 are scaling constants. It can be seen that we propose to include $\tilde{\eta}$ in the last term of (37) in addition to $\tilde{\varepsilon}$ as in the cross term introduced in (35), to make the contribution also hold close to an equilibrium point. Also, we multiply with the quadratic term $\tilde{\omega}^{\top}\tilde{\omega}$ because we consider large angular velocity as in the third case, and the negative sign is used also in accordance with the third case, because we want to move away from the closest initial equilibrium point for large initial kinetic energy. The scaling constants k_1, k_2 and k_3 need be to tuned based on the system parameters such as moment of inertia and controller gains, to be able to do best possible prediction of the preferable equilibrium. Optimizing these constants can be done either manually, by trial and error, or through statistical analysis with an optimization tool. To sum up, we consider the first term in (37) as the potential energy while the two other terms as kinetic.

[1] Note that we don't consider the energy in this study to be of a specific type of unit, *e.g.* joule.

Table 1: Simulation results with different k values.

St.dev	k1	k2	k3	Hits	Percentage
0.01	1	61	56	9944	99.4 %
0.1	1	61	56	8832	88.3 %
1	1	61	56	9388	93.9 %
0.01	1	97	32	9838	98.4 %
0.1	1	97	32	9417	94.2 %
1	1	97	32	9806	98.1 %
0.01	2	51	17	9810	98.1 %
0.1	2	51	17	7246	72.5 %
1	2	51	17	9818	98.2 %
0.01	1	70	22	9963	99.6 %
0.1	1	70	22	9074	90.7 %
1	1	70	22	9823	98.2 %
0.01-1.5	1	70	22	9286	92.9 %
0.01	1	90	33	9878	98.8 %
0.1	1	90	33	9366	93.7 %
1	1	90	33	9784	97.8 %
0.01-1.5	1	90	33	96605	96.6 %

The optimized *k*-values are placed in Table 1 with the related simulation results with 10,000 runs for each row. The "St.dev" column in the Table 1 indicates which standard deviation is used when generating the randomized initial angular velocities. The "Hits" column gives the number of times where the energy function was able to predict the equilibrium which required least energy to reach. The last row contains 100,000 runs of simulations, which are conducted with varying standard deviation in equal steps from 0.01 to 1.5 rad/s in the initial angular velocities.

5.1 Error Analysis

Error analysis is done for the three methods; "InitChoice" is the initial choice equilibrium method introduced by (Kristiansen, 2008), "StatScheme" is the method called statistical scheme by (Schlanbusch, 2012) and the last one "OpEquEnFunc" is the optimized equilibrium energy function developed in this paper. Through 10,000 maneuver situations Table 2 counts the incidents where the prediction is correct. In the situations where the control solution is not leading the rigid body to the optimal equilibrium, the controller will use extra energy. The column "EnergyDiff" gives the percentage rate between the controller's actual used energy, and the energy used if the optimal equilibrium point were chosen all the time. It can be seen that the statistical scheme method and the optimized equilibrium energy function performs much better than the initial choice method, the best method is the one based of the equilibrium energy function.

In Table 3 the simulation results for maneuvers with the use of the best optimized equilibrium energy

Table 2: Energy Table Different methods.

St.dev	Hit	EnergyDiff%	Pred.Method
0.01	9737	0.64%	InitChoice
0.1	5410	31.94%	
1	872	22.55%	
0.01	9959	0.01%	StatScheme
0.1	9089	1.70%	
1	9209	0.62%	
0.01	9888	0.07%	OpEquEnFunc
0.1	9356	0.60%	
1	9773	0.45%	

Table 3: Average Energy Difference.

St.dev	EnergyDiff Hit	EnergyDiff Non-Hit
0.01	0.169	0.016
0.1	0.060	0.004
1	3.366	0.178

function are presented. With 1000 trials for each standard deviation three simulations runs were conducted. The column ”EnergyDiff Hit” gives the difference in the energy consumption of the controller rotating either to the positive or the negative equilibrium in the situations when the method chooses the optimal equilibrium. ”EnergyDiff Non-Hit” is the energy difference when the prediction fails. The result indicates that the prediction method struggles in the situations where the difference in energy is small, thus it can be concluded that in general there is a minimal use of extra energy when the wrong equilibrium is predicted.

6 CONCLUSIONS

In this paper a prediction function for choosing the optimal equilibrium for a rotational sphere is presented. This function is called the equilibrium energy function, and through the initial attitude values of the rotational object, the best equilibrium point is chosen. Simulation results show that this prediction function is able to predict the optimal equilibrium in nearly 97% of the maneuvers.

The energy function is only simulated in a closed-loop system with no perturbing torques, the only acting force on the dynamical model is the controller's attitude torque. It is recommended to investigate how perturbing forces will affect the prediction of the optimal equilibrium.

REFERENCES

Bhat, S. P. and Bernstein, D. S. (2000). A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1):63–70.

Egeland, O. and Gravdahl, J. T. (2002). *Modeling and Simulation for Automatic Control*. Marine Cybernetics AS. ISBN: 82-92356-01-0.

Fjellstad, O.-E. (1994). *Control of Unmanned Underwater Vehicles on Six Degrees of Freedom. A Quaternion Feedback Approach*. NTNU, Dep of Engineering Cybernetics. Dr.ing thesis Report 94-92-W.

Hahn, W. (1967). *Stability of Motion*. Springer-Verlag. ISBN: 9783540038290.

Khalil, H. K. (2002). *Nonlinear Systems*. Prentice Hall. ISBN: 978-01-306-7389-3.

Kristiansen, R. (2008). *Dynamic Synchronization of Spacecraft. Modeling and Coordinated Control of Leader-Follower Spacecraft Formations*. NTNU, Dep of Engineering Cybernetics. Dr.ing thesis 2008:115 ISBN: 978-82-471-8317-5.

Mayhew, C. G., Sanfelice, R. G., and Teel, A. R. (2009). Robust global asymptotic attitude stabilization of a rigid body by quaternion-based hybrid feedback. In *Proceedings of the 48th IEEE Conference on Decision and Control, held jointly with the 28th Chinese Control Conference*, pages 2522–2527, Shanghai, P. R. China.

Paden, B. and Panja, R. (1988). Globally asymptotically stable 'PD+' controller for robot manipulators. *International Journal of Control*, 47(6):1697–1712.

Schlanbusch, R. (2012). *Control of Rigid Bodies with applications to leader-follower spacecraft formations*. NTNU, Dep of Engineering Cybernetics. Dr.ing thesis 2012:49 ISBN: 978-82-471-3365-1.

Sidi, M. J. (1997). *Spacecraft Dynamics & Control. A practical engineering approach*. Cambridge University Press. ISBN: 0-521-78780-7.