A Hybrid Control System for a Tentacle Arm

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Keywords: Tentacle, Weighted Error, Boundary Control.

Abstract: The paper studies the control problem of a class of light- hyper-redundant robots, a tentacle arm, described by hyperbolic Partial Differential Equations with uncertain components. The stability analysis and the resulting controllers are obtained using the concept of boundary geometric control and a spatial weighted error control technique. A hybrid controller with two control components: a PD boundary control and a pneumatic system that controls the locking forces in the joints are discussed. Liapunov techniques are used to prove the control system stability. Numerical simulations and experimental results are also provided to verify the effectiveness of the presented approach.

1 INTRODUCTION

This paper implements a control system for a class of hyper-redundant robots, a tentacle model. The tentacle robots represent one of the most attractive domains of robotics during the last decade. A great deal of research has been conducted using this type of robot and environmental structure. In (Chirikjian, 1990); (Robinson, 1999); (Gravagne, 2000), the kinematic models were analysed. In (Mochiyama, 1999), the problem of controlling the shape of a robot with two-degree-of-freedom joints was also investigated using spatial curves. A controller for continuum robots was developed by (Braganza, 2007). Other researchers derived a new kinematic model by using the differential geometry (Walker, 1999) or introduced a real-time controller for continuum robots (Jones, 2006). In (Kapadia, 2009) it was proposed a sliding controller for extensible robots. The control problem of a class of continuum arms that performs the grasping function by coiling is also discussed in (Ivanescu, 2008). A frequency stability criterion for the grasping control problem is advanced in (Ivanescu, 2010). Several biomimetic robotic prototypes have been developed in (La Spina, 2007); (KeJun, 2010) and continuum robots with multiple, concentric, elastic tubes were analysed and discussed in (Rucker, 2010); (Bailly, 2011). All these research works underline the complexity of control problems, the difficulty in implementing feedback controllers and compensators determined by the dynamic models expressed by partial differential equations (PDE) and by the observability problems in distributed parameter systems. Controller design for these systems is in general based on an approximated finite –dimensional model, by truncating the infinite number of modes and by neglecting the higher frequency modes and by geometric control (Christofides, 1996); (Shang, 2005); (Maidi, 2009); (Maidi, 2010).

Our paper treats the control problem of a class of light tentacle robots. The dynamic model of the arm is described by hyperbolic Partial Differential Equations (PDE) with uncertain components. By using a spatial weighted error control, the infinite dimensional system control becomes a finite dimensional control problem. A robust algorithm with the characteristics of the conventional PD control is proposed. The stability of the system with the respect of weighted error is proven. The paper is structured as follows: section II presents technological model, section III analyses the mathematical model, section IV treats the control problem, section V verifies the control laws by computer simulation, section VI presents the experimental results and section VII is concerned with conclusions.

2 TECHNOLOGICAL MODEL

The technological model basis is a light weight arm. Although the conventional hyper-redundant models

Popescu N., Popescu D. and Ivanescu M..
A Hybrid Control System for a Tentacle Arm.
DOI: 10.5220/0004403200500057
In *Proceedings of the 10th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2013), pages 50-57
ISBN: 978-989-8565-70-9

operate in 3-D space, the motion control will be first inferred from the planar models. The 2D model basis from Fig 1 consists of a chain of vertebrae with a backbone type structure. All the joints of the arm are passive. The driving system of this arm has two components: a DC motor system with cable-tendons ensures the main motion of the arm and a pneumatic system controls the locking forces in the joints. Because the cables do not drive every element, externally attached springs between elements are introduced. The high flexibility of the arm is obtained by these rotational joints associated with the springs, distributed along the arm. The elements of the arm are clustered in segments, each segment having its own pneumatic control system. The pneumatic driven system is composed by a single acting mini-cylinder that develops a variable friction force in the i-joint. For high value of these forces, all the segment joints can be locked. We define this case as "the cluster segment (CS) is locked". If the elements of a CS are locked, the locked joints will not be rotatable and the cluster position is kept. The tendon driving system will rotate only the unlocked joints (Popescu, 2013).

The essence of the arm is the backbone curve C (Fig 2). The independent parameter s is related to the arc-length from origin of the curve C, $s \in \Omega$, $\Omega = [0, l]$, where l is the length of the arm. We denote by q the slope of the curve, $q = q(s)$ is the generalized coordinate. Also, τ represents the equivalent moment at the end of the arm $(s = l)$ exercised by the cable forces F_A. The arm can be assimilated with an ideal hyper-redundant arm, with a distributed mass and damping , with the mass density ρ , the elastic modulus E, the moment of inertia I and the rotational inertial density I_ρ.The position measuring of a cluster segment is obtained by electro-active polymer sensors that are placed on the surface of each segment.

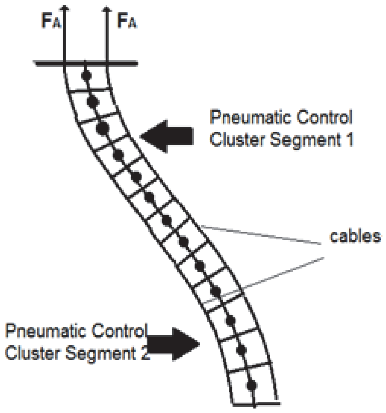

Figure 1: The 2D technological arm.

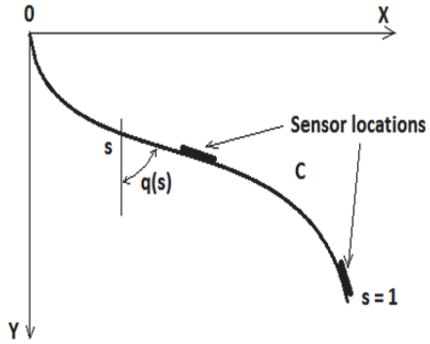

Figure 2: The ideal arm.

The sensors can measure the coordinate $q(t, s)$ at the location $\xi \in \Omega_s$, $\Omega_s \subset \Omega$, where Ω_s denotes the domain of permissible sensor locations.

3 MATHEMATICAL MODEL

The dynamic model can be expressed by the following Partial Differential Equation (PDE) (Gravagne, 2000),

$$I_\rho \, \ddot{q} = EI \frac{\partial^2 q}{\partial s^2} - b \, \dot{q} + c \, q + h \qquad (1)$$

where $q = q(t, s), q \in \Gamma(\Omega) \subset L_2(\Omega), \Omega = \{s | s \in [0, l]\}$, $\dot{q} = \partial q / \partial t$, I_ρ is the rotational inertial density, EI is the arm bending stiffness, b is the equivalent damping coefficient of the arm, c characterizes the elastic behaviour and h defines the nonlinear gravitational term. We assume the following initial and boundary conditions

$$q(0, s) = q_0(s) \qquad (2)$$

$$EI \frac{\partial q}{\partial s}(t, 0) = 0, \qquad EI \frac{\partial q}{\partial s}(t, l) = \tau(t) \qquad (3)$$

where τ is the equivalent moment generated by the cable forces. In (1), the friction is modelled using the viscous component b and neglecting the Coulomb and static friction (Qing, 2006),

$$b = k_B \, B \qquad (4)$$

where k_B is the coefficient of joint geometry and B is the viscous coefficient. The state of the system is defined by the vector $(q \ \dot{q})^T \in \Gamma \subset L_2(\Omega) x \, L_2(\Omega)$. The input is represented by the moment $\tau(t)$ at the boundary of the arm. For the gravitational term $h(t, s)$, that is difficult to be evaluated in a complex motion, we consider that the following constraint is verified (Khopalov, 2010)

$$\|h(., t)\|_2 \le M \|q(., t)\|_2 \qquad (5)$$

where M is a positive constant. We consider a desired state $q^d(s), q^d \in L_2(0,l)$, that satisfies (1) with boundary conditions (3) and we denote by

$$e(t,s) = q^d(s) - q(t,s) \quad (6)$$

the distributed error variable, $e \in L_2(0,l)$.

Definition 1 (Popescu, 2013)**.** The Weighted Error with respect to a sensor $(s = \xi_j)$, (ξ_jW-Error), is the spatial weighted value of the distributed error variable (6),

$$\tilde{e}^{\xi_j}(t) = \int_0^l \chi(s,\xi_j) e(t,s) ds, \tilde{e}^{\xi_j} \in C^2 \quad (7)$$

where $\chi(s,\xi_j)$ is the spatial weighting function that satisfies the following equation

$$\frac{d^2\chi(s,\xi_j)}{ds^2} = -\Lambda_j \chi(s,\xi_j) \quad (8)$$

with boundary conditions

$$\chi(0,\xi_j) = 0, \ \frac{d\chi(l,\xi_j)}{ds} = 0, j = 1,2,\dots,N \quad (9)$$

where Λ_j is a positive constant. We chose a solution of (8)-(9) as

$$\lim_{K\to\infty} \int_0^l (\chi(s,\xi_j) - \sum_{i=1}^{K} p_i^{\xi_j} w_i(s))ds = 0 \quad (10)$$

where $w_i(s), i = 1,2,..K$, are the eigenfunctions of the Sturm-Liouville problem

$$\frac{\partial^2 w_i(s)}{\partial s^2} = -\lambda_i w_i(s) \quad (11)$$

$$w_i(0) = 0, \quad \frac{\partial w_i(l)}{\partial s} = 0 \quad (12)$$

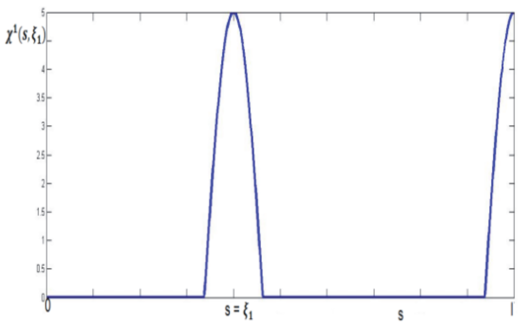

Figure 3: The weighting function, $\chi^1(s,\xi_1)$.

$$\chi^j(s,\xi_j) = A\sin\left(\frac{\pi}{\rho}\left(s - \xi_j + \frac{\rho}{2}\right)\right), \quad s \in \Omega_{sj}$$

$$\chi^j(s,\xi_j) = 0, \quad s \in \overline{\Omega_{sj}} \quad (13)$$

where $\Omega_{sj} = \{|s - \xi_j| \le \rho/2\}, \xi_j$ represents the sensor position, A is the magnitude and ρ defines the function characteristic (Fig 3). This function verifies the boundary conditions (9) and the set $\{(w_i), i = 1,2,..,K\}$ forms a complete set. So we can use the approximation

$$\int_0^l (\chi^j(s,\xi_j) - \sum_{i=1}^{K} p_i^{\xi_j} w_i(s))ds = 0 \quad (14)$$

where

$$p_i^{\xi_j} = \frac{\int_0^l \chi^j(s,\xi_j) w_i(s) ds}{\int_0^l w_i(s)^2 ds} \quad (15)$$

and $w_i(s)$ can be obtained from (11) - (12) as

$$w_i(s) = \sin(2i+1)\frac{\pi}{2l}s, \ \lambda_i = ((2i+1)\frac{\pi}{2l})^2, i = 1,2,\dots,, K$$

The boundary conditions (8) are satisfied by $w_i(s)$ and the constant Λ_j can be evaluated from (8) and (14).

4 CONTROL

4.1 Control System

The control problem consists of finding the control law $\tau(t)$ such that the ξ_jW-Error (7) converges to zero.

Definition 2. The W-Error control system is stable if

$$\lim_{t\to\infty} \tilde{e}^{\xi_j}(t) = 0 \quad (16)$$

In terms of this definition we can synthesize a PD - ξ_i W-Error controller that guarantees stability in the closed loop system.

Theorem 1. A ξ_iW -Error control of the system (1)-(3) is stable (in the sense of Definition 2) if the control law is

$$\Delta\tau^{\xi_j}(t) = -\frac{1}{\sum_{i=1}^{k} p_i^{\xi_j} w_i(l)}((EI\sum_{i=1}^{K} p_i^{\xi_j}\frac{\partial w_i(0)}{\partial s}(q^d(0) - q(t,0)) + k_1\int_0^l \sum_{i=1}^{K} p_i^{\xi_j} w_i(s)(q^d(s) - q(t,s))ds + k_2\int_0^l \sum_{i=1}^{K} p_i^{\xi_j} w_i(s)(-\dot{q}(t,s))ds \quad (17)$$

where k_1, k_2 are the control coefficients, that verify the conditions:

$$k_1 > 0, k_2 > 0$$
$$(b - \propto + k_2)(\propto (EI\Lambda_j) + \propto k_1) - \frac{1}{4}(\propto b - (M + k_1) - \propto (M + k_2))^2 > 0 \quad (18)$$

$$\Delta\tau^{\xi_j}(t) = \tau^d - \tau^{\xi_j}(\tau) \quad (19)$$

and τ^d is the desired moment applied at $s = l$.
Proof. See Appendix.

4.2 Control Strategy

The control system is presented in Fig 4. It is a hybrid driving system that controls the cable motors and pneumatic system in order to ensure sequential locked or unlocked cluster segments**.** The desired trajectory is obtained sequentially by concatenation of the locked or unlocked cluster segment effects. Let $(q_1^d(l_1), q_2^d(l_2), q_3^d(l_3)..q_N^d(l))$ be the desired trajectory defined by the position sensors along each cluster segment .

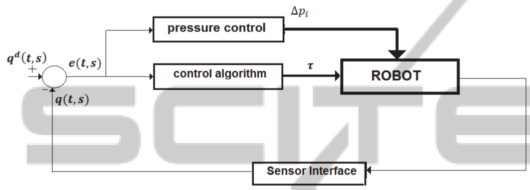

Figure 4: Control system.

Step 1. The position sensor, $s = \xi_1 = l_1$ associated to the first CS, is activated (the desired position $q_1^d(l_1)$) . The control algorithm (17)-(19) is applied at the cable driving system. All arm is bending (Fig 5a).

Step 2. When $\tilde{e}^{\xi_1}(t) = 0$, the pneumatic control is activated and the CS 1 is locked.

Step 3. The position sensor, $s = \xi_2 = l_2$ associated to the second CS, is activated (the desired position$q_2^d(l_2)$) .

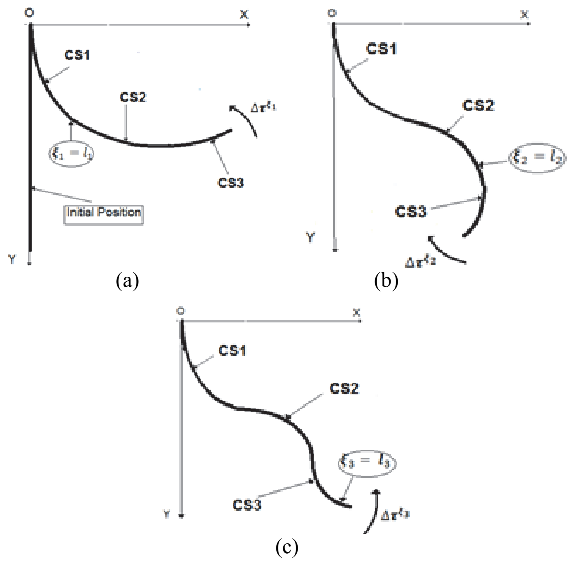

Figure 5: Control strategy.

The control algorithm (17)-(19) is applied at the cable driving system (Fig 5b).

Step 4. When $\tilde{e}^{\xi_2}(t) = 0$, the pneumatic control is activated and the CS 2 is locked.

Step 5. The procedure is repeated for the cluster segments 3, 4… N.

Consequently, we can control the arm motion, sequentially, step-by-step, by altering the locked and unlocked CS configuration. The whole procedure for an arm with three cluster segments is shown in Figure 5.

5 NUMERICAL SIMULATION

Consider the dynamic model of a tentacle robotic arm described by (1) where the length of the arm is $l = 1\,m$, the rotational inertial density is $I_\rho = 0.001$ kg m^2, the bending stiffness $EI = 15$, the viscous coefficient is $b = 12\,Nms/rad$ and the elastic coefficient is $c = 1.5$. These constants are scaled to realistic ratios for a long thin arm. The initial and boundary conditions are: $q_0(s) = 0$, $q_s(t,0) = 0$; $EIq_s(t,l) = \tau$, where τ is the torque applied at the top of the arm $(s = l)$. The uncertain term h(t, s) defines the gravitational components, $h(s) = \rho g A \int_0^s \sin(q)\,ds$, where ρ is the linear density, g is gravitational acceleration and A is the section area. For the characteristic values of these parameters $(\rho = 0.8\ kg/m, g = 10\ m/s^2, A = 4\,10^{-4} m^2)$, associated to this thin long arm, the inequality (5) is satisfied for M = 2. The arm contains two equal cluster segments.

A spatial weighting function (13) is selected for $\xi_1 = 0.4m, \xi_2 = 0.9m$ and an approximation (14) with K=100 is used. The constants $\Lambda_1 = 4.5$ and $\Lambda_2 = 6.8$ are determined. A control law (17) with $\alpha = 0.2, k_1 = 4, k_2 = 20$ is implemented. These coefficients verify the stability conditions (18).

Step 1. The desired state is $q^d(s) = 1.8\cos(1.5\,s)$ and represents the objective of the first stage. The arm (both segments) is bending to the desired position (as in Fig 5a). The position sensor ξ_1 is used for position control. The control law (17) with $\xi_j = \xi_1$ is implemented. The dynamic of the distributed error $e(t,s)$ is presented in Fig 6.

Step 2. When $\tilde{e}^{\xi_1}(t) = 0$, the pneumatic control is activated and CS 1 is locked.

Step 3. A desired state for the 2nd segment $q^d(s) = -1.8\cos(1.5\,s)$ is imposed. The position sensor ξ_2 is used for position control. The control law (17) with $\xi_j = \xi_2$ is implemented. The dynamic of the distributed error $e(t, s)$ is presented in Fig 7.

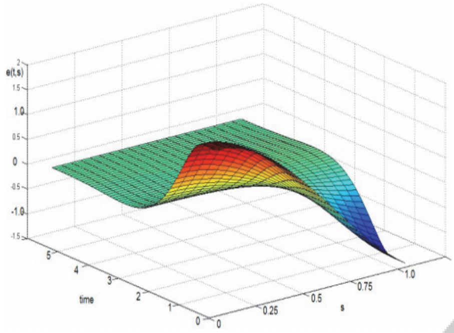

Figure 6: Error dynamics, e(t, s)- Cluster Segment 1.

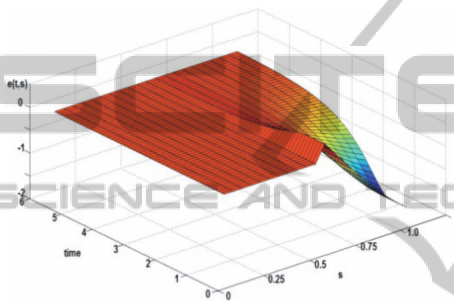

Figure 7: Error dynamics, e(t, s)- Cluster Segment 2.

We remark that the 1st CS is locked, the distributed error is $e(t, s) = 0, for\ s \in [0, \frac{l}{2}]$ and the control position is obtained for the 2nd CS, $s \in [\frac{l}{2}, l]$, (Fig 5b).

Step 4. When $\tilde{e}^{\xi_2}(t) = 0$, the pneumatic control is activated and CS 2 is locked. The good performances of the proposed control algorithm can be concluded from the graphics.

5 EXPERIMENTAL RESULTS

In order to verify the suitability of the control algorithm, a platform with a 2D tentacle arm has been employed for testing. The arm consists of two cluster segments, each segment having six links serially connected by revolute joints in a chain. All the joints are passive. A pair of antagonistic cable actuators connected at the terminal point $s = l = 0.4$ m ensures the actuation system. The force in each cable is determined by the DC motors and a transmission system. The "state of locking" of each joint is obtained by a pneumatic mini-linear actuator. A polymer thick film layer is placed on the upper element of each segment. A sensor exhibits a decrease in resistance when an increase of the film curvature is used. A Wheatstone bridge system is used to measure the variation of the resistance. The arm in the initial position, a vertical one is shown in Fig. 8.

Figure 8: The arm positions.

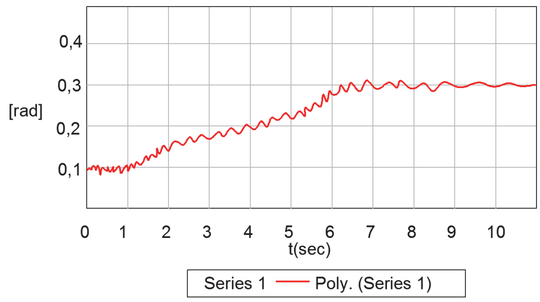

Figure 9: Tracking position- Cluster Sensor1.

A Quanser based platform is used for control and signal acquisition. A control law (17) with $\theta^d(s) = 1.8\cos(1.5\,s)$ is implemented. The new positions of the arm after steps 1 and 2 are presented in Fig. 8. The sensor information on the first segment is shown in Fig. 9.

Now, the cluster segment 1 is locked and a new actuation is obtained by bending the segment 2 for a new desired position $\theta^d(s) = 2.7\cos(1.5\,s)$. This position is illustrated in Fig 8 and the sensor data are presented in Fig 10. An analyse of this experimental result confirms the algorithm performance.

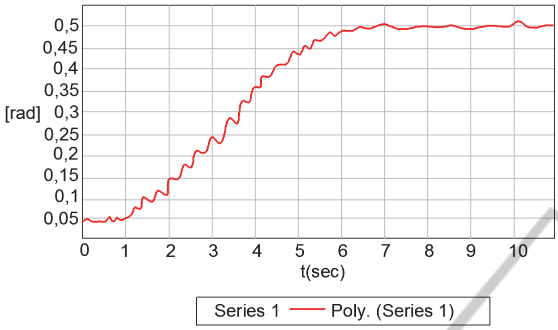

Figure 10: Tracking position- Cluster Sensor2.

6 CONCLUSIONS

In this paper, the control problem that is related to a class of tentacle arms has been discussed. The model basis consists of a chain of vertebrae periodically spaced, each element having a special joint that ensures the rotation, elastic contact and a controllable friction force with the following element. All the joints are passive. We propose a hybrid controller with two control components: a PD boundary control and a pneumatic system that controls the locking forces in the joints. For the dynamic model described by a hyperbolic PDE with uncertain components, a robust algorithm based a spatial weighted error technique is discussed. Liapunov methodes are used to prove the control system stability. Numerical simulations and experimental results verify the effectiveness of the presented algorithms and techniques.

ACKNOWLEDGEMENTS

This work has been supported by PPCA 150/2012 Romania.

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APPENDIX

For the desired state $(q^d(t,s))$ the W-Error dynamics can be obtained from (1), (7), (3), (9), as (in order to simplify the notation, the index ξ_j is omitted),

$$\begin{aligned} I_\rho\, \ddot{\tilde{e}}(t) &= -b\dot{\tilde{e}}(t) - (EI\,\Lambda - c)\tilde{e}(t) + \tilde{h}(t) \\ &+ \Delta\tau \sum_{i=1}^{K} p_i w_i\,(l) \\ &- EI\Big(\sum_{i=1}^{K} p_i \frac{\partial w_i(0)}{\partial s}\Big(q^d(t,0) - q(t,0)\Big) \end{aligned} \quad \text{(A1)}$$

$\tilde{e}(0)=0$ where p_i are determined by (15), $\tilde{h}(t)$ is obtained from the relation

$$\tilde{h} = \int_0^l \sum_{i=1}^{K} p_i\, w_i(h - h^d)ds$$

and the constraint (5) becomes,

$$|\tilde{h}| \leq M\|(\tilde{e}\ \dot{\tilde{e}})^T\|_2$$

Let us consider the Liapunov function

$$V = V(t) = \frac{1}{2} I_\rho \dot{\tilde{e}}^2 + \frac{1}{2}(EI\,\Lambda - c)\tilde{e}^2 + \alpha\,\tilde{e}\,\dot{\tilde{e}}$$

where α is a positive constant that satisfies the condition

$$\alpha^2 < 4\, I_\rho(EI\Lambda - c)$$

This inequality ensures that V is a positive definite function (Silvester's Theorem (Krstic,2006)).The time derivative will be

$$\dot{V} = I_\rho\, \dot{\tilde{e}}\ddot{\tilde{e}} + (EI\Lambda - c)\tilde{e}\,\dot{\tilde{e}} + \alpha\dot{\tilde{e}}^2 + \alpha\,\tilde{e}\ddot{\tilde{e}} \quad \text{(A2)}$$

By evaluating (A2) along with the solutions of (A1),with the control law (17), we obtain

$$\begin{aligned} \dot{V} = -(b-\alpha)\,\dot{\tilde{e}}^2 - \alpha(EI\Lambda - c)\tilde{e}^2 + \tilde{h}\,\dot{\tilde{e}} + \alpha\tilde{h}\,\tilde{e} \\ -\alpha b\tilde{e}\,\dot{\tilde{e}} + \tau_1\,\dot{\tilde{e}} + \alpha\tau_1\tilde{e} \end{aligned} \quad \text{(A3)}$$

or

$$\begin{aligned} \dot{V} \leq -(b-\alpha)\,\dot{\tilde{e}}^2 - \alpha(EI\Lambda - c)\tilde{e}^2 - \alpha b\tilde{e}\,\dot{\tilde{e}} - \\ k_2\dot{e}^2 - \alpha k_1\tilde{e}^2 + \left|\dot{\tilde{e}}\big(\tilde{h} - k_1\tilde{e}\big)\right| + \left|\alpha\tilde{e}\big(\tilde{h} - k_2\,\dot{\tilde{e}}\big)\right| \end{aligned} \quad \text{(A4)}$$

From (A4), we infer that

$$\left|\dot{\tilde{e}}\big(\tilde{h} - k_1\tilde{e}\big)\right| \leq |\dot{\tilde{e}}|(|\tilde{h}| + k_1|\tilde{e}|) \leq (M + k_1)|\tilde{e}||\dot{\tilde{e}}|$$

$$\left|\alpha\tilde{e}\big(\tilde{h} - k_2\,\dot{\tilde{e}}\big)\right| \leq \alpha|\tilde{e}|(|\tilde{h}| + k_2|\dot{\tilde{e}}|) \leq \alpha(M + k_2)|\tilde{e}||\dot{\tilde{e}}|$$

Using these inequalities, (A4) can be rewritten as

$$\dot{V} \leq -\begin{bmatrix}\dot{\tilde{e}} \\ \tilde{e}\end{bmatrix}^T \begin{bmatrix} (b - \alpha + k_2) & \frac{1}{2}(\alpha b - (M + k_1) - \alpha(M + k_2)) \\ \frac{1}{2}\big(\alpha b - (M + k_1) - \alpha(M + k_2)\big) & \alpha(EI\Lambda - c) + \alpha k_1 \end{bmatrix} \begin{bmatrix}\dot{\tilde{e}} \\ \tilde{e}\end{bmatrix}$$

or

$$\dot{V} \leq -\begin{bmatrix}\dot{\tilde{e}} \\ \tilde{e}\end{bmatrix}^T Q \begin{bmatrix}\dot{\tilde{e}} \\ \tilde{e}\end{bmatrix}$$

The stability condition requires as the matrix Q to be positive definite,

$$(b-\alpha+k_2)\left(\alpha(EI\Lambda-c)+\alpha k_1\right)-\frac{1}{4}\left(\alpha b-(M+k_1)-\alpha(M+k_2)\right)^2>0$$

that corresponds to the condition (18) of Theorem 1.