Analytical Forward Kinematics to the 3 DOF Congruent Spherical Parallel Robot Manipulator

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Keywords: Robot Manipulator, Parallel Platform, Forward Kinematics, Analytical Solution.

Abstract: This paper studies the kinematics of a special three degree-of-freedom (3 DOF) spherical parallel robot manipulator, where the two pyramids are exactly the same and so it is commonly called the 3 DOF congruent spherical parallel platform. Due to this special structure, the movement of the mobile pyramid can be regarded as the rotation of a rigid body from its base posture to its current status. By use of this special property, the forward kinematics of the parallel robot manipulator is obtained in this paper, and the final solution is a univariate quartic equation, which can be solved analytically without numerical iterations. A numerical example is provided to illustrate the method.

1 INTRODUCTION

Compared with a serial robot manipulator, a parallel robot manipulator has its advantages of higher rigidity and stiffness, simpler structure, better accuracy, and heavier loading. However, its forward kinematics is very complex. A parallel robot manipulator may have 16, even 40 solutions to its forward kinematics. So very few of the parallel robot manipulators have analytical solutions in terms of forward kinematics, and one example of these was presented before (Bruyninckx, 1998). On the other hand, the general 6 DOF spherical robot manipulator was studied by Wohlhart, and its forward kinematics has 16 solutions (Wohlhart, 1994). This paper analyzes a general 3 DOF spherical parallel robot manipulator, as shown in Fig. 1. This robot manipulator has two pyramids, and it has been studied by many researchers (Innocenti and Parenti-Castelli, 1993); (Gosselin et al., 1994a); (Gosselin et al., 1994b); (Huang and Yao, 1999); (Leguay-Durand and Reboulet, 1997); (Zhang et al., 1998).

In this parallel robot manipulator, the two pyramids are connected together by a spherical joint at the point $\boldsymbol{O}$, which is also the origins of the two coordinate systems in the two pyramids. The mobile pyramid $\boldsymbol{O}\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3$ can only rotate at this point $\boldsymbol{O}$, consequently, the parallel robot manipulator can only provide a movement of 3 DOF pure rotation. The structure of the parallel platform is simple, however, the forward kinematics of this general platform, like others, is quite complicated, and the final solution is a univariate eighth polynomial equation (Innocenti and Parenti-Castelli, 1993); (Huang and Yao, 1999), and has to be solved numerically. This paper discusses a special structure of this spherical parallel robot manipulator, where the mobile pyramid is exactly the same with its counterpart, the base pyramid, in shape. So, the structure can be called the 3 DOF congruent spherical parallel platform. In this robot manipulator, the movement of the mobile pyramid, caused by the changes of link lengths, can be regarded as the rotation of a rigid body from its base pyramid to its current status (the mobile pyramid). This idea, from the screw theory (Mavriodis, 1997; Mavriodis, 1998), was used to study some other parallel platforms (Innocenti, 1998); (Bonev et al., 2003); (Li and Xu, 2007); (Guo et al., 2012). By use of this special property, the final forward kinematics to the 3 DOF congruent parallel robot is a univariate quartic equation, which can be solved analytically, instead of an eighth polynomial in the general case, which has to be solved numerically.

Ji P. and Wu H..
Analytical Forward Kinematics to the 3 DOF Congruent Spherical Parallel Robot Manipulator.
DOI: 10.5220/0004399601110115
In *Proceedings of the 10th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2013), pages 111-115
ISBN: 978-989-8565-71-6

2 FORWARD KINEMATICS

2.1 Geometric Structure

A general 3 DOF spherical parallel robot manipulator, as shown in Fig. 1, consists of the base pyramid with three vertices, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ and the point $\boldsymbol{O}$, and the mobile pyramid with three vertices, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and $\boldsymbol{O}$. Here, the point $\boldsymbol{O}$ is the intersection of the two pyramids, and also the origins of the two coordinate systems in their respective pyramids. In the platform, vector $\boldsymbol{L}_k$ (its norm L_k), that is, a link, connects the couple vertices $\boldsymbol{a}_k$ (its norm a_k) and $\boldsymbol{b}_k$ (its norm b_k) (k = 1, 2, 3). Since the two pyramids are the same in this special congruent platform, and the two coordinate systems in the pyramids are also set to be the same, we have:

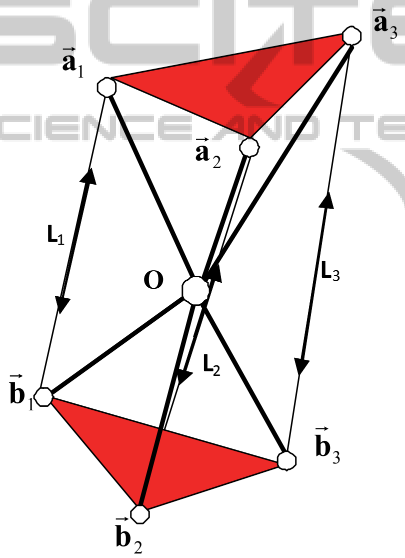

Figure 1: A general 3 DOF spherical parallel platform.

$$\boldsymbol{a}_k = \boldsymbol{b}_k \ (k = 1, 2, 3) \tag{1a}$$

$$a_k = b_k \ (k = 1, 2, 3) \tag{1b}$$

For the convenience, let $\boldsymbol{e}_k$ be the unit vector for vector $\boldsymbol{a}_k$, so,

$$\boldsymbol{a}_k = a_k \, \boldsymbol{e}_k \ (k = 1, 2, 3) \tag{2}$$

The forward kinematics of the congruent platform is to determine the orientation of the mobile pyramid while the lengths of the three links, L_1, L_2 and L_3, are known. From the geometric relationship in Fig. 1, we have

$$\boldsymbol{L}_k = [\boldsymbol{R}] \, \boldsymbol{a}_k - \boldsymbol{b}_k \ (k = 1, 2, 3) \tag{3}$$

Due to Eq. (1a), Eq. (3) becomes

$$\boldsymbol{L}_k = ([\boldsymbol{R}] - \boldsymbol{I}) a_k \ (k = 1, 2, 3) \tag{4}$$

Here, [$\boldsymbol{I}$] is a unit 3 × 3 matrix, and [$\boldsymbol{R}$] is the transformation matrix between the two coordinate systems, or the pyramids. Obviously, Eq. (4) can be rewritten as follows:

$$L_k^2 = 2a_k^2 - 2\boldsymbol{a}_k^{\mathrm{T}}[\boldsymbol{R}]\boldsymbol{a}_k \ (k = 1, 2, 3) \tag{5}$$

If we set a_k be 1 in Eq. (5), that is, L_k stands for the ratio between the link length L_k and the vertex a_k (k = 1, 2, 3), Eq. (5) becomes

$$L_k^2 = 2 - 2\boldsymbol{e}_k^{\mathrm{T}}[\boldsymbol{R}]\boldsymbol{e}_k \ (k = 1, 2, 3) \tag{6}$$

2.2 The Transformation Matrix

By defining $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^{\mathrm{T}}$, a unit vector in space, then the transformation matrix [$\boldsymbol{R}$] can be written as follows (Angeles, 1997):

$$\begin{aligned}[\boldsymbol{R}] &= e^{[\lambda]\theta} \\ &= \cos(\theta)\boldsymbol{I} + \sin(\theta)[\boldsymbol{\lambda}] + [1 - \cos(\theta)]\boldsymbol{\lambda\lambda}^{\mathrm{T}}\end{aligned} \tag{7}$$

where θ is the rotation angle around the unit vector (axis) **λ**, and [**λ**] is a skew-symmetry matrix generated by the unit vector **λ**. In fact,

$$[\lambda] = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix} \tag{8a}$$

Besides,

$$\boldsymbol{e}_k^{\mathrm{T}}[\boldsymbol{\lambda}]\boldsymbol{e}_k = 0 \tag{8b}$$

$$\boldsymbol{e}_k^{\mathrm{T}}(\boldsymbol{\lambda\lambda}^{\mathrm{T}})\boldsymbol{e}_k = \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{e}_k\,\boldsymbol{e}_k^{\mathrm{T}})\boldsymbol{\lambda} \tag{8c}$$

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\lambda} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \tag{8d}$$

Obviously, Eq. (7) can be rewritten as:

$$[\boldsymbol{R}] = (1 - V_\theta)\boldsymbol{I} + S_\theta[\boldsymbol{\lambda}] + V_\theta\boldsymbol{\lambda\lambda}^{\mathrm{T}} \tag{9}$$

where $V_\theta = 1 - \cos(\theta)$, $S_\theta = \sin(\theta)$, $C_\theta = \cos(\theta)$. Consequently, Eq. (6) becomes:

$$L_k^2 = 2V_\theta - 2V_\theta\,(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_k)^2 \ (k = 1, 2, 3) \tag{10}$$

That is,

$$2V_\theta[\boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{I} - \boldsymbol{e}_k\boldsymbol{e}_k^{\mathrm{T}})\boldsymbol{\lambda}] = L_k^2 \ (k = 1, 2, 3) \tag{11}$$

Eq. (11) is symmetrical to **λ**, that is, if **λ** is the solution to Eq. (11), –**λ** is also the solution. Furthermore, it is symmetrical to θ, too.

2.3 The Solution

In order to get the final solution of the forward kinematics, V_θ is firstly eliminated in Eq. (11). From Eq. (11), we have

$$2V_\theta = \frac{L_k^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_k\right)^2} \quad (k = 1, 2, 3) \tag{12}$$

So,

$$\frac{L_1^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_1\right)^2} = \frac{L_2^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_2\right)^2} \tag{13a}$$

$$\frac{L_2^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_2\right)^2} = \frac{L_3^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_3\right)^2} \tag{13b}$$

$$\frac{L_3^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_3\right)^2} = \frac{L_1^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_1\right)^2} \tag{13c}$$

That is,

$$L_2^2 - L_1^2 = L_2^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_1\right)^2 - L_1^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_2\right)^2 \tag{14a}$$

$$L_3^2 - L_2^2 = L_3^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_2\right)^2 - L_2^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_3\right)^2 \tag{14b}$$

$$L_1^2 - L_3^2 = L_1^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_3\right)^2 - L_3^2\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_1\right)^2 \tag{14c}$$

Since $\boldsymbol{\lambda}$ is a unit vector, that is, Eq. (8d), Eqs. (14a), (14b) and (14c) have a homogenous form on $\boldsymbol{\lambda}$, that is,

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{W}_k\boldsymbol{\lambda} = 0 \ (k = 1, 2, 3) \tag{15}$$

where

$$\boldsymbol{W}_1 = L_3^2\boldsymbol{e}_2\boldsymbol{e}_2^{\mathrm{T}} - L_2^2\boldsymbol{e}_3\boldsymbol{e}_3^{\mathrm{T}} - (L_3^2 - L_2^2)\boldsymbol{I} \tag{16a}$$

$$\boldsymbol{W}_2 = L_1^2\boldsymbol{e}_3\boldsymbol{e}_3^{\mathrm{T}} - L_3^2\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (L_1^2 - L_3^2)\boldsymbol{I} \tag{16b}$$

$$\boldsymbol{W}_3 = L_2^2\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - L_1^2\boldsymbol{e}_2\boldsymbol{e}_2^{\mathrm{T}} - (L_2^2 - L_1^2)\boldsymbol{I} \tag{16c}$$

Set

$$t_1 = \frac{L_1^2}{L_3^2} \quad \text{and} \quad t_2 = \frac{L_2^2}{L_3^2} \tag{17}$$

Now $\boldsymbol{W}_k$ (k = 1, 2, 3) can be simplified as:

$$\boldsymbol{W}_1 = \boldsymbol{e}_2\boldsymbol{e}_2^{\mathrm{T}} - t_2\boldsymbol{e}_3\boldsymbol{e}_3^{\mathrm{T}} - (1 - t_2)\boldsymbol{I} \tag{18a}$$

$$\boldsymbol{W}_2 = t_1\boldsymbol{e}_3\boldsymbol{e}_3^{\mathrm{T}} - \boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (t_1 - 1)\boldsymbol{I} \tag{18b}$$

$$\boldsymbol{W}_3 = t_2\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - t_1\boldsymbol{e}_2\boldsymbol{e}_2^{\mathrm{T}} - (t_2 - t_1)\boldsymbol{I} \tag{18c}$$

By defining

$$x = \frac{\lambda_1}{\lambda_2} \quad \text{and} \quad y = \frac{\lambda_2}{\lambda_3} \tag{19}$$

Eq. (15) becomes:

$$f_1(x, y) = [x, y, 1]^{\mathrm{T}}\ \boldsymbol{W}_1[x, y, 1] = 0 \tag{20a}$$

$$f_2(x, y) = [x, y, 1]^{\mathrm{T}}\ \boldsymbol{W}_2[x, y, 1] = 0 \tag{20b}$$

$$f_3(x, y) = [x, y, 1]^{\mathrm{T}}\ \boldsymbol{W}_3[x, y, 1] = 0 \tag{20c}$$

Among f_1, f_2 and f_3, only two of them are independent and the other one is dependent. Using any two of them, we can obtain a fourth polynomial in variable x (or y) as follows:

$$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 = \sum_{k=0}^{4} c_kx^k = 0 \tag{21}$$

Eq. (21) can be solved analytically without numerical iterations and it has at most four real roots. Once x and y are found, that is, $\boldsymbol{\lambda}$ is obtained, then by Eq. (12), the rotation angle θ can be obtained:

$$\theta = \cos^{-1}\left[1 - \frac{0.5L_k^2}{1-\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{e}_k\right)^2}\right] \tag{22}$$

In Eq. (22), k can be 1 or 2 or 3. However, a verification for a valid θ for different k (k = 1, 2, 3) is required since three θ values may not the same. If the three θ values differ from each other, the superfluous θ should be discarded. The reason for the existence of a superfluous θ is that θ is related with all of the three absolute link lengths L_k (k = 1, 2, 3) directly while x, y and $\boldsymbol{\lambda}$ are obtained from the given t_1, t_2 (relative values of link lengths) only, as defined in Eq. (17). Besides, if θ is a real solution, either is -θ. This is obvious due to the cosine function in Eq. (11). Now, both $\boldsymbol{\lambda}$ and θ are known, from Eq. (7), we can obtain at most eight transformation matrices $\boldsymbol{R}$. Four of them are from θ, and the other four are from –θ, which are actually the transposes of the respective $\boldsymbol{R}$ from θ.

3 NUMERICAL EXAMPLE

The closed-form forward kinematics (21) and (22) suggest that at most eight real solutions exist to the 3 DOF congruent spherical parallel robot manipulator. A numerical example is presented here to show the above method. The geometric structure data of a 3 DOF congruent spherical parallel robot manipulator are as follows: $\boldsymbol{e}_1$ = {0.707107, 0.0, 0.707107}, $\boldsymbol{e}_2$ = {-0.353553, 0.612372, 0.707107}, $\boldsymbol{e}_3$ = {-0.353553, -0.612372, 0.707107} for the vertex unit vectors of the two platforms in their own frames, and the link length ratio are L_1/a_1 = 1.30, L_2/a_2 = 1.42, and L_3/a_3 = 1.44. In this case, the matrices $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ in Eqs. (18a) and (18b) become:

$$W_1 = \begin{bmatrix} -0.05005 & -0.88551 & -0.01430 \\ -0.88551 & -0.03575 & 1.77102 \\ -0.01430 & 1.77102 & -0.02860 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} -0.44195 & 0.365896 & -1.4593 \\ 0.365896 & 1.01735 & -0.731791 \\ -1.4593 & -0.731791 & 0.1918 \end{bmatrix}$$

So, f_1 and f_2 are:

$f_1 = -0.0137924 - 0.0137924x - 0.0241368x^2 + 1.70816y - 0.854081xy - 0.0172405y^2 = 0$

$f_2 = 0.0924961 - 1.40750x - 0.213132x^2 - 0.705817y + 0.352909xy + 0.490620y^2 = 0$

And Eq. (21) is:

$0.122476 - 2.11581x + 1.71067x^2 - 0.182711x^3 - 0.0784458x^4 = 0$

Finally, its four roots are:

$x_1 = -6.40053$;
$x_2 = 0.060861$;
$x_3 = 1.890762$;
$x_4 = 2.119774$.

The following table lists all the solutions:

Table 1: Final solutions.

(x, y)	$\boldsymbol{\lambda}$	θ
(-6.4005, 0.1274)	(-0.9878, 0.0196, 0.1543)	±107.141
(0.0609, 0.0088)	(0.0607, 0.0088, 0.9981)	±157.375
(1.8908, 2.6451)	(0.5558, 0.7775, 0.2939)	±108.817
(2.1198, -2.8442)	(0.5751, -0.7717, 0.2713)	±108.467

4 CONCLUSIONS

The forward kinematics of the 3 DOF congruent spherical parallel robot manipulator was first represented as three quadric equations of three parameters, then they were rewritten as an fourth polynomial in one variable by eliminating the other two variables, which provides a direct analytical solution without numerical iterations. A numerical example was presented to show the method developed in the paper.

ACKNOWLEDGEMENTS

The authors would like to thank Department of Industrial and Systems Engineering, The Hong Kong Polytechnic University for the financial support.

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