A Fuzzy Dynamic Belief Logic

Xiaoxin Jing and Xudong Luo*
Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou, China

Keywords: Belief Revision, Fuzzy Logic, Dynamic Logic.

Abstract: This paper introduces a new logic approach to reason about the dynamic belief revision by extending well-known Aucher's dynamic belief revision approach to fuzzy environments. In our system, propositions take a valuation in linguistic truth term set, and a belief revision is also in a qualitative way. Moreover, we reveal some properties of our system in epistemic style and do a comparison between our fuzzy belief system with famous AGM postulats.

1 INTRODUCTION

The problem of belief change is an active topic in logic (Aucher, 2006; van Ditmarsch, 2005; Shapiro et al., 2011) and artificial intelligence (Sardina and Padgham, 2011; Casali,Godob and Sierra, 2011). Its focus is to understand how an agent should change his belief in the light of new information. Researchers have developed various models for belief revision. Epistemic plausibility model is one of main approaches to model the dynamics of the belief. It adds the plausibility ordering in the epistemic model for each agent, i.e., a pre-order $w \leq v$ that says agent i considers world w at least as plausible as v.

In order to reason about this structure, the epistemic language is extended with a conditional belief operator. van Benthem (2007) developed a dynamic belief revision frame based on the dynamic epistemic logic and then adds the conditional belief operator in it. Baltag (2006) uses the epistemic plausibility model for conditional belief in a multi-agent epistemic environment, and introduces plausibility pre-order on actions, notated as action plausibility models to display the dynamic setting, which is somehow the extension of well-known Aucher's method in (Aucher, 2006). Another typical approach for belief revision is epistemic probability model (Baltag and Smets, 2006), which is closed to the epistemic plausibility model with probability measures in the place of plausibility ordering. In these models, various arithmetical formulas are used to compute the probability of belief of the output-states from the probabilities of the input-states and the probabilities of actions.

In the logic above, the beliefs are all crisp. However in real-life, it is not always the case (Wu and Zhang, 2012). In fact, our belief is often fuzzy. For example, "I think the temperature will be high tomorrow". Here "high" is a fuzzy conception because there is not any crisp division between "high" and "low". And suppose that we want to book an air ticket, we thought ticket A is good at first and then when we know that the flight would be delayed, we need to revise our previous beliefs about the ticket. On the one hand, "the ticket is good" is a fuzzy proposition because there are no crisp standards for a good ticket. It will depend on many factors such as the price, the departure time and service, and so on. On the other hand, we might not be able to decide whether or not to change our previous belief completely. Rather, we can only say the ticket is better or worse than before. How can we handle this kind of fuzzy belief revision?

Fuzzy theory can provide a powerful tool to handle this situation, which is studied in various areas, such as mathematics, logic and computer science. Fuzzy logic began with the 1965 proposal of fuzzy set theory by Zadeh (1965) and it is a form of many-valued logic (Zadeh, 1975). It deals with reasoning that is approximate rather than accurate. In traditional logic, a proposition is usually *true* or *false*, while fuzzy logic proposition can have a truth value in-between 0 and 1. In this sense, fuzzy logic is a better way for us to handle uncertain reasoning for fuzzy belief revision.

The structure of the paper is listed as follows. Section 2 briefs the fuzzy theory, which we will use for developing our logic. Section 3 defines our fuzzy system based on the dynamic belief logic. Section 4 studies some properties of our logic. Section 5 gives a comparison between our fuzzy logic system with AGM postulats. Section 6 discusses the related work to confirm that our work has advanced the state of art.

*Corresponding author.

Jing X. and Luo X..
A Fuzzy Dynamic Belief Logic.
DOI: 10.5220/0004257302890294
In *Proceedings of the 5th International Conference on Agents and Artificial Intelligence* (ICAART-2013), pages 289-294
ISBN: 978-989-8565-39-6

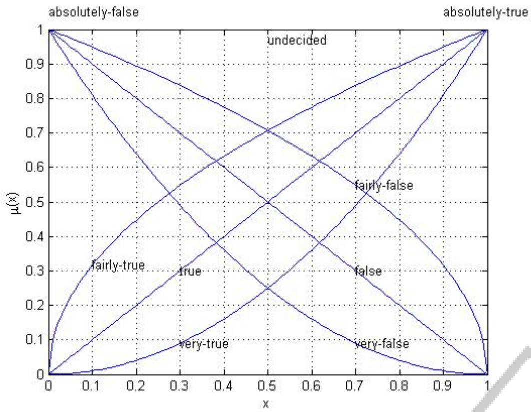

Figure 1: Linguistic truth.

Section 7 concludes the paper with future work.

2 PRELIMINARIES

This section recaps fuzzy set and logic theory. Let $X = \{x\}$ be a space of points with a generic element, denoted as x, of X. A fuzzy set A in X is characterised by a membership function of $\mu(x)$, which associates with each point in X a real number in interval $[0,1]$, with the values of $\mu(x)$ at x representing the membership degree of x in A. In fuzzy logic, truth value comes in continuous degrees. There are different forms for truth value. By a linguistic variable we mean a variable whose values are words or sentences in a natural language (Zadeh, 1975; Luo et al., 2002).

Defininition 1 (Linguistic Truth)**.** *A proposition can take truth on linguistic truth rather than classical proposition truth* $\{0,1\}$*, i.e.,*

$$\begin{aligned} LTTS \;=\; \{&absolutely\text{-}true, very\text{-}true, true,\\ &fairly\text{-}true, undecided, fairly\text{-}false,\\ &false, very\text{-}false, absolutely\text{-}false\}. \end{aligned} \quad (1)$$

For convenience, we denote

$$LTTS_t = \{absolutely\text{-}true, very\text{-}true, true, fairly\text{-}true\},$$
$$LTTS_f = \{absolutely\text{-}false, very\text{-}false, false, fairly\text{-}false\}.$$

Thus, we have

$$LTTS = LTTS_t \cup LTTS_f \cup \{undecided\}.$$

Semantics of the terms in this term set are defined as shown in Table 1 (Luo et al., 2002).

The operation on linguistic variables are defined according to the corresponding operation on numerical variables by using the extension principle as follows:

Defininition 2 (Extension Principle)**.** *Suppose f is a function with n arguments $x_1, \ldots, x_n$, denoted by $\vec{x}$. Let $\mu_i(x_i)$ be the membership function of argument x_i ($1 \leq i \leq n$). Then*

$$\mu(y) = sup\{\mu_1(x_1) \wedge \ldots \wedge \mu_n(x_n) \mid f(\vec{x}) = y\}, \quad (2)$$

where sup *denote the supremum operation on a set, $\wedge$ means the conjunction of the items. Let the fuzzy set corresponding to μ be B, and let the fuzzy set corresponding to μ_i be A_i. For convenience, we denote the operation of the extension principle as $\otimes$. i.e.,*

$$B = \otimes(A_1, \ldots, A_n, f). \quad (3)$$

If an operation on some linguistic terms is not closed in the predefined linguistic term set, a linguistic approximation technique in necessary in order to find a term in the term set, whose meaning (membership function) is the closest to that of the result of the operation.

Defininition 3 (Linguistic Approximation)**.** *The most straight forward approach, the BEST FIT, uses the Euclidean Distance (ED) as follows:*

$$ED(A,B) = \sqrt{\sum\{(\mu_A(x) - \mu_B(x))^2 \mid x \in [0,1]\}} \quad (4)$$

between fuzzy sets A and B defined on $[0,1]$*, to evaluate which one in the term set is the closest to the set being approximated. Namely, $\tau \in LTTS$, being the closest to τ'' should satisfy*

$$\forall \tau' \in LTTS, ED(\tau, \tau'') \leq ED(\tau', \tau''). \quad (5)$$

For convenience, we denote the above operation of linguistic approximation as $\odot$, i.e.

$$\tau = \odot(\tau'') \quad (6)$$

3 FUZZY DYNAMIC BELIEF LOGIC

This section will present our logic, which will be defined on the linguistic truth set and combines with the belief dynamic system of (van Ditmarsch, 2005).

3.1 Language

We define the language of our fuzzy dynamic belief logic as follows:

Defininition 4 (Language of belief-epistemic logic)**.**

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid K_j\phi \mid B_j\phi \mid [*\phi]\psi$$

where p is an atomic formula as usual, $\neg\phi$ and $\phi \wedge \psi$ are the Boolean combination of the propositional formulas, $K_j\phi$ means agent j knows proposition ϕ, $B_j\phi$ means agent j believes proposition ϕ, and $[*\phi]\psi$ expresses that proposition ψ holds after revising angent's belief with formula ϕ.

Table 1: Linguistic truth.

$\mu_{absolutely-true}(x)=\begin{cases}1 & \text{if } x=1\\ 0 & \text{otherwise}\end{cases}$	$\mu_{absolutely-false}(x)=\begin{cases}1 & \text{if } x=0\\ 0 & \text{otherwise}\end{cases}$
$\mu_{very-true}(x)=\mu^2_{true}(x)$	$\mu_{very-false}(x)=\mu^2_{false}(x)$
$\mu_{true}(x)=x, \forall x\in[0,1]$	$\mu_{false}(x)=1-x, \forall x\in[0,1]$
$\mu_{fairly-true}(x)=\mu^{1/2}_{true}(x)$	$\mu_{fairly-false}(x)=\mu^{1/2}_{false}(x)$
$\mu_{undecided}(x)=\begin{cases}1 & \text{if } x\in(0,1)\\ 0 & \text{otherwise}\end{cases}$	

3.2 Semantics

The semantics of our system is completely different from the one in the classical two-valued logic. The truth value of the proposition is taken on the language truth term set, $LTTS$, given by (1). The functions of different linguistic truths' membership degrees have been listed in Table 1 and their figure have been showed in Fig.1 correspondingly.

Then we will give the semantics of the formulas in three steps. First, we will give the semantics of the simple formulas, (i.e., the no-epistemic formulas), then the epistemic formulas with epistemic operator K, B, and finally we will define the semantics of the belief revision.

First in the simplest situation, the semantics of the non-epistemic formulas (i.e., the ones in the classical predicate logic). We can obtain the value of the formulas from estimates for the valuation of its subformulas. For example, if P is a Boolean combination of P_1 and P_2, $Val(P_1)=\tau_1\in LTTS$, $Val(P_2)=\tau_2\in LTTS$, then $Val(P)=\tau\in LTTS$, and we can obtain τ by doing some operation on τ_1 and τ_2. Specifically, we apply the extension principle $\otimes$ and the linguistic approximation technique $\odot$ on the valuation of the sub-formulas, and at last obtain the result we expected.

Defininition 5. *Let $Val(P)=\tau\in LTTS$. Let $\mu_\tau(x)$ denotes the membership degree of a proposition on the linguistic value $LTTS$ when given a value x between $[0,1]$. Then*

1. $Val(\neg P)=\tau_{\neg P}$, *and* $\mu_{\tau_{\neg P}}(x)=\mu_{\tau_P}(1-x), \forall x\in[0,1]$;
2. $Val(P_1\wedge P_2)=\odot(\otimes(\tau_{P_1},\tau_{P_2}),\triangle)$;
3. $Val(P_1\vee P_2)=\odot(\otimes(\tau_{P_1},\tau_{P_2}),\bigtriangledown)$; *and*
4. $Val(P_1\to P_2)$
$$=\begin{cases}\textit{absolutely-true} & \text{if } Val(P_1)\le Val(P_2),\\ \odot(1-\mu(x)+\mu(y)) & \text{if } Val(P_1)>Val(P_2).\end{cases}$$

According to (Bonissone and Decker, 1986), $\triangle$ denotes the operator given by one of $\triangle_1$, $\triangle_2$ and $\triangle_3$ defined as follows:

$$\triangle_1(x,y)=\max\{0,x+y-1\},$$
$$\triangle_2(x,y)=x\times y,$$
$$\triangle_3=\min\{x,y\};$$

and $\bigtriangledown$ denotes the operator given by one of $\bigtriangledown_1$,$\bigtriangledown_2$ and $\bigtriangledown_3$ defined as follows:

$$\bigtriangledown_1(x,y)=\min\{1,x+y\},$$
$$\bigtriangledown_2(x,y)=x+y-x\times y,$$
$$\bigtriangledown_3(x,y)=\max\{x,y\}.$$

As we can see in the literatures of dynamic epistemic logic (van Ditmarsch, 2005; Sardina and Padgham, 2011), the semantic of the epistemic formulas are interpreted in the style of modal logic. Therefore, before we take the second step to define the truth value of the epistemic formulas, we need to define the belief-epistemic logic model in advance.

Defininition 6. *A belief-epistemic model is a 4-tuple of (W,R,V,κ_j), where*

1. W is the set of the possible worlds;
2. R is the accessibility relation between the possible worlds;
3. V is an function from the propositions to $LTTS$ (i.e., we associate each proposition with a set of possible worlds, meaning the valuation of the formulas in these possible worlds); and
4. κ is a function from the possible worlds in W to natural numbers.

Given an actual state, with which we associate a factual description of the world, an agent may be uncertain about which of a set of different states is actually the case. This is the set of plausible states. Any of those plausible states may have its own associated set of plausible states, relative to that state. In dynamic epistemic logic, term 'plausible' means 'accessible'. In classical modal logic, the accessibility relation is crisp, there are exactly two cases: $R(w,w')$ and $\neg R(w,w')$. $R(w,w')$ means that starting with world w, world w' is accessible using R, while $\neg R(w,w')$ means that starting with world w, world w' is not accessible using R. Now we want to get more expressive power and so we replace the crisp relation by a "soft" accessibility relation of $R : W \times W \to LTTS$, where $LTTS$ is given by (1).

Given two plausible states, the agent may think that one is more likely to be the actual state than the other. In other words, the agent has a preference among states. The difference between plausible and implausible states is: preferences among plausible states are comparable, some plausible states are preferred over other plausible states, while the implausible states are all equally implausible. Preferences are assumed to be partially ordered and we write $<$ for an agents preference relation on the set of plausible states given an actual world. Here we take κ to represent a preference degree of the possible worlds for an agent; the less κ the more an agent prefers that possible world. Then, we can give the semantics of the epistemic formulas as follows:

Defininition 7. *Given model M and any possible world w in M,*

1. $Val(K_j\phi, M, w) = \inf\{Val(\phi, M, w') \mid w, w' \in W, R(w, w') = absolutely\text{-}true\}$;
2. $Val(B_j\phi, M, w) = sup\{Val(\phi, M, w') \mid w, w', w'' \in W, R(w, w'), R(w, w'') \neq absolutely\text{-}false$, *and for all* $w', w'' \in W, \kappa(w') \leq \kappa(w'')\}$.

In the above definition, item 1 expresses that the valuation of $K_j\phi$ in possible world w under model M is the infimum of the valuations of ϕ in worlds w', where $R(w, w')$ is *absolutely-true*. Item 2 expresses the valuation of $B_j\phi$ in possible world w in model M is the supremum of the valuations of ϕ in world w', where $R(w, w')$ and $R(w, w'')$ are not $absolutely\text{-}false$, and the preference degree of w' is not less than that of any other possible world w''.

Defininition 8. *Given belief-epistemic logic model* $M = (W, R, V, \kappa)$ *and a belief-epistemic logic model* $M' = (W', R', V', \kappa')$*, which is an arbitrary belief-epistemic model for a set of atoms P, a set of agents N, if*

$$Val([*\phi]\psi, M, w) = Val(\psi, M', w'),$$

where $(M', w') : (M, w) \parallel [*\phi] \parallel (M', w')$*, then* M' *is the new model by revising the original model M with* $[*\phi]$*, and after the revision we transfer from the possible world w to a new possible world* w'.

Formula $[*\phi]\psi$ reads as "after revision with ϕ, ψ holds". The semantics that we propose is typical for a dynamic modal operator: a state transformer $[*\phi]$ induces a binary relation between belief-epistemic states. That is, if we revise belief-epistemic state w in model $M = (W, R, V, \kappa)$ with formula $[*\phi]$, then we can get a new belief-epistemic state, denoted as w', in a new model, denoted as $M' = (W', R', V', \kappa')$. Here we get $W = W'$ and $V = V'$ because the possible worlds in these two models are the same and the facts in possible worlds do not change, but for an agent the preference for the possible worlds (the result of function κ) has been changed by revision with $[*\phi]$. In this case, the definition becomes:

$$Val([*\phi]\psi, (W, R, V, \kappa), w) = Val(\psi, (W, R', V, \kappa'), w').$$

There are different methods for belief revision. Here we proposed a revision method based on Aucher's (Aucher, 2006), which is called *successful minimal belief revision*.

Defininition 9 (Revision Method)**.** *If we revise a model M with a formula* ϕ*, then*

$$\kappa^*(w) = \begin{cases} \kappa(w) - Min\{\kappa(v) \mid Val(\phi, M, v) \in LTTS_t\} \\ \qquad\qquad if\, Val(\phi, M, w) \in LTTS_t, \\ \kappa(w) + 1 - Min\{\kappa(v) \mid Val(\neg\phi, M, v) \in LTTS_t\} \\ \qquad\qquad otherwise. \end{cases}$$

Defininition 10 (Validity)**.**

1. $\Sigma \models_\tau \phi$ *iff for any model M and world w in M, if* $\forall B \in \Sigma, Val(B) \geq \tau$*, then* $Val(\phi) \geq \tau$.
2. $\Sigma \models \phi$ *iff* $\forall \tau \in LTTS$*,* $\Sigma \models_\tau \phi$.

4 PROPERTIES

According to the validity condition we defined above, we can prove some theorems for our fuzzy belief logic system.

Theorem 1. $K_j\phi \models \phi$.

This theorem is coincide with the intuitions of human beings: if we know a proposition ϕ, then ϕ is true in our world.

Theorem 2. $K_j\phi \models B_j\phi$.

The theorem means if agent j knows proposition ϕ, then he must believe it.

Theorem 3. *If the frame is transitive then* $K_j\phi \models K_jK_j\phi$.

5 COMPARISON WITH THE AGM POSTULATES

The AGM postulates (named after the names of their proponents, Alchourrn, Gärdenfors, and Makinson) are properties that an operator that performs revision should satisfy in order for that operator to be considered rational. The belief set of an agent is denoted by a theory set of K, which is deductively closed set for formulas in the logical language, ϕ is the revision formula. The postulates are listed as follows:

1. $K * \phi$ is a theory   type
2. $\phi \in K * \phi$ success
3. $K * \phi \subseteq K + \phi$ upper bound
4. if $\neg\phi \in K$, then $K + \phi \subseteq K * \phi$ lower bound
5. $K * \phi = K\bot$ iff ϕ is inconsistent   triviality
6. if ϕ is equivalent to ψ then $K * \phi = K * \psi$ extensionality
7. $K * (\phi \wedge \psi) \subseteq (K * \phi) + \psi$ iteration upper bound
8. if$\neg\psi \in K * \phi$, then $(K * \phi) + \psi \subseteq K * (\phi \wedge \psi)$ iteration lower bound

In this section we will check whether or not the AGM postulates are fulfilled after an update in our logic system. As a preliminary step, we need to define belief sets, and decide the type of the revision formulas. In the reminder, let (M,s) be a belief state, where M is defined in Definition 6; let $\Sigma = \{\phi \mid M,s \models B\phi\}$ be a belief set, and ψ be a revision formula; and we restrict the revision formulas on propositional formulas. Then formally, we have:

Defininition 11.
$K = \{\chi \mid Val(B\chi, M, s) \neq absolutely\text{-}false\}$;
$K * \phi = \{\chi \mid Val([*\phi]B\chi, M, s) \neq absolutely\text{-}false\}$;
$K + \phi = K \cup \{\phi\}$.

Now we are going to check whether or not the 8 postulates can be verified. Formally, we have the following theorem:

Theorem 4. *In our fuzzy belief logic system, AGM postulates*

- K_1, K_5, K_6 *are verified;*
- K_3, K_4, K_7, K_8 *are undecided; and*
- K_2 *is weakly satisfied, i.e., if we revise our previous belief with a proposition of* ϕ *and* $Val(\phi) = \{\tau \mid \tau \in LTTS_t\}$ *in the revision method proposed in Definition 9, then we will believe proposition* ϕ *to an extent of* τ' *such that* $\tau' \in LTTS_t$ *and* $\tau' = Val(\phi, M, v)$, *where* $\kappa(v) \leq \kappa(v')$ *and* $Val(\phi, M, v), Val(\phi, M, v') \in LTTS_t$.

6 RELATED WORK

Generally speaking, so far few people have study the fuzzy form of dynamic belief logic although fuzzy modal logic is the foundation of the fuzzy belief logic and fuzzy dynamic logic because they are actually the extension of modal logic.

Researchers started to study fuzzy modal logic in about 1970 (Schotch, 1975; Zadeh, 1965; Zadeh, 1975). Moreover, some scholars have developed a complete system of fuzzy modal logic (Mironov, 2005). In most of the literature on fuzzy modal logic, they deal the fuzzy environment with the many-value logic, while in this paper a new method is proposed to depict the fuzziness, i.e. the linguistic variable terms ($LTTS$). On one hand, $LTTS$ can give the propositions' truth value in a continuous degree, like Lukasiewicz many-value logic. On the other hand, $LTTS$ is more closer to our daily language and it may play an more important part in application.

In the study of fuzzy belief logic (Zhang and Liu, 2012), researchers formalize reasoning about fuzzy belief and fuzzy common belief, reduce the belief degrees to truth degrees, and finally prove the completeness of the fuzzy belief and fuzzy common belief logic. However, they have just studied the fuzzy version of belief and knowledge in the state situation. While in daily life, actually if our belief or knowledge have changed, there must be a new actions or an event happened. Like Public Announcement Logic (van Benthem, 2002), we may know ϕ after someone has declared proposition ϕ or some other proposition ψ related to ϕ. We can say that the actions or events actually cause to the belief or knowledge changing. So, it is really important to consider the dynamics when we reason about the beliefs. However, the existing work did this little.

In addition, the dynamic fuzzy logic can handle the fuzzy environments (Hughes,Esterline and Kimiaghalam, 2012). However, belief revision has not be handled like what we did in this paper. On the other hand, although dynamic epistemic logic (van Benthem, 2002; Ditmarsch,Hoek and Kooi, 2007) and dynamic belief logic (van Ditmarsch, 2005; van Benthem, 2007) are studied vastly, there are few literature to expand the dynamic epistemic logic and dynamic belief logic into the fuzzy realm. However in this paper, we give a new approach to reason about dynamic belief revision in fuzzy environments.

7 CONCLUSIONS

This paper is a fuzzy extension of dynamic belief revision with quantitative method depicting belief revision. We give the syntax and semantics of the fuzzy dynamic logic and use the Linguistic Truth Term Set as the truth values for the fuzzy propositions. Then we expose some properties for the logic. Our fuzzy method to deal with dynamic belief logic is not only of great importance for theory study like epistemic logic and belief logic, but also of great application in Artificial Intelligence and Multi-agent Systems.

In the future, we will give the complete axiomatic system and prove the soundness and completeness of our logic,[2] and we will also integrate the fuzzy logic for other dynamic belief logic besides Aucher's theory.

ACKNOWLEDGEMENTS

This paper is supported by National Natural Science Foundation of China (No. 61173019), Bairen plan of Sun Yat-sen University, National Fund of Philosophy and Social Science (No. 11BZX060), Office of Philosophy and Social Science of Guangdong Province of China (No. 08YC-02), Fundamental Research Funds for the Central Universities in China, and major projects of the Ministry of Education (No. 10JZD0006). We also appreciate reviewers' valuable comments, which helped us to improve the paper significantly.

REFERENCES

Aucher, G. (2006) A Combined System for Update Logic and Belief Revision. In *Intelligent Agents and Multi-Agent Systems, Intelligent Agents and Multi-Agent Systems*.3371(132), 1-18

Baltag, A., Smets, S. (2006) Dynamic Belief Revision over Multi-Agent Plausibility Models. In *Proceedings of the LOFT'06, University of Liverpool*.11-24.

Bonissone P.P., Decker K.S. (1986) Selecting Uncertainty Calculi and Granularity: An Experiment in Trading-off Precision and Complexity. In *Uncertainty in Articficial Intelligenc* 217-247

Casalia, A., Godob, L., Sierra, C. (2011) A Graded BDI Agent Model to Represent and Reason about Preferences. In *Artificial Intelligence* 175(7-8), 1468-1478

Ditmarsch H., Hoek, W., Kooi, B. (2007) Dynamic Epistemic Logic, *Synthesis Library*, 337, Springer.

Hughes, J., Esterline, A., Kimiaghalam, B. (2012): Fuzzy Dynamic Logic, in *http://phiwumbda.org/~jesse/papers/fuzzy.ps*

Luo, X., Zhang, C., and Jennings N. R. (2002) A Hybrid Model for Sharing Information Between Fuzzy, Uncertain and Default Reasoning Models in Multi-Agent Systems. In *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 10(4), 401-450.

Mironov, A.M. (2005) Fuzzy Modal Logics. In *Journal of Mathematical Sciences* 128(6), 3461-3483.

Rott, H. (2012) Bounded Revision, Two-Dimensional Belief Change Between Conservative and Moderate Revision. In *Journal of Philosophical Logic*41(1), 173-200.

[2]The proof of the completeness is very complicated, beyond the scope of this paper. We will discuss the issue in a seperate paper.

Schotch, P.K. (1975) Fuzzy Modal Logic. In *Proceeding of the 5th Internation Symposial on Multiple-Valued Logic* 176-182.

Shapiro, S., Pagnucco, M., Lesperance, Y., Levesque, H.J. (2011) Iterated Belief Change in the Situation Calculus. In *Artificial Intelligence* 175(1), 165-192.

Sardina, S., Padgham, L. (2011) A BDI Agent Programming Language with Failure Handling, Declarative Goals, and Planning. In *Autonomous Agents and Multi-Agent Systems* 23(1), 18-70

van Ditmarsch, H. (2005) Prolegomena to Dynamic Logic for Belief Revision. In *Synthese* 147(2), 229-275

van Benthem, J. (2007) Dynamic Logic for Belief Revision. In *Journal of Applied NonClassical Logics* 17(2), 129-155

van Benthem, J. (2002) 'One is a Lonely Number': On the Logic of Communication. In *Proceedings of Logic Colloquium '02* 96-129

Wu, X., Zhang, J. (2012) Rough set models based on random fuzzy sets and belief function of fuzzy sets. In *International Journal of General Systems* 41(2), 123-141.

Zadeh, L.A. (1965) Fuzzy sets. In *Information and Control* 8(3), 338-353

Zadeh, L.A. (1975) The concept of a linguistic variable and its application to approximate reasoning. In *Information Sciences* 8(3),199-249

Zhang, J., Liu, X. (2012) Fuzzy belief measure in random fuzzy information systems and its application to knowledge reduction, In *Neural Computing & Applications*