A Fluid Limit for the Engset Model
An Application to Retrial Queues

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Abstract: We represent the classical Engset-loss model by the stochastic process counting the number of customers in the system. A fluid limit for this process is established for all the possible values of the various parameters of the system, as the number of servers tends to infinity along with the number of sources. Our results are derived through a semi-martingale decomposition method. A numerical application is provided to illustrate these results. Then, we represent a finite-source retrial queue considering in addition the number of sources in orbit. Finally, we extend the fluid limit results to a retrial queueing system, discussing different cases.

1 INTRODUCTION

In many real-life queueing systems of finite capacity, a customer may find a full system upon arrival. In several finite-source models, this request can return to the source and stay there for a randomly distributed time until it tries again to reach a server. The Engset model represents a loss queueing system having this input mechanism for several finite sources producing Poisson processes of the same intensity (see, e.g., (Engset, 1918)). We suppose that the system has no buffer, hence a request is either immediately served or immediately lost, whenever no server is available upon arrival.

Such a model has been applied to a variety of realistic computer and telecommunication systems and networks. For exemple, an Engset system is adequate to represent a radio-mobile network in which the radio sources emit messages only if no message of the same source is currently in service. One could think that the radio sources re-emit the same message as long as the latter is refused due to the fact that all channels are busy, and wait to re-issue a new message whenever the previous message is in treatment.

This model has a wide field of applications, so it has been studied extensively through analytical and algorithmic methods as well. However, when the system becomes very large, several complexity problems may appear. The fluid limit technique offers the possibility to approximate the exact values of some characteristics of the system, when one or more parameters tend to infinity. In our case, the number of servers tends to infinity along with the number of sources. Such techniques have been applied fruitfully to many queueing systems (Robert, 2000; Asmussen, 2003; Anisimov, 2007; Decreusefond and Moyal, 2012). Recently, (Feuillet and Robert, 2012) constructed exponential martingales for the Engset model, allowing to derive asymptotic estimates for several hitting times of interest. We build on these results to derive the fluid limit of an Engset model having a single server (Section 3), and then several servers (Section 4). Simulations are presented in Section 5.

In a finite source retrial queue, the messages which could not reach a server are sent to the so-called *orbit*, from which they are re-emitted on and on, at a rate that is possibly higher than the original one. It is then easily seen that the Engset model in nothing but a particular case of a retrial queueing system for which the two emission rates are equal. Based on this observation, in Section 6 we investigate some applications of our initial result to derive the fluid approximation of a retrial queue, under various conditions on the system parameters.

2 THE ENGSET MODEL

We consider an Engset system with S ($S \geq 1$) servers. There are K ($K > S$) independent Poisson sources emitting requests with intensity λ. The service times

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of the requests are exponentially distributed of parameter μ. Whenever a request finds all servers busy, it is immediately lost. If not, the request enters service and the corresponding source remain inactive during its treatment. As soon as the latter service has been completed, the source becomes active again, and re-emit jobs according to a Poisson process of intensity λ that is independent from the past. In particular, there is no dependence between the holding times and idle periods of the sources.

Let $X_S := (X_S(t); t \geq 0)$ denote the process counting the number of customers in the system (i.e., the number of busy servers) at current time. X_S is a Markov process, whose stationary measure is well-known and easily derived.

We first examine the Engset queueing model with a single server, and denote $X := X_1$ the corresponding process.

2.1 Semi-martingale Decompositions

Recall (Jacod and Shiryaev, 2003; Decreusefond and Moyal, 2012) that for a given Feller Markov process Z of state space E and infinitesimal generator $\mathcal{A}$ defined for all bounded $f : E \to \mathbb{R}$ by

$$\mathcal{A} f(i) = \lim_{h\to 0} \frac{1}{h}\Big(\mathbf{E}\left[f(X(h)) \mid X(0) = i\right] - f(i)\Big), i \in E,$$

the process

$$M_f : t \to f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{A} f(Z(s))\, ds \quad (1)$$

is a martingale w.r.t. the natural filtration of Z.

For $n \geq 1$ fixed, consider an Engset system M/M/n/n/nK, and add a superscript ^{n} to all the parameters involved. It is easily seen that the infinitesimal generator $\mathcal{A}^n$ of X^n reads for all bounded $f : \mathbb{R} \to \mathbb{R}$ and all $i \in \{0,...,n\}$,

$$\mathcal{A}^n f(i) = \\ = \begin{cases} \lambda(K-i)\left(f(i+1) - f(i)\right) + \mu i\left(f(i-1) - f(i)\right), \\ \qquad\qquad\qquad\qquad i \in \{1,...,n-1\}; \\ \mu n\left(f(n-1) - f(n)\right), i = n. \end{cases}$$

So, taking f as the identity function of $\{0,...,n\}$ in (1), we get

$$\mathcal{A}^n f(i) = \lambda(nK - i)\mathbf{1}_{\{i<n\}} - i\mu,$$

which leads to the following semi-martingale decomposition :

$$X^n(t) = X^n(0) - \mu\int_0^t X^n(s)\, ds \\ + \lambda \int_0^t (nK - X^n(s))\, \mathbf{1}_{\{X^n(s)<n\}}\, ds + M^n(t),$$

where M^n is a martingale and $X^n(0) \in [0,n]$.

2.2 The Free Process

The *free* process describes an Engset model without limitation in the number of servers, hence an infinite server queues M/M/∞/∞/nK, having the same input mechanism. As above, the process Y^n counting the number of customers in the system, satisfies the semi-martingale decomposition

$$\begin{aligned} Y^n(t) &= Y^n(0) - \mu \int_0^t Y^n(s)\, ds \\ &\quad + \lambda \int_0^t (nK - Y^n(s))\, ds + P^n(t) \\ &= Y^n(0) - (\lambda+\mu)\int_0^t Y^n(s)\, ds + \lambda nKt + P^n(t), \end{aligned} \quad (2)$$

where P^n is a martingale.

3 FLUID LIMIT

We are interested in the asymptotic behavior of X^n (properly rescaled), as the number of servers goes to infinity together with the number of sources. We obtain hereafter a fluid limit that coincides with that of a loss Erlang system M/M/n/n, and proceed as in Section 6.7 of (Robert, 2000).

We normalize the various processes as follows. For all $t \geq 0$,

$$\bar{X}^n(t) = \frac{X^n(t)}{n}; \bar{Y}^n(t) = \frac{Y^n(t)}{n}.$$

Assume that the deterministic initial condition satisfies

$$\bar{X}^n(0) \underset{n\to\infty}{\longrightarrow} x,$$

where $x \in [0,1]$ fixed.

We easily check that the semi-martingale equation (2) is similar to that of an M/M/∞ system of arrival intensity λnK and service durations of parameter $\lambda+\mu$. Whenever $\bar{Y}^n(0) \underset{n\to\infty}{\longrightarrow} x$, it then follows from Theorem 6.13 of (Robert, 2000) that for all $T \geq 0$,

$$\mathbf{E}\left[\sup_{0\leq t\leq T} |\, \bar{Y}^n(t) - Y^*(t)\,|\right] \underset{n\to\infty}{\longrightarrow} 0, \quad (3)$$

where, for all $t \geq 0$,

$$Y^*(t) = \alpha + (x - \alpha)\, e^{-(\lambda+\mu)t}, \quad (4)$$

setting

$$\alpha = \frac{\lambda K}{\lambda+\mu}.$$

Let the hitting times

$$\begin{aligned} \tau^n &:= \inf\{t \geq 0; \bar{X}^n(t) = 1\} \\ &= \inf\{t \geq 0; X^n(t) = n\} \\ &= \inf\{t \geq 0; Y^n(t) = n\} \end{aligned} \quad (5)$$

and

$$\tau = \inf\{t \geq 0; Y^*(t) = 1\}. \tag{6}$$

3.1 Heavy Traffic

First, we examine the case $\alpha > 1$. Then, we get

$$\tau = \frac{1}{\lambda+\mu}\log\left(\frac{\alpha - x}{\alpha - 1}\right). \tag{7}$$

The following lemma follows from Proposition 6 of (Feuillet and Robert, 2012).

Lemma 1. *In heavy traffic, the following convergence in probability holds for the hitting time (5) :*

$$\tau^n \xrightarrow[n\to\infty]{\mathbf{P}} \tau, \tag{8}$$

where τ is defined by (7).

Theorem 1. *For all $\varepsilon > 0$ and all $T \geq 0$, we have*

$$\mathbf{P}\left(\sup_{0\leq t\leq T} |\bar{X}^n(t) - X^*(t)| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0,$$

where, for all $t \geq 0$,

$$X^*(t) = 1 \wedge \left(\alpha + (x-\alpha)e^{-(\lambda+\mu)t}\right).$$

Proof. We have

$$\begin{aligned}\mathbf{P}&\left(\sup_{0\leq t\leq T} |\bar{X}^n(t) - X^*(t)| > \varepsilon\right)\\ &\leq \mathbf{P}\left(\sup_{0\leq t\leq T\wedge\tau^n} |\bar{Y}^n(t) - X^*(t)| > \varepsilon\right)\\ &\quad + \mathbf{P}\left(\inf_{\tau^n\leq t\leq T} \bar{X}^n(t) < 1 - \frac{\varepsilon}{2}\right)\\ &\quad + \mathbf{P}\left(\sup_{\tau^n\leq t\leq T} |1 - X^*(t)| > \frac{\varepsilon}{2}\right). \end{aligned} \tag{9}$$

By the continuity of $X^*(.)$, the first and third term on the r.h.s. of (9) vanish, respectively in view of (3) and (8). The second term vanishes as it is less than $\mathbf{P}(\tilde{\tau}^n \leq t)$, where $\tilde{\tau}^n$ is the hitting time of $\lfloor\frac{n\varepsilon}{2}\rfloor + 1$ by Z^n, the congestion process of an M/M/1 queue with arrival intensity $n\mu$ and service rate $\lambda n(K-1)$. As $n\mu < \lambda n(K-1)$, this queue is stable, so it is a classical result that $\tilde{\tau}^n$ is of the order of $\left(\frac{\lambda(K-1)}{\mu}\right)^{\lfloor\frac{n\varepsilon}{2}\rfloor+1}$. □

3.2 Light Traffic

Assume now that $\alpha < 1$.

Theorem 2. *In light traffic, for all $\varepsilon > 0$ and all $T \geq 0$, we have*

$$\mathbf{P}\left(\sup_{0\leq t\leq T} |\bar{X}^n(t) - X^*(t)| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0,$$

where in that case, for all $t \geq 0$,

$$X^*(t) = Y^*(t) = \alpha + (x-\alpha)e^{-(\lambda+\mu)t}. \tag{10}$$

Proof. Notice that

$$\begin{aligned}\mathbf{P}&\left(\sup_{0\leq t\leq T} |\bar{X}^n(t) - X^*(t)|\right)\\ &\leq \mathbf{P}(\tau^n \leq T) + \mathbf{P}\left(\sup_{0\leq t\leq T} |\bar{Y}^n(t) - Y^*(t)| > \varepsilon\right),\end{aligned}$$

and apply (3) together with Markov inequality. The first term vanishes as τ^n is asymptotically of the order of $(n\alpha^n)^{-1}$, as can be shown along the lines of (Feuillet and Robert, 2012). □

3.3 Critical Case

Suppose that $\alpha = 1$. We reason as above :

Theorem 3. *In the critical case, τ^n is of the order of $\log\sqrt{n}$, hence the same convergence as in Theorem 2 holds true.*

4 MULTISERVER ENGSET MODEL

We now check that the results of Section 3 still hold true for an Engset queue with S identical servers and the process X_S counting the busy servers. We consider the Engset model M/M/nS/nS/nK. Easily, we obtain the infinitesimal generator $\mathcal{A}_S^n$ of X_S^n

$$\mathcal{A}_S^n f(i) = \lambda(nK - i)\mathbf{1}_{\{i<nS\}} - i\mu,$$

and the corresponding semi-martingale decomposition

$$\begin{aligned}X_S^n(t) &= X_S^n(0) - \mu\int_0^t X_S^n(s)\,ds\\ &\quad + \lambda\int_0^t (nK - X_S^n(s))\,\mathbf{1}_{\{X^n(s)<nS\}}\,ds + M_S^n(t),\end{aligned}$$

where M_S^n is a martingale and $X_S^n(0) \in [0, nS]$. The free process Y_S^n is given by

$$Y_S^n(t) = Y_S^n(0) - (\lambda+\mu)\int_0^t Y_S^n(s)\,ds + \lambda nKt + P_S^n(t),$$

where P_S^n is a martingale. For all $t \geq 0$, we consider the normalized processes $\bar{X}_S^n(t)$ and $\bar{Y}_S^n(t)$ over n and the limit of the initial condition :

$$\bar{X}_S^n(0) \xrightarrow[n\to\infty]{} x_S,$$

where $x_S \in [0, S]$ fixed. Moreover, we observe that (3) holds in the case of S servers with $Y_S^*(t) = Y^*(t)$ and

$$\tau_S^n = \inf\{t \geq 0; Y_S^n(t) = nS\},$$
$$\tau_S = \inf\{t \geq 0; Y_S^*(t) = S\}.$$

Consequently, the hitting time τ_S^n converges in probability to

$$\tau_S = \frac{1}{\lambda+\mu}\log\left(\frac{\alpha - x_S}{\alpha - S}\right).$$

Theorem 4. *For all* $\varepsilon > 0$ *and all* $T \geq 0$, *we have*

$$\mathbf{P}\left(\sup_{0\leq t\leq T} |\overline{X}_S^n(t) - X_S^*(t)| > \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0,$$

where, for all $t \geq 0$, *we consider the following cases :*
if $\alpha > S$,

$$X_S^*(t) = S \wedge \left(\alpha + (x_S - \alpha)\, e^{-(\lambda+\mu)t}\right); \qquad (11)$$

if $\alpha \leq S$,

$$X_S^*(t) = \alpha + (x_S - \alpha)\, e^{-(\lambda+\mu)t}. \qquad (12)$$

As a consequence, the fluid limit for the Engset system can be derived for any arbitrary, but fixed, values of the number of sources and servers.

5 NUMERICAL RESULTS

In this section, we present a numerical example for the multiserver Engset model concerning the different cases discussed in Section 4. Consider a M/M/S/S/K queueing system with parameters $S = 500$, $\lambda = 0.1$ and $\mu = 0.5$. The critical value for the number of sources is, therefore, $K = 3000$. Setting various values for the number of sources, we may obtain the heavy traffic ($\alpha > 500$), the light traffic ($\alpha < 500$) or the critical case ($\alpha = 500$). We set the initial value $x_S = 100$ for the busy servers.

In the following figures, we present a realization of the process X_S^1 of the model described above in the time interval $[0, 50]$, along with the fluid limit, for the three cases as given in (11) and (12).

Notice that, in Figure 1, the hitting time of S servers for the realization (4.6381) is fairly close to the theoretical value of τ_S (5.3648).

6 APPLICATION TO SINGLE SERVER RETRIAL QUEUES

Retrial queues follow the following scenario : when a customer arrives with all servers and waiting positions (if any) being busy, he leaves the service area but after some randomly distributed time repeats his demand. For a review of the main results on the topic see (Falin and Templeton, 1997) and the references therein.

The general queueing system with retrials is described more precisely as follows. There are finitely many identical independent fully available servers at which requests arrive. Each source can generate a request with rate λ. If an arriving request finds at least one server free, it immediately occupies the server

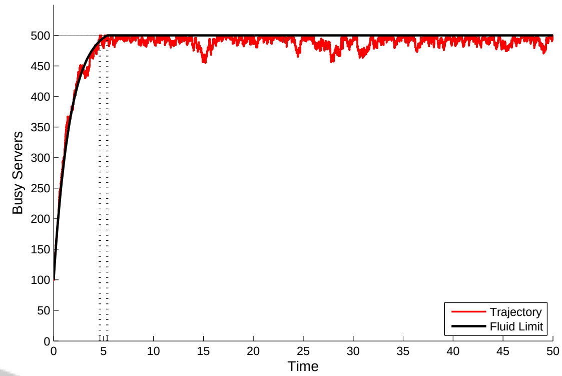

Figure 1: Heavy traffic ($K > 3000$).

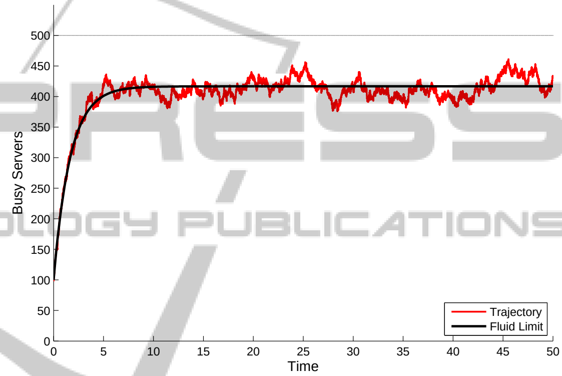

Figure 2: Light traffic ($K < 3000$).

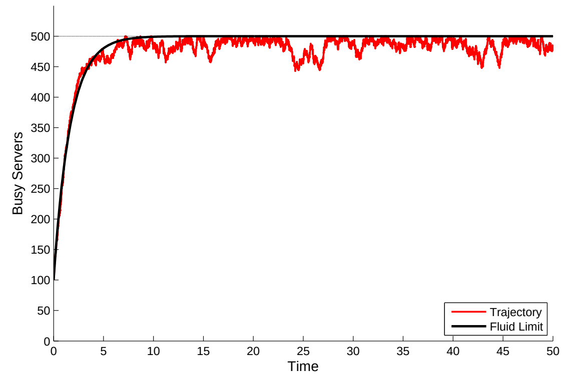

Figure 3: Critical case ($K = 3000$).

and leaves it after completion of service. The rate of service time is denoted by μ. If all servers are busy at arrival time, then the source goes into the orbit (a secondary queue of infinite size) and starts the generation of requests with rate ν until it finds a free server. After completion of service, the source returns to the initial state and it can generate a new request, while the server may serve a new request. All the times involved in the model are assumed to be mutually independent of each other.

The presence of the orbit makes the retrial queueing model more flexible than the Engset one, once the source may generate a message with a different rate from the rate of the initial state. Actually, an Engset queue is a special case of the certain finite-source retrial system under the condition $\nu = \lambda$. Because of this fact, we can apply the results of the previous section to a retrial queue, and, therefore, investigate different cases under diverse system parameters.

Consider a single-server retrial queue with finitely many sources. Let $E = \{0,1,\dots,K-1\}$ and $F = \{0,1\}$. Let $N := (N(t); t \geq 0)$ be the number of sources of in orbit and $C := (C(t); t \geq 0)$ the number of busy servers, with state spaces E and F respectively. The system state at time t can be described by the coupled process $A(t) := (N(t), C(t))$. The process $A := (A(t); t \geq 0)$ is a continuous-time Markov process with finite state space $E \times F$. Since the state space of the process A is finite, the process is ergodic with stationary measure $\tilde{\pi}(\cdot,\cdot)$ defined as follows

$$\tilde{\pi}(i,j) = \lim_{t\to\infty} \mathbf{P}\left(N(t) = i, C(t) = j\right), i \in E,\ j \in F.$$

We consider a sequence of retrial systems with the n-th one having n servers and nK sources. Let $E^n = \{0,1,\dots,nK-1\}$ and $F^n = \{0,1,\dots,n\}$. Let $A^n = (N^n, C^n)$ be the corresponding process for the n-th system, with state space $E^n \times F^n$.

For all $i \in E^n$, $j \in F^n$, we consider the functions $f(i,j) := i$ and $g(i,j) := j$. Then, the infinitesimal operator $\mathcal{Q}^n$ of A^n applied to f and g reads

$$\mathcal{Q}^n f(i,j) = -i\nu \mathbf{1}_{\{j<n\}} + (nK - i - n)\lambda \mathbf{1}_{\{j=n\}}$$

and

$$\begin{aligned}\mathcal{Q}^n g(i,j) = {} & [i\nu + (nK - i - j)\lambda - j\mu]\,\mathbf{1}_{\{j<n\}} \\ & - n\mu \mathbf{1}_{\{j=n\}}.\end{aligned}$$

which yields the following semi-martingale decompositions :

$$\begin{aligned} N^n(t) = {} & N^n(0) - \nu \int_0^t N^n(s)\,ds \\ & + (\nu - \lambda)\int_0^t N^n(s)\mathbf{1}_{\{C^n(s)=n\}}\,ds \\ & + \lambda n(K-1)\int_0^t \mathbf{1}_{\{C^n(s)=n\}}\,ds + M_1^n(t); \\ \langle M_1^n \rangle_t = {} & \nu \int_0^t N^n(s)\mathbf{1}_{\{C^n(s)<n\}}\,ds \\ & + \lambda \int_0^t \left(nK - N^n(s) - n\right)\mathbf{1}_{\{C^n(s)=n\}}\,ds \end{aligned}$$

and

$$\begin{aligned} C^n(t) = {} & C^n(0) + (\nu - \lambda)\int_0^t N^n(s)\mathbf{1}_{\{C^n(s)<n\}}\,ds \\ & + \lambda \int_0^t (nK - C^n(s))\mathbf{1}_{\{C^n(s)<n\}}\,ds \\ & - \mu \int_0^t C^n(s)\,ds + M_2^n(t); \\ \langle M_2^n \rangle_t = {} & \int_0^t \left(nK - N^n(s) - C^n(s)\right)\mathbf{1}_{\{C^n(s)<n\}}\,ds \\ & + \int_0^t \nu N^n(s)\mathbf{1}_{\{C^n(s)<n\}}\,ds \\ & + n\mu \int_0^t \mathbf{1}_{\{C^n(s)=n\}}\,ds. \end{aligned}$$

6.1 Law of Large Numbers

We apply the same normalization as in Section 3:

$$\bar{N}^n(t) = \frac{N^n(t)}{n};\ \bar{M}_1{}^n(t) = \frac{M_1^n(t)}{n};$$
$$\bar{C}^n(t) = \frac{C^n(t)}{n};\ \bar{M}_1{}^n(t) = \frac{M_1^n(t)}{n}.$$

Assume that

$$\bar{N}^n(0) \underset{n\to\infty}{\longrightarrow} n_0;\ \bar{C}^n(0) \underset{n\to\infty}{\longrightarrow} c_0,$$

where $n_0 \in [0,K]$ and $c_0 \in [0,1]$. Thus, for all $t \geq 0$, we obtain

$$\begin{aligned} N^*(t) = {} & n_0 + \lambda(K-1)\int_0^t \mathbf{1}_{\{C^*(s)=1\}}\,ds \\ & + (\nu - \lambda)\int_0^t N^*(s)\mathbf{1}_{\{C^*(s)=1\}}\,ds \\ & - \nu \int_0^t N^*(s)\,ds; \end{aligned} \tag{13}$$

$$\begin{aligned} C^*(t) = {} & c_0 + (\nu - \lambda)\int_0^t N^*(s)\mathbf{1}_{\{C^*(s)<1\}}\,ds \\ & + \lambda \int_0^t (K - C^*(s))\mathbf{1}_{\{C^*(s)<1\}}\,ds \\ & - \mu \int_0^t C^*(s)\,ds. \end{aligned} \tag{14}$$

Theorem 5. *The following weak convergence holds :*

$$(\bar{N}^n, \bar{C}^n) \Rightarrow (N^*, C^*),$$

where the deterministic functions N^* and C^* are the unique solutions of (13) and (14), respectively.

Proof. We follow the classical steps for proving weak convergence of processes. The increasing processes $\langle \bar{M}_1^n \rangle$ and $\langle \bar{M}_2^n \rangle$ vanish uniformly on any compact interval as n goes large. Moreover, the Aldous-Reboledo tightness criterion for semi-martingales (see, e.g. (Joffe and Mtivier, 1986)) is easily met by both $\bar{N}^n$ and $\bar{C}^n$. Thus, the sequence $\left(\bar{N}^n(\cdot), \bar{C}^n(\cdot)\right)$ is tight, and any subsequential limit $(N^*(\cdot), C^*(\cdot))$ reads for all $t \geq 0$, the equations (13) and (14). The weak limit is then a solution of this system. The uniqueness of the latter is easily checked by showing that the underlying mapping is locally-Lipschitz continuous. □

6.2 Discussion

We discuss applications of the latter result in several cases.

(i) If $\lambda = \nu$, from (4) and (14), C^* coincides with the fluid limit Y^* of Section 3, in the various cases. Setting, whenever τ is finite (i.e. in the heavy traffic case),

$$\tau_0 = n_0 e^{-\lambda\tau},$$

from the equation (13), we obtain that

$$N^*(t) = n_0 e^{-\lambda t}\mathbf{1}_{\{t<\tau\}} + (K - 1 + [\tau_0 - (K-1)]e^{-\lambda(t-\tau)})\mathbf{1}_{\{t\geq\tau\}}.$$

We check as well the intuitive result that asymptotically, the orbit is either full (heavy traffic case) or empty (light traffic/ critical case).

(ii) Suppose now that $K\lambda \leq \lambda + \mu$, and fix the initial condition $n_0 = 0$. Let

$$\rho^n = \inf\{t \geq 0; N^n(t) > 0\}.$$

Up to ρ^n, no arrival occurs from the orbit, so the value of ν is irrelevant. It is easily seen that

$$\rho^n = \tau^n \text{ in distribution,}$$

where τ^n corresponds to the previous hitting time for an Engset model. So, from the results of the previous section in the light traffic and critical cases, a proof similar to that of Theorem 2 shows that, for all $t \geq 0$,

$$N^*(t) = 0;\ C^*(t) = Y^*(t),$$

where $Y^*(t)$ is given in (10).

(iii) All the same, if finally $\lambda < \nu$ and $K\lambda > \nu + \mu$, we show by stochastic comparison of Markov processes that

$$\rho^n \leq_{st} \tau^n,$$

where τ^n corresponds to a heavily loaded Engset model of arrival rate λ. So, Theorem 1 entails that for some $0 < \rho \leq \tau$, where τ is defined by (6),

$$C^*(t) = 1, \text{ for all } t \geq \rho,$$

i.e. the server is busy all the time after ρ, at the fluid level.

7 CONCLUSIONS

We have derived the fluid limit of an Engset queueing system with several servers. After discussing the connection between the Engset queue and retrial queueing models, we present several fluid limit results for retrial queues. The generalization of the fluid limit for all possibles values of the parameters λ, μ and ν of the retrial model, and the numerical confirmation of their accuracy is a challenging problem that is currently under investigation.

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