Constrained Predictive Control of MIMO System
Application to a Two Link Manipulator

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Abstract: In the paper application of constrained predictive control to multi input, multi output system is presented. The method is based on feedback linearization and LQ control. Constraints of the system are implemented by interpolation of reference trajectory. Finding solution is a compromise between the unconstrained LQ control and a constrained feasible control and is executed by minimization of one variable. The application of the method to a two link manipulator is used to present advantages and limitations of the algorithm.

1 INTRODUCTION

Feedback linearization is a powerful technique that allows to obtain linear model with exact dynamics (Isidori, 1985), (Slotine and Li, 1991). Linear quadratic control is well known optimal control method and with its dynamic programming properties can be also easily calculated (Anderson & Moore, 1990). The combination of feedback linearization and LQ control has been used in many algorithms in Model Predictive Control applications for many years and it is used also in present papers (He De-Feng et al., 2011), (Margellos and Lygeros, 2010). Another problem apart from finding the optimal solution on a given horizon (finite or infinite) is the constrained control. A method which use the advantages of feedback linearization, LQ control and applying signals constraints was proposed in (Poulsen et al., 2001). It rely in every step on interpolation between the LQ optimal control and a feasible solution – the solution that fulfils given constraints. A feasible solution is obtain by taking calculated from LQ method optimal gain for a perturbed reference signal. The compromise between the feasible and optimal solution is calculating by minimization of one variable – the number of degrees of freedom in prediction is reduced to one variable.

2 THE TWO LINK MANIPULATOR SYSTEM

The considered system is the two link manipulator (fig.1). It consists of two rigid links and two one degree-of-freedom wrists, whose motion is in the vertical axis. The objective of control is to move the clutch of the manipulator from one position in two dimensional space to the other. The output variables are the two angles $y_1=x_1$ and $y_2=x_2$. The coordinates of the clutch can be obtained from kinematics equations (1)

$$
\begin{aligned}
a &= d_1 cos x_1 + d_2 \cos(x_1 + x_2) \\
b &= d_1 sin x_1 + d_2 \sin(x_1 + x_2)
\end{aligned}
\tag{1}
$$

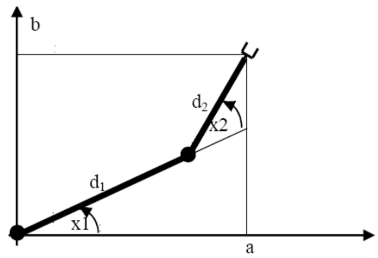

Figure 1: The two link manipulator system.

The dynamics of the system is represented by below equations (2)

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -N_{11}(V_1 + T_1) - N_{12}(V_2 + T_2) + N_{11}u_1 + N_{12}u_2 \\
\dot{x}_4 &= -N_{12}(V_1 + T_1) - N_{22}(V_2 + T_2) + N_{12}u_1 + N_{22}u_2
\end{aligned}
\tag{2}
$$

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where x_1 and x_2 are the angles, variables x_3 and x_4 are the respective angular velocities, d_1, d_2 are the lengths of the links. The input variables u_1 and u_2 are the moments of force in the wrists. Furthermore

$$\begin{aligned} V_1 &= -m_2 d_1 (x_{c2} sinx_2 + y_{c2} cosx_2)(2x_3 x_4 + x_4^2) \\ V_2 &= m_2 d_1 (x_{c2} sinx_2 + y_{c2} cosx_2) x_3^2 \end{aligned} \tag{3}$$

represents Coriolis and centrifugal forces and

$$\begin{aligned} T_1 &= s_1 \tanh(k_t x_3) + f_1 x_3 \\ T_2 &= s_2 \tanh(k_t x_4) + f_2 x_4 \end{aligned} \tag{4}$$

are the friction forces approximated by smooth functions. The approximation is used to fulfil conditions of feedback linearization method.
Masses and inertial forces are represented by **M** matrix, where

$$\begin{gathered} \begin{bmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}^{-1} \\ M_{11} = I_1 + I_2 + m_2 d_1^2 + 2 m_2 d_1 (x_{c2} cosx_2 - y_{c2} sinx_2) \\ M_{12} = m_2 d_1 (x_{c2} cosx_2 - y_{c2} sinx_2) \\ M_{22} = I_2 \end{gathered} \tag{5}$$

The values of coefficients which appeared in equations (3-5) are listed in tab.1.

Table 1: Coefficients of manipulator system.

	link 1	link 2
I_i[kg*m^2]	1	1
m_i[kg]	10	5
d_i[m]	1	0.7
x_{ci}[m]	0.5	0.35
y_{ci}[m]	0	0
s_i[Nm]	0.1	0.1
f_i[kg*m^2/s]	0.01	0.01

In considered system input variables u_1 and u_2 are constrained by -1 and 1 Nm

$$\begin{gathered} \min(u_i) = -1[\mathrm{Nm}] \\ \max(u_i) = 1[\mathrm{Nm}] \end{gathered} \tag{6}$$

2 CONTROL ALGORITHM

2.1 Feedback Linearization

Nonlinear equations of manipulator system are smooth and the system has full relative degree. The system has the same number of inputs and outputs. Feedback linearization of the system can be accomplished with diffeomorphism

$$\mathbf{z} = \varphi(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} \tag{7}$$

and the new input variables

$$\begin{aligned} v_1 &= N_{11}(u_1 - V_1 - T_1) + N_{12}(u_2 - V_2 - T_2) \\ v_2 &= N_{12}(u_1 - V_1 - T_1) + N_{22}(u_2 - V_2 - T_2) \end{aligned} \tag{8}$$

The inputs of the nonlinear system (2) are nonlinear functions of v_1, v_2 and the state **x** obtained from (8)

$$\begin{aligned} u_1 &= \psi_1(v_1, v_2, \mathbf{x}) \\ u_2 &= \psi_2(v_1, v_2, \mathbf{x}) \end{aligned} \tag{9}$$

Consequently we obtain two identical linear systems

$$\begin{matrix} \dot{z}_1 = z_2 & & \dot{z}_3 = z_4 \\ \dot{z}_2 = v_1 & \text{and} & \dot{z}_4 = v_2 \\ y_1 = z_1 & & y_2 = z_3 \end{matrix} \tag{10}$$

for which the theory of linear control can be applied.

2.2 Linear Quadratic Control

Each of the two linear systems is discretized with sampling interval T_s. In order to track the change of set point the state is augmented by new variable with included reference signal w_t and each system is described in form

$$\begin{gathered} \mathbf{z}_{t+1} = \begin{bmatrix} \mathbf{A}_d & 0 \\ -\mathbf{C}_d & 1 \end{bmatrix} \mathbf{z}_t + \begin{bmatrix} \mathbf{B}_d \\ 0 \end{bmatrix} v_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t \\ y_t = \begin{bmatrix} \mathbf{C}_d & 0 \end{bmatrix} \mathbf{z}_t \end{gathered} \tag{11}$$

The linear quadratic cost function can be written as

$$J_t = \sum_{k=t}^{\infty} \mathbf{z}_k^T \mathbf{Q} \mathbf{z}_k + R v_k^2, \tag{12}$$

and the solution

$$v_t = L_y w_t - \mathbf{L} \mathbf{z}_t, \tag{13}$$

where **L** is the optimal gain obtained from Riccati equation and $L_y = \mathbf{L}\begin{bmatrix} \mathbf{C}_d & 0 \end{bmatrix}^T$.

2.3 Constrained Predictive Control

The system equation (11) can be used as model prediction equation to calculate the state for the samples t+1=(k,…,k+H-1), on horizon H in the time instant k. Constraints will be included into control law by interpolation method in every predicted step. It rely on using perturbed reference trajectory

$$\widetilde{w}_{t|k} = w_{t|k} + \alpha_k p_{t|k} \tag{14}$$

In place of reference trajectory in equations (11 and 13). Then the prediction equation:

$$\hat{\mathbf{z}}_{t+1|k} = \begin{bmatrix} \mathbf{A}_d & 0 \\ -\mathbf{C}_d & 1 \end{bmatrix} \hat{\mathbf{z}}_{t|k} + \begin{bmatrix} \mathbf{B}_d \\ 0 \end{bmatrix} \hat{v}_{t|k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_{|kt} + \alpha_k \begin{bmatrix} 0 \\ 1 \end{bmatrix} p_{t|k} \tag{15}$$

And the control predicted values:

$$\hat{v}_{t|k} = \mathbf{L}_y w_{t|k} + L_y p_{t|k} - \mathbf{L}\hat{\mathbf{z}}_{t|k}, \tag{16}$$

Variable α_k is calculated in instant k and is the same for every predicted states and inputs on H, $p_{t|k}$ for $t=k,\ldots,k+H-1$ forms a vector $\mathbf{p}_{H|k}$. α_k can adopt values from 0 to 1 and $\mathbf{p}_{H|k}$ is chosen in that manner so the perturbed reference trajectory with $\alpha_t=1$ provides feasible, satisfying constraints solution for the considered system. Whereas $\alpha_t=0$ corresponds with unconstrained control. The aim is to minimize variable α_t on the horizon with respect to system equations (15,16) and constraints (6). Since the two systems (11) are considered, constrained values are the functions of two variables α_k^{I} and α_k^{II} through nonlinear equations (9) and (7) (linear in this example, but nonlinear in general).

2.4 Feasible Trajectory

The perturbation vector $\mathbf{p}_{H|k}$ providing feasible solution can be obtain from previous *k-1* step by

$$\mathbf{p}_{H|k} = \alpha_{k-1}\mathbf{p}_{H|k-1.} \tag{17}$$

With calculated α_t for the n=3 dimensional system (11) we can express the prediction equation from (15) with used (16) in form:

$$\hat{\mathbf{z}}_{t+1|k} = \mathbf{\Phi}\hat{\mathbf{z}}_t + \mathbf{\Gamma}(w_{t|k} + \alpha_k p_{t|k}), \tag{18}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{A}_d - \mathbf{B}_d\begin{bmatrix} L_1 & L_2 \end{bmatrix} & -\mathbf{B}_d L_3 \\ -\mathbf{C}_d & 1 \end{bmatrix},$$

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{B}_d L_y \\ 1 \end{bmatrix}.$$

The initial perturbation $\mathbf{p}_{H|i}$ for i at the beginning of control application is calculated by using zero as the reference signal and the initial state corresponding to the step of original reference signal. The method presented in (Poulsen et al., 2001) of obtaining initial feasible perturbation provided too large absolute values of control v at the beginning of the predicted vector. The alternative method is used in the paper with minimization of v_t as a function of p_t in (20).

For the state equation

$$\mathbf{z}_{t+1} = \mathbf{\Phi} z_t + \mathbf{\Gamma} p_t \tag{19}$$

and initial $\mathbf{z}_0 = -(\mathbf{z}_f - \mathbf{z}_k)$,where $\mathbf{z}_f$ – final stable state z for w_{k+1}, $\mathbf{z}_k$ – initial state for control system, additional cost function is used

$$J_t = \sum_{k=t}^{\infty} \mathbf{z}_k{}^T \mathbf{Q}_p \mathbf{z}_k + R_p v_k{}^2 \tag{20}$$

where

$$v_t = -\mathbf{L}\mathbf{z}_t + L_y p_t \tag{21}$$

then the cost function has form

$$J_t = \sum_{k=t}^{\infty} z_k{}^T Q_j z_k + R_p p_k{}^2 + 2 z_k{}^T N_j p_k \tag{22}$$

with

$$Q_j = Q_p + L^T R L,\quad R_j = L_y R L_y,\quad N_j = -L^T R L_1 \tag{23}$$

The optimal gain $\mathbf{K}$ obtained by minimization (22) is used to calculate initial perturbations $p_{t|i}$, t=1,...H-1

$$p_{t|i} = -\mathbf{K}\mathbf{z}_{t|i}. \tag{24}$$

Now we can describe the predicted variable $\mathbf{z}_{k+1}$ and predicted control law v_{k+l} as a functions of initial and final state, reference trajectory and one variable α_k. In equations $k=i+1$ is used:

$$\hat{\mathbf{z}}_{k+l|k} = \mathbf{\Phi}^l \mathbf{z}_k + \mathbf{\Lambda}\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+l-1} \end{bmatrix} + \alpha_k \mathbf{\Lambda}\begin{bmatrix} \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^0 \\ \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^1 \\ \vdots \\ \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^{l-1} \end{bmatrix}(\mathbf{z}_f - \mathbf{z}_k) \tag{25}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Phi}^{l-1}\mathbf{\Gamma} & \cdots & \mathbf{\Phi}\mathbf{\Gamma} & \mathbf{\Gamma} \end{bmatrix}$$

$$\begin{aligned} \hat{v}_{k+l|k} = &-\mathbf{L}\mathbf{\Phi}^l \mathbf{z}_k + -\mathbf{L}\begin{bmatrix} \mathbf{\Lambda} & \begin{bmatrix} -\mathbf{C}_d \\ 0 \end{bmatrix} \end{bmatrix}\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+l} \end{bmatrix} + \\ &-\alpha_k \mathbf{L}\begin{bmatrix} \mathbf{\Lambda} & \begin{bmatrix} -\mathbf{C}_d \\ 0 \end{bmatrix} \end{bmatrix}\begin{bmatrix} \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^0 \\ \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^1 \\ \vdots \\ \mathbf{K}(\mathbf{\Phi}-\mathbf{\Gamma})^l \end{bmatrix}(\mathbf{z}_f - \mathbf{z}_k) \end{aligned} \tag{26}$$

The linearized system of manipulator example (2) consists of two linear equations (10) therefore in the algorithm two prediction equations (25) and law equations (26) are used. To avoid problems with multivariable minimization it is assumed that α_k is equal to both subsystems, $\alpha_k^{I}=\alpha_k^{II}$.

3 PERFORMANCE OF THE ALGORITHM

3.1 Simulations

Simulations was performed for the change of output y_1 from 1.0489 to -0.0716 [rad] and y_2 from 0.9626 to 1.9284 [rad]. This is equivalent to the change of coordinates (a,b) from (0.2,1.5) to (0.8,0.6). The weight matrices in cost function (12) for both subsystems was chosen as **Q**=[0 0 0; 0 0 0; 0 0 1], R=0.001 hence the emphasize in the minimization of the difference between the set point and the output. Remaining variables of the vector **z** are weighted with 0, since constraints are coped while minimization of α. R has to be positive define hence the small value was chosen. The resulted trajectories satisfied constraints, variables are changing fast and without significant overshoot. Oscillations on inputs charts are the effect of small R.

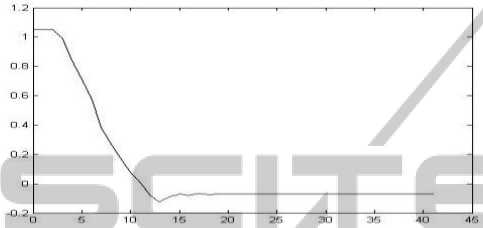

Figure 2: Output variable y_1.

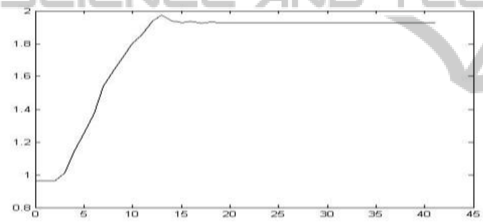

Figure 3: Output variable y_2.

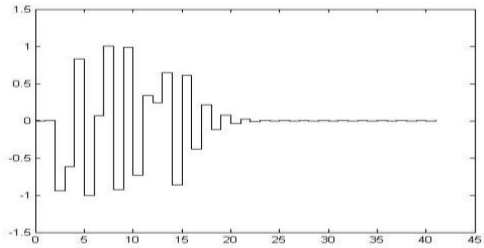

Figure 4: Input variable u_1.

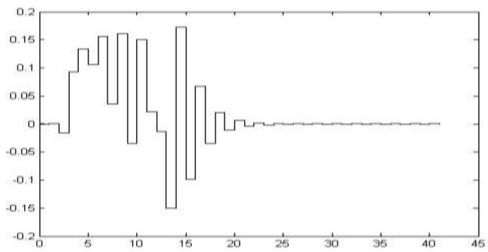

Figure 5: Input variable u_2.

3.2 Variables as Functions of α

In simulations only constraints of the two inputs values was considered. In this section it can be seen that for the remaining variables (the variables of state **x**) can be considered constraints. The idea of the algorithm is that by decreasing α the absolute variables of inputs and consequently variables of state are higher, the possibility of violating constraints is greater. On figures (6-11) this dependence of state variables and input on α is presented.

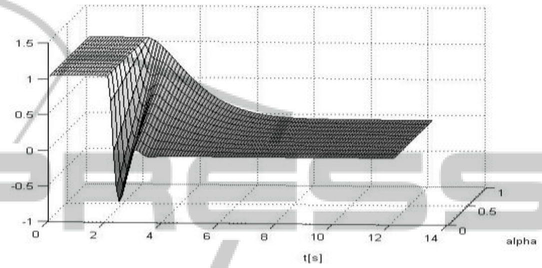

Figure 6: Output variable y_1 in dependence on α.

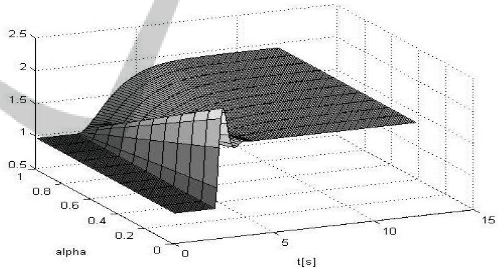

Figure 7: Output variable y_2 in dependence on α.

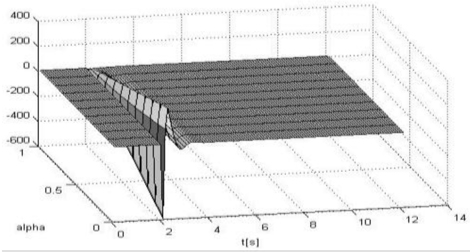

Figure 8: Input variable u_1 in dependence on α.

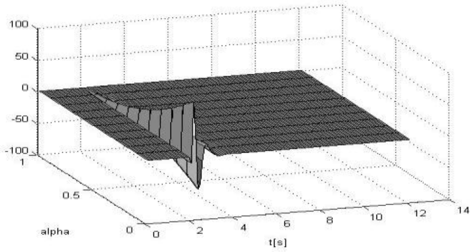

Figure 9: Output variable u_2 in dependence on α.

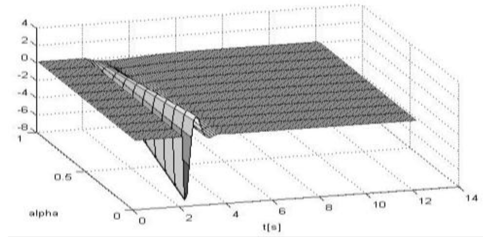

Figure 10: State variable x_3 in dependence on α.

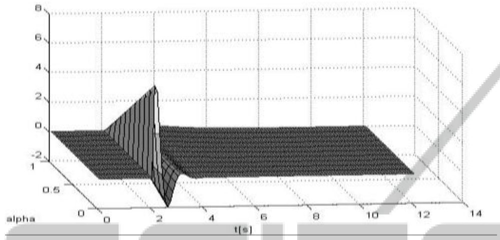

Figure 11: State variable x_4 in dependence on α.

3.3 System with Coordinates as Outputs

In order to present that the algorithm is not valid for every feedback linearizable system of smooth function the implementation of manipulator system with different outputs was prepared. In this section the outputs represents coordinates of the system that is $y_1=a$, $y_2=b$. Then the system equations

$$\begin{gathered}\dot{x}_1 = x_3\\ \dot{x}_2 = x_4\\ \dot{x}_3 = -N_{11}(V_1+T_1) - N_{12}(V_2+T_2) + N_{11}u_1 + N_{12}u_2\\ \dot{x}_4 = -N_{12}(V_1+T_1) - N_{22}(V_2+T_2) + N_{12}u_1 + N_{22}u_2\\ \dot{a} = -d_1 x_3 sinx_1 - d_2(x_3+x_4)\sin(x_1+x_2)\\ \dot{b} = d_1 x_3 cosx_1 + d_2(x_3+x_4)\cos(x_1+x_2)\end{gathered} \tag{27}$$

The feedback linearization will be accomplished by diffeomorphism

$$\begin{gathered}z_1 = a\\ z_2 = -d_1 x_3 sinx_1 - d_2(x_3+x_4)\sin(x_1+x_2)\\ z_3 = b\\ z_4 = d_1 x_3 cosx_1 + d_2(x_3+x_4)\cos(x_1+x_2)\end{gathered} \tag{28}$$

And new inputs $v_1 = \dot{z}_2$, $v_2 = \dot{z}_4$.

Two linear systems are obtained as in (10). The system (21) has relative degree r=4, therefore there are 4 variables the linear system and two additional variables $[z_5\ z_6]^T=T_i(x)$ has to be chosen. They have to satisfy (Isidori, 1985), (Slotine & Li, 1991) equation

$$\frac{\partial}{\partial x}[T_i(x)]g(x) = 0 \tag{29}$$

One of the possible choice is

$$\begin{gathered}z_5 = x_1\\ z_6 = x_1 + x_2\end{gathered} \tag{30}$$

The system (21) with performed linearization (22-24) and used presented algorithm is not working properly. The reason for this is that the dependence of some variables on α is not monotonic as can be seen on figures (12-17).

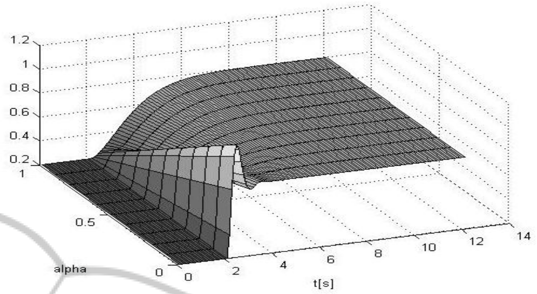

Figure 12: Output variable y_1 in dependence on α.

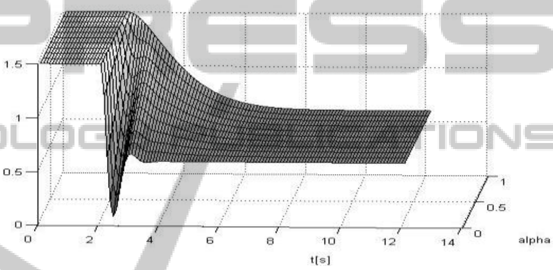

Figure 13: Output variable y_2 in dependence on α.

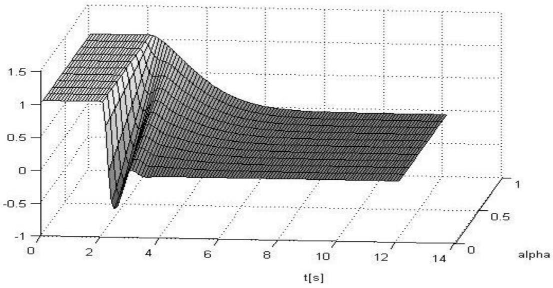

Figure 14: The variable x_1 in dependence on α.

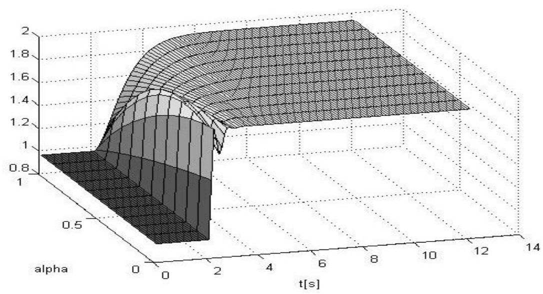

Figure 15: The variable x_2 in dependence on α.

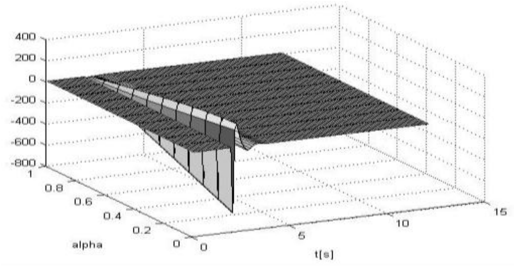

Figure 16: Input variable u_1 in dependence on α.

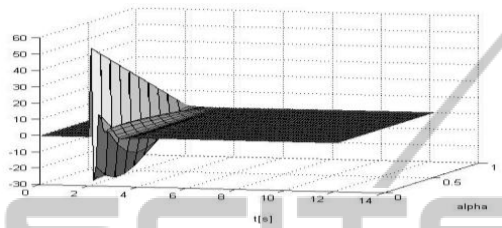

Figure 17: Input variable u_1 in dependence on α.

In this case variables x_2 and u_1 changes in the undesirable manner as α increases. The main reasons for this result is the chosen variable $z_6=x_1+x_2$ and that nonlinear functions described inputs (9) are fractions with nonlinear denominator dependent on α.

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