# Quantitative Estimates of Stability in Controlled GI | D | 1 | ∞ Queueing Systems and in Control of Water Release in a Simple Dam Model

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Abstract: We consider two applied discrete-time Markov control models: waiting times process in GI | D | 1 | ∞ queues with a controlled service rate and water release control in a simple dam model with independent water inflows. The stochastic dynamics of both models is determinated by a sequence of independent and identically distributed random variables with a distribution function $F$. In the situation when an available approximation $\tilde{F}$ is used in place of the unknown $F$, we estimate the deterioration of performance of control policies optimal with respect to the total discounted cost and the average cost per unit of time. For this purpose we introduce a stability index and find uppers bounds for this index expressed in terms of the Prokhorov distance between the distributions functions $F$ and $\tilde{F}$. When $\tilde{F} \equiv \tilde{F}_m$ is the empirical distribution function obtained from a sample of size $m$ in average the stability index is less than a constant times $m^{-1/3}$.



## 1 INTRODUCTION: THE PROBLEM OF CONTINUITY (STABILITY) ESTIMATION IN CONTROL MODELS

Optimization of control policies in queueing systems, inventory and dam models is nowadays important theoretical and engineering issue because of great development of telecommunications and computer networks and increasing difficulties with water supply. In this communication we investigate continuity (or "stability") of optimal dynamic control in the following applied discrete-time Markov control process (MCP's):

*A*. Consecutive waiting times in GI | D | **1** | ∞ queues with *controlled service rates* $a_n \in [\gamma, \bar{\gamma}] \subset (0,\infty)$:

$$X_{n+1} = \max\{0, X_n + a_n - \xi_n\}, \quad n = 0,1,2,\ldots. \quad (1)$$

where $X_n$ is the waiting time of $n$-th job (client), while $\xi_n$ is interarrival time between $n$-th and $(n+1)$-th jobs. (Equations (1) also describe stock level in some inventory models, see e.g. (Anderson et al., 2012), and (Kitaev and Rykov, 1995) for control problem settings in queues).

*B*. Water stocks $\{X_n,\ n = 0,1,2,\ldots\}$ in the following simple model of *water release control* (see, e.g. (Asmussen, 1987), (Bae et al., 2003), (Hernández-Lerma, 1989)):

$$X_{n+1} = \min\{X_n - a_n + \xi_n, M\},\ n = 0,1,2,\ldots \quad (2)$$

(with $n$ used for days, weeks or months). In (2) $M$ is the capacity of reservoir, $\xi_n$ is the water inflow, and $a_n \in [0, X_n]$ is the controlled water consumption in the $n$-th period.

**Remark 1.** *For the sake of brevity (in what follows) we allow the same letters to denote states of distinct processes.*

In equations (1) and (2), $\xi_0, \xi_1, \ldots$ are assumed to be independent and identically distributed (i.i.d) non-negative random variables with a common distribution function (d.f.) $F$.

Let $Q(\pi)$ be a chosen *criterion of optimization* of control policies $\pi$ (or an expected cost of a policy $\pi$, as specified below in sections 2 and 3).

A control policy $\pi = \{a_0, a_1, \ldots\}$ is a sequence of control actions (see e.g. (Dynkin and Yushkevich, 1979), (Hernández-Lerma and Lasserre, 1996) for definitions). The "original task" of a controller is to find (or to approximate) an optimal policy $\pi_*$ (if it exists)

$$Q(\pi_*) = \inf_{\pi} Q(\pi). \quad (3)$$

However, in our stability (continuity) estimation setting this goal can not be accomplished (at least directly) since we suppose that the d.f. $F$ is *unknown* (at least partly) to the controller, and it is replaced by an *available approximation* $\tilde{F}$ obtained either from some theoretical considerations or from statical estimators of unknown parameters, (or the whole $F$). One possible way to go around is to consider instead of (1),

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(2) the corresponding "approximating" Markov control processes:

$$\tilde{X}_{n+1} = \max\{0, \tilde{X}_n + a_n - \tilde{\xi}_n\}, \quad n = 1,2,.. \quad (4)$$

$$\tilde{X}_{n+1} = \min\{\tilde{X}_n - a_n + \tilde{\xi}_n, M\}, \quad n = 1,2,.. \quad (5)$$

where $\tilde{\xi}_0, \tilde{\xi}_1, \ldots$ are i.i.d random variables with the d.f. $\tilde{F}$.

Suppose that using the same criterion $Q$ one can find (at least "theoretically") an optimal control policy $\tilde{\pi}_*$ for process (4) (or for process (5)). Nevertheless we insist that the "*original*" process is given by equations (1) (or by (2)) and therefore the controller aims to find $\tilde{\pi}_*$ only in order to apply it to control process (1) (or (2), respectively). This means that $\tilde{\pi}_*$ is used as an available approximation to an unavailable policy $\pi_*$

The below stability (continuity) index $\Delta$ (see e.g. (Gordienko and Salem, 2000), (Gordienko et al., 2009)) gives a quantitative measure of the accuracy of such approximation:

$$\Delta := Q(\tilde{\pi}_*) - Q(\pi_*) \geq 0. \quad (6)$$

Here $\pi_*$ is the optimal policy satisfying (3). In other words, $\Delta$ expresses the extra cost paid for using $\tilde{\pi}_*$ instead of $\pi_*$.

There are many examples of *unstable problem of control optimization* in MCP's (see e.g. (Gordienko and Salem, 2000), (Gordienko et al., 2009)). In these examples the values of $\Delta$ in (6) are greater than a positive constant (sometimes large enough), in spite of the fact that $\tilde{F}$ approaches $F$ in certain strong sense (for example, in the total variation metric).

For the above mentioned applied MCP's we establish two (for two different optimization criteria) *stability inequalities* of the form:

$$\Delta \leq K\pi(F, \tilde{F}), \quad (7)$$

where $K$ is an explicitly calculated constant, and $\pi$ is the Prokhorov (sometimes called "Levy-Prokhorov") metric on the space of d.f's. The convergence in $\pi$ is equivalent to the *weak convergence* ( convergence in distribution). Inequality (7) asserts that if $\tilde{F}$ is a "good" approximation to $F$ (in sense of closeness in $\pi$), then if one uses the policy $\tilde{\pi}_*$ then he/she pays "almost the same amount" as applying the optimal policy $\pi_*$. Note that, in spite of lack of complete information on $F$, sometimes it is possible to find an upper bound for values of $\pi(F, \tilde{F})$. When $\tilde{F} = \tilde{F}_m$ is the empirical distribution function obtained from the sample $X_1, X_2, \ldots, X_m$, of size $m$, then $E\pi(F, \tilde{F}_m) \leq const \cdot m^{-1/3}$, provided that $\int_0^\infty x^{3/2} F(dx) < \infty$.

It is worth noting that in the theory of stochastic processes (controlled or not) the term "stability" has had a variety of meanings (see, for instance, (MacPhee and Müller, 2006), (Meyn and Tweedie, 2009)). We use this term since we did not found anything better. The study of quantitative estimation of continuity in noncontrollable queues (under perturbation of "governing" d.f.'s) is a rather developed topic (see, eg. (Zolotarev, 1976), (Kalashnikov, 1983), (Abramov, 2008), (Gordienko and Ruiz de Chavez, 1998)). For controlled Markov processes the "stability" was studied (in the another framework), for instance, in (Van Dijk, 1988), (Van Dijk and Sladky, 1999). "Stability inequalities" for MCP's (using probability metrics distinct from the Prokhorov one) were found in several relevant papers (see for instance (Gordienko and Salem, 2000), (Gordienko et al., 2009)). Surely, "stability" of control policies as it is considered here in some way is connected with the sensitivity theory, and with the profound theory of robust control (which, in particular, are successfully used in dam models, see e.g. (Barbu and Sritharan, 1998), (Litrico and Georges, 2001)).

The findings in this communication are new theoretical results on estimation of "stability" of applied control process (in terms of the Prokhorov metric.) Stability estimation in queueing models may play a role in design of "reasonable" control policies under uncertainties about probability distributions. The water release model considered here is quite elementary, and so has restricted applications. However, our approach can be extended to more realistic dam models.

# 2 STABILITY ESTIMATION WITH RESPECT TO THE EXPECTED TOTAL DISCOUNTED COST

Equations (1) represent the MCP on the state space $\mathfrak{X} = [0, \infty)$ and the sets of admissible control actions $A(x) = [\gamma, \bar{\gamma}]$, $x \in \mathfrak{X}$ (with $\gamma < \bar{\gamma}$ being the assigned extreme values of a service rate). In turn, equations (2) define the MCP on the state space $\mathfrak{X} = [0, M]$ with the sets of admissible actions $A(x) = [0, x]$, $x \in \mathfrak{X}$.

Suppose that for each of the above mentioned processes a measurable real-valued *one-step cost function* $c(x, a)$, $x \in \mathfrak{X}$, $a \in A(x)$ is specified. Thus, at stage $t$ the controller "pays" $c(x_t, a_t)$ if the process occurs in the state $x_t$ and the control action $a_t$ is selected. For example, for the controlled queue (1) $c(x, a)$ can be increasing in the variable $a$ representing a service rate, and for process (2) $c(x, a)$ can be decreasing as a function of water consumption $a$. In fact we do not need such detailed specification. We will only suppose throughout the rest of the paper that the function *c is bounded* and it *satisfies the Lipschitz condition* in both arguments.

Let $\alpha \in (0,1)$ be a given *discount factor*. For any fixed *initial state* $x \in \mathfrak{X}$ of process (1) or (2), and for any chosen control policy $\pi$, the *total expected discounted cost* is defined as follows:

$$V(x,\pi) := E_x^\pi \sum_{n=0}^{\infty} \alpha^n c(X_n, a_n), \tag{8}$$

where $E_x^\pi$ stands for the expectation corresponding the probability measure (on the space of trajectories) generated by application of the policy $\pi$ with the initial state $x$ of the process. Similarly, denoting by $\tilde{E}_x^\pi$ the respective expectation for process (4) (or 5) we define by the expression similar to (8) the expected discounted cost $\tilde{V}(x,\pi)$.

The control policy $f \equiv (f_1, f_2, \dots)$ is called *stationary* if on each stage $t = 0,1,2,\dots$ this policy prescribes to choose the control action $a_t = f(X_t)$, where

$$f:\ \mathfrak{X} \to A := \bigcup_{x \in \mathfrak{X}} A(x)$$

is a measurable function such that $f(x) \in A(x)$, $x \in \mathfrak{X}$. Along with the aforementioned restrictions on the one-step function, we assume that the d.f.'s $F$ and $\tilde{F}$ are continuos.

Fixing any pair of MCP's either (1)-(4) or (2)-(5) we can prove the following assertion.

**Proposition 1.** *There exist stationary optimal policies $f_*$ and $\tilde{f}_*$. That is:*

$$V(x,f_*) := \inf_\pi V(x,\pi), \quad x \in \mathfrak{X};$$
$$\tilde{V}(x,\tilde{f}_*) := \inf_\pi \tilde{V}(x,\pi), \quad x \in \mathfrak{X}.$$

In view of this proposition the stability index in (6) (now with $Q = V$) can be rewritten as follows:

$$\Delta(x) = V(x,\tilde{f}_*) - V(x,f_*) \geq 0, \quad x \in \mathfrak{X}.$$

**Theorem 1.** *For both pairs of MCP's (1)-(4) and (2)-(5) we obtain that*

$$\sup_{x\in\mathfrak{X}} \Delta(x) \leq K\pi(F,\tilde{F}), \tag{9}$$

*where $\pi$ is the Prokhovov metric, and $K$ is an* explicitly calculated *constant (depending only on $\alpha$ and on the characteristics of the one-step function c.)*

This theorem is proved applying the technique of contractive operators.

# 3 STABILITY ESTIMATION WITH RESPECT TO THE AVERAGE EXPECTED COST

In this section we consider another classical optimization criteria – *long-run average expected cost per unit of time*: For $x \in \mathfrak{X}$ and the policy $\pi$ it is defined as follows:

$$J(x,\pi) := \limsup_{n\to\infty} n^{-1} E_x^\pi \sum_{t=0}^{n-1} c(X_t, a_t). \tag{10}$$

Using $Q = J$ in (6) we offer an upper bound for this stability index $\Delta$ *only* for the MCP's defined by equations (2) and (5) (since the ergodicity conditions needed to establish such bound are not easy to guarantee for the processes (1) and (4). In any case this requires further investigations.)

Apart from the conditions on the one-step function $c$ pointed in section 2, in the present section we make much more restrictive assumptions about properties of the distribution functions $F$ and $\tilde{F}$. We suppose that both d.f.s $F$ and $\tilde{F}$ have continuously differentiable densities (denoted by $g$ and $\tilde{g}$) with bounded supports. Moreover it is assumed that $g$ and $\tilde{g}$ are strictly positive on some open interval $(0,\Gamma) \supset (0,M]$.

**Remark 2.** *We use the above conditions only to outline the idea. In fact these assumptions can be significantly relaxed (for instance, removing the assumption about the finiteness of supports).*

Similarly to (10) (defined for process (2)) we write the average cost for the approximating control process (5) as follows:

$$\tilde{J}(x,\pi) := \limsup_{n\to\infty} n^{-1} \tilde{E}_x^\pi \sum_{t=0}^{n-1} c(\tilde{X}_t, a_t).$$

**Proposition 2.** *Under the assumption made we obtain that:*

*(i) The minimal average costs*

$$\inf_\pi J(x,\pi) \equiv J_* \quad \text{and} \quad \inf_\pi \tilde{J}(x,\pi) \equiv \tilde{J}_*$$

*do not depend on the initial states x of the processes.*

*(ii) There exist stationary optimal control policies $f_*$ and $\tilde{f}_*$ (for (2) and (5), respectively):*

$$J(x,f_*) \equiv J(f_*) = J_*,$$
$$\tilde{J}(x,\tilde{f}_*) \equiv \tilde{J}(\tilde{f}_*) = \tilde{J}_*.$$

Thus for $Q = J$ the stability index in (6) is expressed in the following way:

$$\Delta = J(\tilde{f}_*) - J(f_*) \geq 0.$$

We are ready to state our second result.

**Theorem 2.** *Under the assumption made for the MCP's (2) and (5) we obtain:*

$$\Delta \leq K_* \pi(F,\tilde{F}), \tag{11}$$

*where, again, $\pi$ is the Prokhovov distance between the d.f.'s $F$ and $\tilde{F}$, and $K_*$ is an explicitly calculated constant (depending on characteristics of the densities $g$, $\tilde{g}$ and of the function c).*

**Remark 3.** *The proof of inequality (11) also uses methods of contractive operators, but in a much more sophisticated way than in case of the proof of (9). The needed contractive properties of dynamics programming operator are demonstrated using suitable ergodic features of processes. (For this we need the above mentioned restrictions on densities.)*

The last stability inequality we offer is more rough than (11), but it is expressed in terms of more transparent distance (the total variation metric).

**Corollary 1.** *Under assumption made*

$$\Delta \leq \overline{K} \int_0^\infty |g(s) - \tilde{g}(s)|\, ds.$$

## 4 CONCLUSIONS

In this talk we study two important applied control models which involve a not completely know distribution function $F$ of independent and identically distributed random variables. Supposing that $F$ is approximated by an available distribution function $\tilde{F}$ (obtained from statistical estimations or theoretical simplifications) we evaluate quantitatively the performance worsening when an approximating control policy is used in place of an unavailable optimal policy.

For such evaluation we introduce the so called stability index, and prove inequalities bounding this index in terms of the Prokhorov distance between $F$ and $\tilde{F}$. These bounds can be useful to measure quality of service rates control in computer networks and of control procedures in certain elements of water supply systems.

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