TETRACHROMATIC COLOUR SPACES
Spherical and Toroidal Hue Spaces

Alfredo Restrepo Palacios
Dept. Ing. Electrica y Electronica, Laboratorio de Señales, Universidad de los Andes, Cra. 1 No. 18A- 12, ML-427, Bogotá, Colombia

Keywords: Tetrachromacy, Colour Space, Hypercube, Spherical Hue, Toroidal Hue.

Abstract: From a 4-hypercube colour space, spaces of the type hue-saturation-luminance are derived. The hue component may have the topology of a 2-sphere, a 2-torus or a 3-sphere, in several possibilities we consider.

1 INTRODUCTION

Analogously to RGB colour space and the spaces of the type hue-saturation luminance you can derive from it, we start from a tetrachromatic colour hypercube and derive several spaces of the type hue-colourfulness-luminance.

2 TETRACHROMATIC COLOUR SPACES

Four types of a photoreceptor in an eye of an animal, four spectral bands in an imaging system or four light sources with different but perhaps overlapping spectra that shine over the same spot, give rise to a four-dimensional Cartesian cube of possible responses, or stimuli.

2.1 The WXYZ Colour Hypercube

The hypercube I^4, where $I = \{0\} \cup (0,1) \cup \{1\}$, has 16 vertices, 32 edges, 24 square faces and, in addition to its inner 4-cell, 8 solid (i.e. three-dimensional) cubes. The union of the solid cubes is the polytope $\{4\ 3\ 3\}$, in Scläfli notation, which is a topological 3-sphere that is the boundary of the hypercube. The 14 vertices that remain after dropping out the vertices [0000] and [1111], are the vertices of an icositetrahedron of 24 triangles that result from dividing 12 of the square faces along certain diagonals (Restrepo, 2012). This *chromatic icositetrahedron* is used analogously to the *chromatic hexagon* of the RGB cube (Restrepo, 2011) to define a tetrachromatic hue; each triangle corresponds to one of the 24 possible permutations of the tetrad $[wxyz]$. In addition, 16 of the 24 square faces can be chosen that form a piecewise linear (PL) torus; the torus in turn can be used to define tetrachromatic hue in a different way.

2.2 A Polytopal, 4D HSL Space

The 24 possible orderings of the coordinate tetrad $[wxyz]$ determine 24 families of *hue*. To each point in the hypercube there corresponds the range ρ and the midrange μ of its coordinates $[w,x,y,z]$. The range $\rho = \max\{w,x,y,z\} - min\{w,x,y,z\}$ is a measure of the distance from the point $[w,x,y,z]$ to the *achromatic segment* $\{[t,t,t,t] \in I^4 : t \in [0,1]\}$ and is taken to be the *chromatic saturation*. The midrange $\mu = \frac{1}{2}(\max\{w,x,y,z\} + min\{w,x,y,z\})$ is a measure of the distance to the *black point* [0,0,0,0] and is taken to be the *luminance* of the point.

The pair (μ,ρ) lives on an isosceles triangle of base 1 and height 1. Each point not in the achromatic segment is assigned a unique *hue point* **d** on the chromatic icositetrahedron; **d** results from an initial move, roughly towards [0000], along a direction parallel to the achromatic segment, followed by an expansion from the point [0000] until the chromatic icositetrahedron is reached. Let $\mathbf{p} = [p_1, p_2, p_3, p_4] = [w,x,y,z]$ be a point in the hypercube with $\rho \neq 0$, and let $m = min\{w,x,y,z\}$ and $M = max\{w,x,y,z\}$; then, $\mathbf{d} = \frac{1}{\rho}[w,x,y,z] - \frac{m}{\rho}[1,1,1,1]$. Now, **p** lies on the *chromatic triangle* having vertices [0 0 0 0], [1 1 1 1] and **d**, and each of the points on this triangle, not on the achromatic segment, are assigned the same hue point **d**. To name the hue point **d**, give the vertices of a triangle in the chromatic icositetrahedron that contains

Restrepo Palacios A..
TETRACHROMATIC COLOUR SPACES - Spherical and Toroidal Hue Spaces.
DOI: 10.5220/0003846604230426
In *Proceedings of the International Conference on Computer Vision Theory and Applications* (VISAPP-2012), pages 423-426
ISBN: 978-989-8565-03-7

Table 1: Vertices of triangles in the chromatic dodecahedron and the corresponding ordering of coordinates. In the fourth column, it is indicated the positions of the coordinates that are fixed in the three vertices. Note that each pair of consecutive triangles (including the last and the first) share 2 vertexes.

#	order	vertex 1	vertex 2	vertex 3	fcs
1	wxyz	0001	0011	0111	1, 4
2	xwyz	0001	0011	1011	2, 4
3	xywz	0001	1001	1011	2, 4
4	xyzw	1000	1001	1011	1, 2
5	xzyw	1000	1010	1011	1, 2
6	xzwy	0010	1010	1011	2, 3
7	xwzy	0010	0011	1011	2, 3
8	wxzy	0010	0011	0111	1, 3
9	wzxy	0010	0110	0111	1, 3
10	zwxy	0010	0110	1110	3, 4
11	zxwy	0010	1010	1110	3, 4
12	zxyw	1000	1010	1110	1, 4
13	zyxw	1000	1100	1110	1, 4
14	zywx	0100	1100	1110	2, 4
15	zwyx	0100	0110	1110	2, 4
16	wzyx	0100	0110	0111	1, 2
17	wyzx	0100	0101	0111	1, 2
18	ywzx	0100	0101	1101	2, 3
19	yzwx	0100	1100	1101	2, 3
20	yzxw	1000	1100	1101	1, 3
21	yxzw	1000	1001	1101	1, 3
22	yxwz	0001	1001	1101	3, 4
23	ywxz	0001	0101	1101	3, 4
24	wyxz	0001	0101	0111	1, 4

it, together with barycentric coordinates with respect to these vertices; see columns 3 to 5 of Table 1. To find this icositetrahedron triangle face, consider the ordering of the coordinates of the point and use the second column of Table 1. By construction, at least one of the coordinates i_1 of the hue point $\mathbf{d}$ has value 0 ($p_{i_1} = m$) and at least one of the coordinates i_4 has value 1 ($p_{i_4} = M$); the remaining two coordinates can be similarly labeled so that $p_{i_2} \leq p_{i_3}$. The triangle that contains the hue point has vertexes $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ with $q_{i_1} = r_{i_1} = s_{i_1} = 0$, $q_{i_4} = r_{i_4} = s_{i_4} = 1$, and $q_{i_2} = 0, q_{i_3} = 0$, $r_{i_2} = 0, r_{i_3} = 1$ and $s_{i_2} = 1, s_{i_3} = 1$. The hue point can now be expressed as $\mathbf{d} = \alpha\mathbf{q} + \beta\mathbf{r} + \gamma\mathbf{s}$, with $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma = 1$; then, $\gamma = \frac{\mathbf{p}_{i_2} - m}{\rho}$, $\beta = \frac{\mathbf{p}_{i_3} - \mathbf{p}_{i_2}}{\rho}$, and $\alpha = 1 - \gamma - \beta = \frac{M - \mathbf{p}_{i_3}}{\rho}$. The colour attributes of the point $\mathbf{p}$ in this space are thus the hue, the chromatic saturation and the luminance, given by $\mathbf{d}$, ρ and μ, respectively.

2.3 A Double-cone Type Space

The polytopal space of Section 2.2 is now transformed to a double-cone type, 4D space, in the sense that the chromatic icositetrahedron is rounded up to a 2-sphere. The hue point is now named with spherical coordinates, with a latitude angle ϕ and an azymuth angle θ. The chromatic icositetrahedron is made to correspond to this hue sphere, where the vertex [0 1 1 1] corresponds to the north pole and the vertex [1 0 0 0] corresponds to the south pole, by assigning the 14 vertexes of the chromatic icositetrahedron to 14 points uniformly distributed over the sphere, as indicated in Table 2. With $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi) \cup \{*\}$ ($\phi = 0$ and $\phi = \pi$ respectively correspond to the *north and south poles* where θ is ”left undefined as $*$”) the 14 points uniformly distributed on the sphere are those with $(\theta, \phi) \in \{(*, 0), (*, \pi)\} \cup \{(\frac{\pi}{3}, 0), (\frac{\pi}{3}, \frac{\pi}{3}), (\frac{\pi}{3}, \frac{2\pi}{3}), (\frac{\pi}{3}, \frac{3\pi}{3}), (\frac{\pi}{3}, \frac{4\pi}{3}), (\frac{\pi}{3}, \frac{5\pi}{3})\} \cup \{(\frac{2\pi}{3}, 0), (\frac{2\pi}{3}, \frac{\pi}{3}), (\frac{2\pi}{3}, \frac{2\pi}{3}), (\frac{2\pi}{3}, \frac{3\pi}{3}), (\frac{2\pi}{3}, \frac{4\pi}{3}), (\frac{2\pi}{3}, \frac{5\pi}{3})\}$.

At the 6+6 triangles with a vertex assigned to a pole, the computation of the spherical coordinates is slightly different than at the remaining 12 triangles where the barycentric coordinates are plainly used with the help of Table 2. For the afore mentioned triangles with a vertex [0111] or [1000], the barycentric coordinates α, β γ are transformed to $\eta = 1 - \gamma$ (distance to ”vertex γ”) and $\xi = 1 - \frac{\beta}{1-\gamma}$ (angular measure from line through ”vertex γ” and hue point, and line through ”vertexes γ and β”), β is chosen so that the angle increases when the hue point moves from vertex β to vertex α, and γ is the barycentric coordinate corresponding to the pole. For example, to the hue point in triangle with vertices $\mathbf{q}$ = [0010], $\mathbf{r}$ =[0110] and $\mathbf{s}$ =[1110] and corresponding barycentric coordinates $\alpha = 0.6$, $\beta = 0.3$, and $\gamma = 0.1$, there corresponds $(\theta, \phi) = \alpha(2\pi, \pi/3) + \beta(5\pi/3, \pi/3) + \gamma(5\pi/3, 2\pi/3)$ = $(5.6\pi/3, 1.1\pi/3)$ and to hue point in triangle with vertices $\mathbf{q}$ = [0100] and $\mathbf{r}$ = [0110] $\mathbf{s}$ = [0111] (north pole) , and corresponding barycentric coordinates $\alpha = 0.6$, $\beta = 0.3$, and $\gamma = 0.1$, there corresponds $\eta = 0.9$, $\xi = 1/3$ and $(\theta, \phi) = (\xi\pi/3 + 5\pi/3, \eta\pi/3) = (5.25\pi/3, 0.4\pi/3)$. In this way, for the double-cone type space, the colour components are the luminance μ, the chromatic saturation ρ, the *azymuth hue* θ and the *latitude hue* ϕ.

To convert back angular hue coordinates to icositetrahedron coordinates, consider 3 cases: the spherical triangle has no vertex at a pole, or it has a vertex at the north pole ($\phi = 0$), or it has the south pole ($\phi = \pi$) as a vertex. In the first case, we compute barycentric coordinates of the point with respect to the vertices (θ_i, ϕ_i) and then get the point (w_i, x_i, y_i, z_i) by computing the corresponding barycentric combination using the corresponding vertices of the chromatic icositetrahedron. For triangles with a vertex at the north pole, we use the map $(\theta, \phi) \mapsto (\theta\frac{\phi}{\pi/3}, \phi)$, and compute the barycentric coordinates of the mapped point with respect to the triangle with vertices $(0,0)$, $((n-1)\pi/3, \pi/3)$, and $(n\pi/3, \pi/3)$. For the south pole we use the map $(\theta,\phi) \mapsto (\theta\frac{\pi-\phi}{\pi/3}, \phi)$ and the triangle with vertices $(0,\pi)$, $((n-1)\pi/3, 2\pi/3)$, and $(n\pi/3, 2\pi/3)$.

Table 2: The 14 vertices of the chromatic dodecahedron are made to correspond to 14 points uniformly distributed over the sphere S^2. The points on the sphere are indicated with spherical coordinates $\theta \in [0,2\pi)$ and $\phi \in [0.\pi]$.

#	vertex	θ	ϕ
1	0111	$*$	0
2	0010	0	$\pi/3$
3	0011	$\pi/3$	$\pi/3$
4	0001	$2\pi/3$	$\pi/3$
5	0101	$3\pi/3$	$\pi/3$
6	0100	$4\pi/3$	$\pi/3$
7	0110	$5\pi/3$	$\pi/3$
8	1010	0	$2\pi/3$
9	1011	$\pi/3$	$2\pi/3$
10	1001	$2\pi/3$	$2\pi/3$
11	1101	$3\pi/3$	$2\pi/3$
12	1100	$4\pi/3$	$2\pi/3$
13	1110	$5\pi/3$	$2\pi/3$
14	1000	$*$	π

2.4 Runge Space

A "round" space is obtained by deforming the hypercube into the 4-ball $\{(w',x',y',z') \in \mathbf{R}^4 : w^2+x^2+y^2+z^2 \leq 1\}$. Let $[w,x,y,z]$ be a point in the hypercube, shift the hypercube so that intermediate gray ends up at the origin of 4-space R^4 and rescale so that the maximum values of the coordinates is 1 and the minimum is -1. Let $[w',x',y',z'] = 2[w-0.5, x-0.5, y-0.5, z-0.5]$ be the coordinates of the resulting hypercube $[-1,1]^4$.

The *lightness* in this space is given by the angle with the achromatic axis: $\lambda = arcos\frac{w'+x'+y'+z'}{2\sqrt{w'^2+x'^2+y'^2+z'^2}}$ $= arcos\frac{w+x+y+z-2}{2\sqrt{w^2+x^2+y^2+z^2+1-(w+x+y+z)}}$. Rather than using a chromatic saturation measure i.e. a distance measure to the achromatic line segment, we use a distance measure from the central point of intermediate gray and we obtain a measure of colourfulness in the sense that it is a measure of "ungrayness". The main difference with chromatic saturation is that any primary, say [1000], is as colourful as the black and the white points [0000] and [1111]; in fact, any point on the boundary of the hypercube is fully colourful. Let $\Lambda = max\{|w'|,|x'|,|y'|,|z'|\}$; if $\Lambda \neq 0$, the point on the boundary of the hypercube that is in the same direction is $\frac{1}{\Lambda}[w',x',y',z']$ (at least one of its coordinates has value of 1); let $d = \frac{1}{\Lambda}\sqrt{w'^2+x'^2+y'^2+z'^2}$ and normalize by this length (with the result that the hypercube is deformed into a 4-ball), getting the point $\frac{1}{d}[w',x',y',z']$ whose distance from the center of the ball is
$\kappa = \frac{\sqrt{w'^2+x'^2+y'^2+z'^2}}{\Lambda^{-1}\sqrt{w'^2+x'^2+y'^2+z'^2}} = \Lambda$. Thus
$\kappa = max\{2w-1, 2x-1, 2y-1, 2z-1\}$ is the *colourfulness* of the point $[w,x,y,z]$.

The hue is as in double cone space.

2.5 A Toroidal Hue Space

The eight cubes of the boundary of the hypercube can be grouped into two solid tori in three ways; consider for example the solid tori $\mathbf{YZ} = \{z=0\} \cup \{y=1\} \cup \{z=1\} \cup \{y=0\}$ and $\mathbf{WX} = \{w=1\} \cup \{x=0\} \cup \{w=0\} \cup \{x=1\}$. Each cube of solid torus **YZ** intersects each cube of solid torus **WX** at a square face; for example, the square $\{y=0\} \cap \{w=1\}$; there are 16 such square faces where the solid tori touch and together they form the PL *chromatic 2-torus*. For each point $\mathbf{p} = [w,x,y,z]$ in the hypercube, different from intermediate gray, find a corresponding *hue point* **h** on the chromatic 2-torus; consider the intersection **e** of the ray emanating from intermediate gray and going through **p** and the boundary of the hypercube; call **e** the *exit point* from the hypercube. Let $[w',x',y',z'] := [w-0.5, x-0.5, y-0.5, z-0.5]$ and let $\chi = \frac{1}{2}\frac{1}{max\{|w'|,|x'|,|y'|,|z'|\}}$; the coordinate(s) of $\mathbf{f} := \chi[w',x',y',z'] + [0.5,0.5,0.5,0.5]$ in the set $\{0,1\}$ indicate the solid cube(s) in the boundary of the hypercube **e** is; for example, the cube $\{w=0\}$ when $f_1 = 0$. Then find the point **h** in the boundary of the corresponding solid torus (**WX** in the example), which is the chromatic 2-torus, that is closest to **e**. The hue point is now best specified by the remaining coordinate that is largest in magnitude; in the example, take $max\{|y'|,|z'|\}$ (otherwise, if the exit point is in the torus YZ, take $max\{|w'|,|x'|\}$); this indicates which face of the chromatic torus **e** is closest to. In the example, if $max\{|y'|,|z'|\} = |z'|$ and $z' < 0$, then the closest face is $\{w=0\} \cap \{z=0\}$, if $max\{|y'|,|z'|\} = |z'|$ and $z' > 0$, the closest face is $\{w=0\} \cap \{z=1\}$. Once a face in the chromatic torus that contain the hue point is identified, the point **p** is assigned hue coordinates ω and η (angles mod-1), as indicated in Table 3. The quantity $\rho := \chi^{-1}$ measures the distance of the point **p** from intermediate gray, call it the *chromaticity*. To this measure of distance of the max, there corresponds *balls* centered at intermediate gary and with a shape that is a scaled version of the hypercube. (When the max is taken without shifting the hypercube, a measure of distance to the point black results, and the balls are *corners* of a hypercube.) The quantity $\tau = -2max\{|w'|,|x'|\}$ or $\tau = 2max\{|w'|,|x'|\}$, depending on whether the exit point is in solid torus **YZ** or **WX**, respectively, is a signed measure of the distance of the exit point to the chromatic torus; $\tau = 0$ if $\mathbf{e} = \mathbf{h}$; $|\tau|$ measures the distance from the center of the corresponding solid cube the ray leaves the hypercube. For example, the center of the solid cube $\{w = 0\}$ has equal contributions from the primaries x, y and z and none of the primary w; the point is "a dark *xyz*", on the other hand, the center of the solid cube $\{w = 1\}$ is a "desaturated w".

Table 3: Values of the coordinates ω and η depending on the solid torus the hue point is.

#	cube	η	ω
1	$\{x=0\}\cap\{y=1\}$	z	w
2	$\{x=0\}\cap\{z=1\}$	$0.25(y+1)$	w
3	$\{x=0\}\cap\{y=0\}$	$0.25(z+2)$	w
4	$\{x=0\}\cap\{z=0\}$	$0.25(y+3)$	w
5	$\{w=0\}\cap\{y=1\}$	z	$0.25(x+1)$
6	$\{w=0\}\cap\{z=1\}$	$0.25(y+1)$	$0.25(x+1)$
7	$\{w=0\}\cap\{y=0\}$	$0.25(z+2)$	$0.25(x+1)$
8	$\{w=0\}\cap\{z=0\}$	$0.25(y+3)$	$0.25(x+1)$
9	$\{x=1\}\cap\{y=1\}$	z	$0.25(w+2)$
10	$\{x=1\}\cap\{z=1\}$	$0.25(y+1)$	$0.25(w+2)$
11	$\{x=1\}\cap\{y=0\}$	$0.25(z+2)$	$0.25(w+2)$
12	$\{x=1\}\cap\{z=0\}$	$0.25(y+3)$	$0.25(w+2)$
13	$\{w=1\}\cap\{y=1\}$	z	$0.25(x+3)$
14	$\{w=1\}\cap\{z=1\}$	$0.25(y+1)$	$0.25(x+3)$
15	$\{w=1\}\cap\{y=0\}$	$0.25(z+2)$	$0.25(x+3)$
16	$\{w=1\}\cap\{z=0\}$	$0.25(y+3)$	$0.25(x+3)$

3 CONCLUDING REMARKS

Hering's opponent process model for colour vision explicitly uses the chromatic uniques red and green, and yellow and blue and the achromatic uniques black and white, rather than the primaies RGB. It has the advantage of giving yellow the important role it has in our colour vision and the perception of natural scenes. The transformation $[R,G,B] \mapsto [R-G, 0.5(R+G)-Y, (R+G+B)/3]$ can be seen perhaps as a principal component analysis that enhances the information contents of the code. It is not clear how to carry on the primary-unique dichotomy to the tetrachromatic case; it surely should depend on the application of a tetrachromatic vision system. In principle, there are several possibilities; for example, you could say that $y+z$ is a unique (analogously to the case of unique yellow being an additive mixture of a spectral red and a spectral green; i.e. that the channel M+L is the receptoral basis for perceptual unique yellow). That is, you could propose a collection of *channels* that are derived from receptoral signals w, x, y and z, perhaps linearly.

R^4 can be partitioned into a collection of open 2D *flags* with *pole* the line $w = x = y = z$, and the line itself. For the hypercube, this translates into a partition into a collection of triangles with the achromatic axis on one side and opposite vertex in the chromatic icositetrahedron. The points on each of these triangles, not in the achromatic segment, are said to have the same hue; *hue* being then the orientation with respect to the achromatic axis which in turn is determined in a discrete fashion by the ordering or relative importance of the primaries w, x, y and z.

Alternately, the hypercube can be decomposed into its central point plus a collection of intervals with one extreme at the central point and the other extreme in the boundary of the hypercube, a PL S^3; the boundary of the hypercube is then partitioned into two solid tori joined by a 16-face 2-torus; each point of the S^3 determines a possible hue and this tridimensional hue is characterized in terms of the two angles that describe the 2-torus, plus a linear variable that says how deep within one of the solid tori the point of S^3 is. The trichromatic corresponding analogous case is not commonly used: it corresponds to considering that each point on the 6 faces of the cube is a value of hue; this results in a two-dimensional trichromatic hue, characterized e.g. by latitude and azymuth (the azymuth being a traditional trichromatic hue); the two colour attributes would then be: hue and colourfulness (distance from intermediate gray); pretty much along Runges' model.

Tetrachromatic colour spaces such as these should find applications in the visualization of 4-spectral images (e.g. satellite images) and in 4-spectral computer vision systems. Also, the study of tetracrhomatic vision could benefit, e.g. in the production of visual stimuli, and in the modeling of tetrachromatic perception.

REFERENCES

Restrepo, A. (2011). Colour processing in runge space. In *Electronic Imaging, San Francisco, January 2011*. SPIE.

Restrepo, A. (2012). Tetrachromatic colour space. In *Electronic Imaging, San Francisco, January 2012*. SPIE.