A VACCINATION CONTROL LAW BASED ON FEEDBACK LINEARIZATION TECHNIQUES FOR SEIR EPIDEMIC MODELS

S. Alonso-Quesada[1], M. De la Sen[1] and A. Ibeas[2]
[1]*Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country Campus of Leioa, 48940-Leioa, Bizkaia, Spain*
[2]*Departamento de Telecomunicación e Ingeniería de Sistemas, Escuela Técnica Superior de Ingeniería Universitat Autònoma, Barcelona, Spain*

Keywords: SEIR epidemic models, Vaccination, Nonlinear control, Stability, Positivity.

Abstract: This paper presents a vaccination strategy for fighting against the propagation of epidemic diseases. The disease propagation is described by a SEIR (susceptible plus infected plus infectious plus removed by immunity populations) epidemic model. The model takes into account the total population amounts as a refrain for the illness transmission since its increase makes more difficult contacts among susceptible and infected. The vaccination strategy is based on a continuous-time nonlinear control law synthesized via an exact feedback input-output linearization approach. The control objective is to asymptotically eradicate the infection. Moreover, the positivity and stability properties of the controlled system are investigated.

1 INTRODUCTION

A relevant area in the mathematical theory of epidemiology is the development of models for studying the propagation of epidemic diseases in a host population. The epidemic mathematical models analysed include the most basic ones (De la Sen and Alonso-Quesada, 2010); (Keeling and Rohani, 2008); (Li et al., 1999); (Makinde, 2007); (Mollison, 2003), namely: (i) SI models where only susceptible and infected populations are assumed to be present in the model, (ii) SIR models which include susceptible plus infected plus removed-by-immunity populations and (iii) SEIR models where the infected population is split into two ones, namely, the "infected" (or "exposed") which incubate the disease but they do not still have any disease symptoms and the "infectious" (or "infective") which do have the external disease symptoms. Those models can be divided in two main classes, namely, the so-called "pseudo-mass action models", where the total population is not taken into account as a relevant disease contagious factor and the so-called "true-mass action models", where the total population is more realistically considered as an inverse factor of the disease transmission rates.

There are many variants of the above models as, for instance, the SVEIR epidemic models which incorporate the dynamics of a vaccinated population in comparison with the SEIR models (De la Sen et al., 2011); (Song et al., 2009), the SEIQR-SIS model which adds a quarantine population (Jumpen et al., 2011) and the model proposed in (Safi and Gumel, 2011) which incorporates vaccinated, quarantine and hospitalized populations. Other variant consists of the generalization of such models by incorporating point and/or distributed delays (De la Sen et al., 2010); (Zhang et al., 2009). Another one is concerned with the inclusion of a saturated disease transmission incidence rate for taking into account the inhibition effect from the behavioural change of susceptible individuals when the infectious individual number increases (Xu et al., 2010).

The analysis of the existence of equilibrium points, relative to either the persistence or extinction of the epidemics, the conditions for the existence of a backward bifurcation where both equilibrium points co-exist and the constraints for guaranteeing the positivity and the boundedness of the solutions of such models have been some of the main objectives in the aforementioned papers. Also, the conditions that generate an oscillatory behaviour in such solutions has been dealt with in the literature

Alonso-Quesada S., De la Sen M. and Ibeas A..
A VACCINATION CONTROL LAW BASED ON FEEDBACK LINEARIZATION TECHNIQUES FOR SEIR EPIDEMIC MODELS.
DOI: 10.5220/0003764900760085
In *Proceedings of the International Conference on Bioinformatics Models, Methods and Algorithms* (BIOINFORMATICS-2012), pages 76-85
ISBN: 978-989-8425-90-4
Copyright © 2012 SCITEPRESS (Science and Technology Publications, Lda.)

about epidemic mathematical models (Mukhopadhyay and Bhattacharyya, 2007). Other important aim is that relative to the design of control strategies in order to eradicate the persistence of the infection in the host population (De la Sen and Alonso-Quesada, 2010); (De la Sen et al., 2011); (Makinde, 2007); (Safi and Gumel, 2011). In this context, an explicit vaccination function of many different kinds may be considered, namely: constant, continuous-time, impulsive, mixed constant/impulsive, mixed continuous-time/impulsive, discrete-time and so on.

In this paper, a SEIR epidemic model is considered. The dynamics of susceptible (S) and immune (R) populations are directly affected by a vaccination function $V(t)$, which also has indirectly influence in the time evolution of infected or exposed (E) and infectious (I) populations. In fact, such a vaccination function has to be suitably designed in order to eradicate the infection from the population. This model has been already studied in (De la Sen and Alonso-Quesada, 2010) from the viewpoint of equilibrium points in the controlled and free-vaccination cases. A vaccination auxiliary control law being proportional to the susceptible population was proposed in order to achieve the whole population be asymptotically immune. Such an approach assumed that the SEIR model was of the aforementioned true-mass action type, its parameters were known and the illness transmission was not critical. Moreover, some important issues of positivity, stability and tracking of the SEIR model were discussed. *The present paper proposes an alternative method to obtain the vaccination control law to asymptotically eradicate the epidemic disease. Namely, the vaccination function is synthesized by means of an input-output exact feedback linearization technique. Such a linearization control strategy constitutes the main contribution of the paper. Moreover, mathematical proofs about the epidemics eradication based on such a controlled SEIR while maintaining the non-negativity of all the partial populations for all time are issued.* The exact feedback linearization can be implemented by using a proper nonlinear coordinate transformation and a static-state feedback control. The use of such a linearization strategy is motivated by three main facts, namely: (i) it is a power tool for controlling nonlinear systems which is based on well-established technical principles (Isidori, 1995), (ii) the given SEIR model is highly nonlinear and (iii) such a control strategy has not been yet applied in epidemic models.

2 SEIR EPIDEMIC MODEL

Let $S(t)$, $E(t)$, $I(t)$ and $R(t)$ be, respectively, the susceptible, infected (or exposed), infectious and removed-by-immunity populations at time t. Consider a time-invariant true-mass action type SEIR epidemic model given by:

$$\dot{S}(t) = -\mu S(t) + \omega R(t) - \beta \frac{S(t)I(t)}{N} + \mu N\left[1 - V(t)\right] \tag{1}$$

$$\dot{E}(t) = -(\mu + \sigma)E(t) + \beta \frac{S(t)I(t)}{N} \tag{2}$$

$$\dot{I}(t) = -(\mu + \gamma)I(t) + \sigma E(t) \tag{3}$$

$$\dot{R}(t) = -(\mu + \omega)R(t) + \gamma I(t) + \mu N V(t) \tag{4}$$

subject to initial conditions $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$ and $R(0) \geq 0$ under a vaccination function $V : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$, with $\mathbb{R}_{0+} \triangleq [0, \infty) \cap \mathbb{R}$. In the above SEIR model, $N > 0$ is the total population at any time instant $t \in \mathbb{R}_{0+}$, $\mu > 0$ is the rate of deaths and births from causes unrelated to the infection, $\omega \geq 0$ is the rate of losing immunity, $\beta > 0$ is the transmission constant (with the total number of infections per unity of time at time t being $\beta S(t)I(t)/N$) and, $\sigma^{-1} > 0$ and $\gamma^{-1} > 0$ are, respectively, the average durations of the latent and infective periods. The total population dynamics can be obtained by summing-up (1)-(4) yielding:

$$\dot{N}(t) = \dot{S}(t) + \dot{E}(t) + \dot{I}(t) + \dot{R}(t) = 0 \tag{5}$$

so that the total population $N(t) = N(0) = N$ is constant $\forall t \in \mathbb{R}_{0+}$. Then, this model is suitable for epidemic diseases with very small mortality incidence caused by infection and for populations with equal birth and death rates so that the total population may be considered constant for all time.

3 VACCINATION STRATEGY

An ideal control objective is that the removed-by-immunity population asymptotically tracks the whole population. In this way, the joint infected plus infectious population asymptotically tends to zero as $t \to \infty$, so the infection is eradicated from the population. A vaccination control law based on a static-state feedback linearization strategy is developed for achieving such a control objective. This technique requires a nonlinear coordinate transformation, based on the Lie derivatives Theory (Isidori, 1995), in the system representation.

The dynamics equations (1)-(3) of the SEIR model can be equivalently written as the following nonlinear control affine system:

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases} \tag{6}$$

where $y(t) = I(t) \in \mathbb{R}_{0+}$, $u(t) = V(t) \in \mathbb{R}_{0+}$ and $x(t) = [I(t)\ E(t)\ S(t)]^T \in \mathbb{R}^3_{0+}$ are, respectively, considered as the output signal, the input signal and the state vector of the system $\forall t \in \mathbb{R}_{0+}$ and $R(t) = N - S(t) - E(t) - I(t)$ has been used, with:

$$\begin{aligned} f(x(t)) &= \begin{bmatrix} -(\mu+\gamma)I(t) + \sigma E(t) \\ -(\mu+\sigma)E(t) + \beta_1 I(t)S(t) \\ -\omega\left(I(t)+E(t)\right) + (\mu+\omega)\left(N - S(t)\right) - \beta_1 I(t)S(t) \end{bmatrix} \in \mathbb{R}^3 \\ g(x(t)) &= [0\ \ 0\ \ -\mu N]^T \in \mathbb{R}^3_{0-} \quad ; \quad h(x(t)) = I(t) \in \mathbb{R}_{0+} \end{aligned} \tag{7}$$

where $\beta_1 = \beta/N$ and $\mathbb{R}_{0-} \triangleq (-\infty, 0] \cap \mathbb{R}$. The first step to apply a coordinate transformation based on the Lie derivation is to determine the relative degree of the system. For such a purpose, the following definitions are taken into account: (i) $L_f^k h(x(t)) \triangleq \frac{\partial\left(L_f^{k-1} h(x(t))\right)}{\partial x} f(x(t))$ is the *k*th-order Lie derivative of $h(x(t))$ along $f(x(t))$ with $L_f^0 h(x(t)) \triangleq h(x(t))$ and (ii) the relative degree r of the system is the number of times that the output must be differentiated to obtain the input explicitly, i.e., the number r so that $L_g L_f^k h(x(t)) = 0$ for $k < r-1$ and $L_g L_f^{r-1} h(x(t)) \neq 0$.

From (7), $L_g h(x(t)) = L_g L_f h(x(t)) = 0$ while $L_g L_f^2 h(x(t)) = -\mu\sigma\beta I(t)$, so the relative degree of the system is 3 in $D \triangleq \left\{ x = [I\ \ E\ \ S]^T \in \mathbb{R}^3_{0+} \mid I \neq 0 \right\}$, i.e., $\forall x \in \mathbb{R}^3_{0+}$ except in the singular surface $I = 0$ of the state space where the relative degree is not well-defined. Since the relative degree of the system is exactly equal to the dimension of the state space for any $x \in D$, the nonlinear coordinate change

$$\begin{aligned} \bar{I}(t) &= L_f^0 h(x(t)) = I(t) \\ \bar{E}(t) &= L_f h(x(t)) = [1\ \ 0\ \ 0]\, f(x(t)) = -(\mu+\gamma)I(t) + \sigma E(t) \\ \bar{S}(t) &= L_f^2 h(x(t)) = [-(\mu+\gamma)\ \ \sigma\ \ 0]\, f(x(t)) \\ &= (\mu+\gamma)^2 I(t) - \sigma(2\mu+\sigma+\gamma)E(t) + \sigma\beta_1 I(t)S(t) \end{aligned} \tag{8}$$

allows to represent the model in the called normal form in a neighbourhood of any $x \in D$. Namely:

$$\begin{cases} \dot{\bar{x}}(t) = \bar{f}(\bar{x}(t)) + \bar{g}(\bar{x}(t))u(t) \\ y(t) = h(\bar{x}(t)) \end{cases} \tag{9}$$

where $\bar{x}(t) = \left[\bar{I}(t)\ \ \bar{E}(t)\ \ \bar{S}(t)\right]^T$ and:

$$\begin{aligned} \bar{f}(\bar{x}(t)) &= [\bar{E}(t)\ \ \bar{S}(t)\ \ \varphi(\bar{x}(t))]^T \\ \bar{g}(\bar{x}(t)) &= [0\ \ 0\ \ -\mu\sigma\beta\bar{I}(t)]^T \quad ; \quad h(\bar{x}(t)) = \bar{I}(t) \\ \varphi(\bar{x}(t)) &= (\mu+\omega)\left[\sigma\beta - (\mu+\sigma)(\mu+\gamma)\right]\bar{I}(t) \\ &\quad - (\mu+\omega)(2\mu+\sigma+\gamma)\bar{E}(t) - (3\mu+\sigma+\gamma+\omega)\bar{S}(t) \\ &\quad - \beta_1\left[\omega(\mu+\sigma+\gamma) + (\mu+\sigma)(\mu+\gamma)\right]\bar{I}^2(t) \\ &\quad - \beta_1(2\mu+\sigma+\gamma+\omega)\bar{I}(t)\bar{E}(t) - \beta_1\bar{I}(t)\bar{S}(t) \\ &\quad + \frac{\bar{E}(t)\bar{S}(t)}{\bar{I}(t)} + (2\mu+\sigma+\gamma)\frac{\bar{E}^2(t)}{\bar{I}(t)} \end{aligned} \tag{10}$$

The following result being relative to the input-output linearization of the system is established.

Theorem 1: The state feedback control law

$$u(t) = \frac{-L_f^3 h(x(t)) - \lambda_0 h(x(t)) - \lambda_1 L_f h(x(t)) - \lambda_2 L_f^2 h(x(t))}{L_g L_f^2 h(x(t))} \tag{11}$$

where λ_i, for $i \in \{0, 1, 2\}$, are the controller tuning parameters, induces the linear closed-loop dynamics

$$\dddot{y}(t) + \lambda_2 \ddot{y}(t) + \lambda_1 \dot{y}(t) + \lambda_0 y(t) = 0 \tag{12}$$

around any point $x \in D$.

Proof: The state equation for the closed-loop system

$$\begin{bmatrix} \dot{\bar{I}}(t) \\ \dot{\bar{E}}(t) \\ \dot{\bar{S}}(t) \end{bmatrix} = \begin{bmatrix} \bar{E}(t) \\ \bar{S}(t) \\ \varphi(\bar{x}(t)) - L_f^3 h(x(t)) - \lambda_0 \bar{I}(t) - \lambda_1 \bar{E}(t) - \lambda_2 \bar{S}(t) \end{bmatrix} \tag{13}$$

is obtained by introducing the control law (11) in (9) and taking into account the fact that $L_g L_f^2 h(x(t)) = -\mu\sigma\beta I(t) = -\mu\sigma\beta\bar{I}(t) \neq 0 \quad \forall x \in D$ and the coordinate transformation (8). Moreover, it follows by direct calculations that:

$$\begin{aligned} L_f^3 h(x(t)) &= \left[\sigma\beta(\mu+\omega) - (\mu+\gamma)^3\right] I(t) \\ &\quad + \sigma\left[(\mu+\gamma)^2 + (2\mu+\sigma+\gamma)(\mu+\sigma)\right] E(t) \\ &\quad - \sigma\beta_1\omega I(t)\left[I(t) + E(t)\right] + \sigma^2\beta_1 E(t)S(t) \\ &\quad - \sigma\beta_1(4\mu+\sigma+2\gamma+\omega)I(t)S(t) - \sigma\beta_1^2 I^2(t)S(t) \end{aligned} \tag{14}$$

One may express $L_f^3 h(x(t))$ in the state space defined by $\bar{x}(t)$ via the application of the reverse coordinate transformation to that in (8). Then, it follows directly that $L_f^3 h(x(t)) = \varphi(\bar{x}(t))$. Thus, the state equation of the closed-loop system in the state space defined by $\bar{x}(t)$ can be written as:

SCITEPRESS
SCIENCE AND TECHNOLOGY PUBLICATIONS

$$\dot{\overline{x}}(t) = A\overline{x}(t) \text{ with } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda_0 & -\lambda_1 & -\lambda_2 \end{bmatrix} \quad (15)$$

Furthermore, the output equation of the closed-loop system is $y(t) = C\overline{x}(t)$ with $C = [1 \ 0 \ 0]$ since $y(t) = I(t) = \overline{I}(t)$. From (15) and the closed-loop output equation, it follows that:

$$y^{(\ell)}(t) = CA^{\ell} e^{At}\overline{x}(0) \quad \text{for} \quad \ell \in \{0,\ 1,\ 2,\ 3\} \quad (16)$$

with ℓ denoting the order of the differentiation of $y(t)$. Finally, the dynamics of the closed-loop system (12) is directly obtained from (16).

Remarks 1: (i) The controller parameters λ_i, for $i \in \{0,\ 1,\ 2\}$, will be adjusted such that the roots of the closed-loop system characteristic polynomial $P(s) = \mathrm{Det}(sI_3 - A) = (s+r_1)(s+r_2)(s+r_3)$, with $I_3 \in \mathbb{R}^{3\times 3}$ denoting the identity matrix, be located at prescribed positions. i.e., $\lambda_i = \lambda_i(-r_j)$ for $i \in \{0,\ 1,\ 2\}$ and $j \in \{1,\ 2,\ 3\}$, with $(-r_j)$ denoting the desired roots of $P(s)$. If one of the control objectives is to guarantee the exponential stability of the closed-loop system then $\mathrm{Re}\{r_j\} > 0$ for all $j \in \{1,\ 2,\ 3\}$. Then, the values $\lambda_0 = r_1 r_2 r_3 > 0$, $\lambda_1 = r_1 r_2 + r_1 r_3 + r_2 r_3 > 0$ and $\lambda_2 = r_1 + r_2 + r_3 > 0$ for the controller parameters have to be chosen in order to achieve such a stability result. It implies that the strictly positivity of the controller parameters is a necessary condition for the exponential stability of the closed-loop system.
(ii) The control (11) may be rewritten as:

$$\begin{aligned} u(t) = {} & \frac{(\mu+\omega)\sigma\beta - (\mu+\gamma)^3 + \lambda_0 - \lambda_1(\mu+\gamma) + \lambda_2(\mu+\gamma)^2}{\mu\sigma\beta} \\ & - \frac{\omega}{\mu N}\left[I(t) + E(t)\right] - \frac{(3\mu+\sigma+2\gamma-\lambda_2)}{\mu N} S(t) \\ & + \frac{(\mu+\gamma)^2 + (2\mu+\sigma+\gamma)(\mu+\sigma) + \lambda_1 - \lambda_2(2\mu+\sigma+\gamma)}{\mu\beta}\frac{E(t)}{I(t)} \\ & + \frac{\sigma}{\mu N}\frac{E(t)S(t)}{I(t)} - \frac{\beta}{\mu N^2} I(t)S(t) \end{aligned} \quad (17)$$

by using (8) and (14).

(iii) The control law (11) is well-defined for all $x \in \mathbb{R}^3_{0+}$ except in the surface $I = 0$. However, the infection may be considered eradicated from the population once the infectious population strictly exceeds zero while it is smaller than one individual, so the vaccination strategy may be switched off when $0 < I(t) \le \delta < 1$. This fact implies that the singularity in the control law is not reached. i.e., such a control law is well-defined by the nature of the system. In this sense, the control law

$$u_p(t) = \begin{cases} u(t) & \text{for} \quad 0 \le t \le t_f \\ 0 & \text{for} \quad t > t_f \end{cases} \quad (18)$$

may be used instead of (11) in a practical situation. The signal $u(t)$ in (18) is given by the linearizing control law (11) while t_f denotes the eventual time instant after which the infection propagation may be assumed ended. Formally, such a time instant is

$$t_f \triangleq \mathrm{Min}\left\{t \in \mathbb{R}_0^+ \mid I(t_f) < \delta \ \text{ for some } 0 < \delta < 1\right\} \quad (19)$$

Then, the control action is maintained active while the infection persists in the population and it is switched off once the epidemics is eradicated.

3.1 Control Parameters Choice

The application of the control law (11), obtained from the exact input-output linearization strategy, makes the closed-loop dynamics of the infectious population be given by (12). Such a dynamics depends on the control parameters λ_i, for $i \in \{0,\ 1,\ 2\}$. Such parameters have to be appropriately chosen in order to guarantee the following suitable properties: (i) the stability of the controlled SEIR model, (ii) the eradication of the infection, i.e., the asymptotic convergence of $I(t)$ and $E(t)$ to zero as time tends to infinity and (iii) the positivity property of the controlled SEIR model under a vaccination based on such a control strategy. The following theorems related to the choice of the controller tuning parameter in order to meet such properties are proven.

Theorem 2: Assume that the initial condition $x(0) = [I(0) \ \ E(0) \ \ S(0)]^T \in \mathbb{R}^3_{0+}$ is bounded and all roots $(-r_j)$ for $j \in \{1,\ 2,\ 3\}$ of the characteristic polynomial $P(s)$ associated with the closed-loop dynamics (12) are of strictly negative real part via an appropriate choice of the free-design controller parameters $\lambda_i > 0$, for $i \in \{0,\ 1,\ 2\}$. Then, the control law (11) guarantees the exponential stability of the transformed controlled SEIR model (6)-(10) while achieving the eradication of the infection from the host population as $t \to \infty$. Moreover, the SEIR model (1)-(4) has the following properties: $E(t)$, $I(t)$, $S(t)I(t)$ and $S(t) + R(t) = N - [E(t) + I(t)]$ are bounded for all time, $E(t) \to 0$, $I(t) \to 0$, $S(t)+R(t) \to N$ and $S(t)I(t) \to 0$ exponentially as $t \to \infty$, and $I(t) = o(1/S(t))$.

Proof: The dynamics of the controlled SEIR model (12) can be equivalently rewritten with the state equation (15) and the output equation $y(t) = C\overline{x}(t)$, where $C = [1\ \ 0\ \ 0]$, by taking into account that $y(t) = \overline{I}(t)$, $\dot{y}(t) = \overline{E}(t)$ and $\ddot{y}(t) = \overline{S}(t)$. The initial condition $\overline{x}(0) = \left[\overline{I}(0)\ \ \overline{E}(0)\ \ \overline{S}(0)\right]^T$ in such a realization is bounded since it is related to $x(0)$ via the coordinate transformation (8) and $x(0)$ is assumed to be bounded. The controlled SEIR model is exponentially stable since the eigenvalues of the matrix A are the roots $-r_j < 0$ for $j \in \{1, 2, 3\}$ of $P(s)$ which are assumed to be in the open left-half plane. Then, the state vector $\overline{x}(t)$ exponentially converges to zero as $t \to \infty$ while being bounded for all time. Moreover, $I(t)$ and $E(t)$ are also bounded and converge exponentially to zero as $t \to \infty$ from the boundedness and exponential convergence to zero of $\overline{x}(t)$ as $t \to \infty$ according to the first and second equations of the coordinate transformation (8). Then, the infection is eradicated from the host population. Furthermore, the boundedness of $S(t)+R(t)$ follows from that of $E(t)$ and $I(t)$, and the fact that the total population is constant for all time. Also, the exponentially convergence of $S(t)+R(t)$ to the total population as $t \to \infty$ is derived from the exponential convergence to zero of $I(t)$ and $E(t)$ as $t \to \infty$, and the fact that $S(t)+E(t)+I(t)+R(t) = N \quad \forall t \in \mathbb{R}_{0+}$. Finally, from the third equation of (8), it follows that $S(t)I(t)$ is bounded and it converges exponentially to zero as $t \to \infty$ from the boundedness and convergence to zero of $I(t)$, $E(t)$ and $\overline{x}(t)$ as $t \to \infty$. The facts that $I(t) \to 0$ and $S(t)I(t) \to 0$ as $t \to \infty$ imply directly that $I(t) = o(1/S(t))$.

Remark 2: *Theorem 2* implies the existence of a finite time instant t_f after which the epidemics is eradicated when the vaccination control law (18) is used instead of (11). Concretely, such an existence derives from the fact that $I(t) \to 0$ as $t \to \infty$ via the application of the control law (11).

Theorem 3: Assume an initial condition for the SEIR model satisfying $R(0) \geq 0$, $x(0) \in \mathbb{R}^3_{0+}$, i.e., $I(0) \geq 0$, $E(0) \geq 0$ and $S(0) \geq 0$, and the constraint $S(0)+E(0)+I(0)+R(0) = N$. Assume also that some strictly positive real numbers r_j for $j \in \{1, 2, 3\}$ are chosen such that:

(a) $0 < r_1 < \mu + \text{Min}\{\sigma, \gamma\}$, $r_2 = \mu + \gamma$ and $r_3 > \mu + \text{Max}\{\sigma, \gamma\}$, so that $r_3 > r_2 > r_1 > 0$

(b) r_1 and r_3 satisfy the inequalities:

$$\begin{cases} r_1 + r_3 \geq 2\mu + \sigma + \gamma + \beta - \omega \\ r_1 r_3 \geq (\mu+\sigma)(r_1 + r_3) + (\gamma - \sigma)(2\mu + \sigma + \gamma) - (\mu + \gamma)^2 \\ (r_3 - r_1)(r_3 - \mu - \gamma) \geq \sigma\beta \end{cases}$$

Then:

(i) the application of the control law (11) to the SEIR model guarantees that the epidemics is asymptotically eradicated from the population while $I(t) \geq 0$, $E(t) \geq 0$ and $S(t) \geq 0 \quad \forall t \in \mathbb{R}_{0+}$, and

(ii) the application of the control law (18) guarantees the epidemics eradication after a finite time t_f, the positivity of the controlled SEIR epidemic model $\forall t \in [0, t_f)$ and that $u(t) = V(t) \geq 1 \quad \forall t \in [0, t_f)$ so that $u(t) \geq 0 \quad \forall t \in \mathbb{R}_{0+}$, provided that the controller tuning parameters λ_i, $i \in \{0, 1, 2\}$, are chosen so that $(-r_j)$, $j \in \{1, 2, 3\}$, be the roots of the characteristic polynomial $P(s)$ associated with the closed loop dynamics (12).

Proof: **(i)** On one hand, the epidemics asymptotic eradication is proved by following the same reasoning that in *Theorem 2*. On the other hand, the dynamics (12) of the controlled SEIR model can be written in the state space defined by $\overline{x}(t) = \left[\overline{I}(t)\ \ \overline{E}(t)\ \ \overline{S}(t)\right]^T$ as in (15). From such a realization and taking into account the first equation in (8) and that $(-r_j)$ for $j \in \{1, 2, 3\}$ are the eigenvalues of A, it follows that:

$$I(t) = \overline{I}(t) = y(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} \tag{20}$$

$\forall t \in \mathbb{R}_{0+}$ for some constants c_j for $j \in \{1, 2, 3\}$ being dependent on the initial conditions $y(0)$, $\dot{y}(0)$ and $\ddot{y}(0)$. In turn, such initial conditions are related to the initial conditions of the SEIR model in its original realization, i.e., in the state space defined by $x(t) = [I(t)\ \ E(t)\ \ S(t)]^T$ via (8). The constants c_j

for $j \in \{1, 2, 3\}$ can be obtained by solving the following set of linear equations:

$$\begin{aligned}
\overline{I}(0) &= y(0) = c_1 + c_2 + c_3 = I(0) \\
\overline{E}(0) &= \dot{y}(0) = -(c_1 r_1 + c_2 r_2 + c_3 r_3) = -(\mu+\gamma)I(0) + \sigma E(0) \\
\overline{S}(0) &= \ddot{y}(0) = c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2 \\
&= (\mu+\gamma)^2 I(0) - \sigma(2\mu+\sigma+\gamma)E(0) + \sigma\beta_1 I(0)S(0)
\end{aligned} \tag{21}$$

where (8) and (20) have been used. Such equations can be compactly written as $R_p \cdot K = M$ where:

$$R_p = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix}, \quad K = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \text{and}$$
$$M = \begin{bmatrix} I(0) \\ (\mu+\gamma)I(0) - \sigma E(0) \\ (\mu+\gamma)^2 I(0) - \sigma(2\mu+\sigma+\gamma)E(0) + \sigma\beta_1 I(0)S(0) \end{bmatrix} \tag{22}$$

Then, once the desired roots of the characteristic equation of the closed-loop dynamics have been prefixed the constants c_j for $j \in \{1, 2, 3\}$ of the time-evolution of $I(t)$ are obtained from $K = R_p^{-1} M$ since R_p is a non-singular matrix, i.e., an invertible matrix. In this sense, note that $\mathrm{Det}(R_p) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2) \neq 0$ since R_p is a Vandermonde matrix (Fulton and Harris, 1991) and the roots $(-r_j)$ for $j \in \{1, 2, 3\}$ have been chosen different among them. Namely:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \dfrac{F(r_2, r_3) I(0) + \sigma G(r_2, r_3) E(0) + \sigma\beta_1 I(0)S(0)}{(r_2 - r_1)(r_3 - r_1)} \\ -\dfrac{F(r_1, r_3) I(0) + \sigma G(r_1, r_3) E(0) + \sigma\beta_1 I(0)S(0)}{(r_2 - r_1)(r_3 - r_2)} \\ \dfrac{F(r_1, r_2) I(0) + \sigma G(r_1, r_2) E(0) + \sigma\beta_1 I(0)S(0)}{(r_3 - r_1)(r_3 - r_2)} \end{bmatrix} \tag{23}$$

where $F: \mathbb{R}_+^2 \to \mathbb{R}$ and $G: \mathbb{R}_+^2 \to \mathbb{R}$ are defined as:

$$\begin{aligned}
F(v, w) &\triangleq vw - (\mu+\gamma)(v+w) + (\mu+\gamma)^2 \\
G(v, w) &\triangleq v + w - (2\mu+\sigma+\gamma)
\end{aligned} \tag{24}$$

Note that $c_1 = \dfrac{\sigma(r_3 - \mu - \gamma)E(0) + \sigma\beta_1 I(0)S(0)}{(\mu+\gamma-r_1)(r_3 - r_1)} > 0$ since $I(0) \geq 0$, $E(0) \geq 0$, $S(0) \geq 0$, $F(r_2, r_3) = 0$, $G(r_2, r_3) = r_3 - \mu - \gamma > 0$, $\mu + \gamma - r_1 > 0$ and $r_3 - r_1 > 0$ by taking into account the constraints in (a). On one hand, $I(t) \geq 0$ $\forall t \in \mathbb{R}_{0+}$ is proved directly from (20) as follows. One 'a priori' knows that $c_1 > 0$. However, the sign of both c_2 and c_3 may not be 'a priori' determined from the initial conditions and constraints in (a). The following four cases may be possible: (i) $c_2 \geq 0$ and $c_3 \geq 0$, (ii) $c_2 \geq 0$ and $c_3 < 0$, (iii) $c_2 < 0$ and $c_3 \geq 0$, and (iv) $c_2 < 0$ and $c_3 < 0$. For the cases (i) and (ii), i.e., if $c_2 \geq 0$, it follows from (20) that:

$$\begin{aligned}
I(t) &= c_1 e^{-r_1 t} + c_2 e^{-r_2 t} + \left[I(0) - c_1 - c_2\right] e^{-r_3 t} \\
&= c_1 \left(e^{-r_1 t} - e^{-r_3 t}\right) + c_2 \left(e^{-r_2 t} - e^{-r_3 t}\right) + I(0) e^{-r_3 t} \geq 0
\end{aligned} \tag{25}$$

$\forall t \in \mathbb{R}_{0+}$ where the facts that $I(0) = c_1 + c_2 + c_3 \geq 0$ and, $e^{-r_1 t} - e^{-r_3 t} \geq 0$ and $e^{-r_2 t} - e^{-r_3 t} \geq 0$ $\forall t \in \mathbb{R}_{0+}$ since $r_1 < r_2 < r_3$ have been taken into account. For the case (iii), i.e., if $c_2 < 0$ and $c_3 \geq 0$, it follows that:

$$\begin{aligned}
I(t) &= \left[I(0) - c_2 - c_3\right] e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} \\
&= \left[I(0) - c_3\right] e^{-r_1 t} + c_2 \left(e^{-r_2 t} - e^{-r_1 t}\right) + c_3 e^{-r_3 t} \geq 0
\end{aligned} \tag{26}$$

$\forall t \in \mathbb{R}_{0+}$ by taking into account that $I(0) = c_1 + c_2 + c_3$, $e^{-r_2 t} - e^{-r_1 t} \leq 0$ $\forall t \in \mathbb{R}_{0+}$ since $r_1 < r_2$ and the fact that:

$$\begin{aligned}
&I(0) - c_3 = \\
&\frac{\left[(r_3 - r_1)(r_3 - \mu - \gamma) - \sigma\beta_1 S(0)\right] I(0) + \sigma(\mu + \gamma - r_1)E(0)}{(r_3 - r_1)(r_3 - \mu - \gamma)} \geq 0
\end{aligned} \tag{27}$$

where (23), (24), $G(r_1, r_2) = r_1 - \mu - \gamma < 0$, $F(r_1, r_2) = 0$ and the constraints in (a) and (b) have been used. In particular, the coefficient multiplying to $I(0)$ in (27) is non-negative if r_1 and r_3 satisfy the third inequality of the constraints (b) by taking into account $\sigma\beta_1 S(0) = \sigma\beta S(0)/N \leq \sigma\beta$ and $S(0) \leq N$. This later inequality is directly implied by $I(0) \geq 0$, $E(0) \geq 0$, $S(0) \geq 0$, $R(0) \geq 0$ and $N = I(0) + E(0) + S(0) + R(0)$. Finally, for the case (iv), i.e., if $c_2 < 0$ and $c_3 < 0$, it follows that:

$$\begin{aligned}
I(t) &= \left[I(0) - c_2 - c_3\right] e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} \\
&= I(0) e^{-r_1 t} + c_2 \left(e^{-r_2 t} - e^{-r_1 t}\right) + c_3 \left(e^{-r_3 t} - e^{-r_1 t}\right) \geq 0
\end{aligned} \tag{28}$$

$\forall t \in \mathbb{R}_{0+}$, where $e^{-r_2 t} - e^{-r_1 t} \leq 0$ and $e^{-r_3 t} - e^{-r_1 t} \leq 0$, since $r_1 < r_2 < r_3$, and $I(0) = c_1 + c_2 + c_3 \geq 0$ have been taken into account. In summary, $I(t) \geq 0$ $\forall t \in \mathbb{R}_{0+}$ if all partial populations are initially non-negative and the roots $(-r_j)$, for $j \in \{1, 2, 3\}$, of the closed-loop characteristic polynomial satisfy the constraints in (a) and (b). On the other hand, one

SCITEPRESS
SCIENCE AND TECHNOLOGY PUBLICATIONS

obtains from the reverse coordinate transformation to (8) and (20) that:

$$
\begin{aligned}
E(t) &= \frac{1}{\sigma}\left[\bar{E}(t) + (\mu+\gamma)\bar{I}(t)\right] = \frac{1}{\sigma}\sum_{j=1}^{3} c_j(\mu+\gamma-r_j)e^{-r_j t} \\
S(t) &= \frac{\bar{S}(t) + (\mu+\sigma)(\mu+\gamma)\bar{I}(t) + (2\mu+\sigma+\gamma)\bar{E}(t)}{\sigma\beta_1 \bar{I}(t)} \\
&= \frac{\sum_{j=1}^{3} c_j\left[r_j^2 - (2\mu+\sigma+\gamma)r_j + (\mu+\sigma)(\mu+\gamma)\right]e^{-r_j t}}{\sigma\beta_1 I(t)}
\end{aligned} \tag{29}
$$

from the facts that $\bar{E}(t) = \dot{\bar{I}}(t)$ and $\bar{S}(t) = \ddot{\bar{I}}(t)$. If one fixes the parameter $r_2 = \mu+\gamma$ then:

$$
\begin{aligned}
E(t) &= \frac{1}{\sigma}\left[c_1(\mu+\gamma-r_1)e^{-r_1 t} + c_3(\mu+\gamma-r_3)e^{-r_3 t}\right] \\
S(t) &= \frac{1}{\sigma\beta_1 I(t)}\left\{\begin{aligned} &c_1\left[r_1^2 - (2\mu+\sigma+\gamma)r_1 + (\mu+\sigma)(\mu+\gamma)\right]e^{-r_1 t} \\ &+c_3\left[r_3^2 - (2\mu+\sigma+\gamma)r_3 + (\mu+\sigma)(\mu+\gamma)\right]e^{-r_3 t}\end{aligned}\right\}
\end{aligned} \tag{30}
$$

where the function $H:\mathbb{R}_+ \to \mathbb{R}$ defined as:

$$H(v) \triangleq v^2 - (2\mu+\sigma+\gamma)v + (\mu+\sigma)(\mu+\gamma) \tag{31}$$

is zero for $v = r_2 = \mu+\gamma$ has been used. From the first equation in (30), it follows that $c_3(\mu+\gamma-r_3) = \sigma E(0) - c_1(\mu+\gamma-r_1)$ and then:

$$E(t) = \frac{c_1(\mu+\gamma-r_1)\left(e^{-r_1 t} - e^{-r_3 t}\right) + \sigma E(0)e^{-r_3 t}}{\sigma} \geq 0 \tag{32}$$

$\forall t \in \mathbb{R}_{0+}$ by applying such a relation between c_1 and c_3 in (30) and by taking into account that $c_1(\mu+\gamma-r_1) > 0$, $E(0) \geq 0$ and $e^{-r_1 t} - e^{-r_3 t} \geq 0$ $\forall t \in \mathbb{R}_{0+}$ since $r_1 < r_3$. In this way, the non-negativity of $E(t)$ has been proven. From the second equation in (30), it follows that $c_3 H(r_3) = \sigma\beta_1 I(0)S(0) - c_1 H(r_1)$ and then:

$$S(t) = \frac{c_1 H(r_1)\left(e^{-r_1 t} - e^{-r_3 t}\right) + \sigma\beta_1 I(0)S(0)e^{-r_3 t}}{\sigma\beta_1 I(t)} \geq 0 \tag{33}$$

$\forall t \in \mathbb{R}_{0+}$ by applying such a relation between c_1 and c_3 in (30) and by taking into account that $c_1 H(r_1) > 0$ since $r_1 < \mu + \mathrm{Min}\{\sigma, \gamma\}$, $I(0) \geq 0$, $S(0) \geq 0$, $I(t) \geq 0$ and $e^{-r_1 t} - e^{-r_3 t} \geq 0$ $\forall t \in \mathbb{R}_{0+}$ since $r_1 < r_3$. In this way, the non-negativity of $S(t)$ has been proven. Note that the function $H(v)$ defined in (31) is an upper-open parabola zero-valued for $v_1 = \mu+\sigma$ and $v_2 = \mu+\gamma$ so $H(r_1) > 0$ from the assumption that $r_1 < \mu + \mathrm{Min}\{\sigma, \gamma\}$.

(ii) On one hand, if the control law (18) is used instead of that in (11) then the time evolution of the infectious population is also given by (20) while the control action is active. Thus, $I(t) \to 0$ as $t \to \infty$ in (20) implies directly the existence of a finite time instant t_f at which the control (18) switches off. Obviously, the non-negativity of $I(t)$, $E(t)$ and $S(t)$ $\forall t \in [0, t_f]$ is proved by following the same reasoning used in the part (i) of the current theorem. The non-negativity of $R(t)$ $\forall t \in [0, t_f]$ is proven by using continuity arguments. In this sense, if $R(t)$ reaches negative values for some $t \in [0, t_f]$ starting from an initial condition $R(0) \geq 0$ then $R(t)$ passes through zero, i.e., there exists at least a time instant $t_0 \in [0, t_f)$ such that $R(t_0) = 0$. Then, it follows from (4) that:

$$
\begin{aligned}
\dot{R}(t_0) &= \gamma I(t_0) + \mu N V(t_0) \\
&= \gamma I(t_0) + \frac{\mu\sigma\beta + \lambda_0 - \lambda_1(\mu+\gamma) + \lambda_2(\mu+\gamma)^2 - (\mu+\gamma)^3}{\sigma\beta}N \\
&+ (\lambda_2 + \omega - 3\mu - \sigma - 2\gamma)S(t_0) + \sigma\frac{E(t_0)S(t_0)}{I(t_0)} - \frac{\beta}{N}I(t_0)S(t_0) \\
&+ \frac{(\mu+\gamma)^2 + (2\mu+\sigma+\gamma)(\mu+\sigma) + \lambda_1 - \lambda_2(2\mu+\sigma+\gamma)}{\beta}N\frac{E(t_0)}{I(t_0)}
\end{aligned} \tag{34}
$$

by introducing the control law (18), taking into account that $V(t) = u(t)$ and where the fact that $I(t_0) + E(t_0) + S(t_0) = N$, since $R(t_0) = 0$, has been used. Moreover, the non-negativity of $I(t)$, $E(t)$ and $S(t)$ $\forall t \in [0, t_f]$ as it has been previously proven, implies that $I(t_0) \leq N$, $E(t_0) \leq N$ and $S(t_0) \leq N$. Also, $I(t_0) \geq \delta > 0$ since $t_0 < t_f$ and from the definition of t_f in (19). Then, one obtains:

$$
\begin{aligned}
\dot{R}(t_0) &\geq \gamma I(t_0) + \frac{\mu\sigma\beta + \lambda_0 - \lambda_1(\mu+\gamma) + \lambda_2(\mu+\gamma)^2 - (\mu+\gamma)^3}{\sigma\beta}N \\
&+ (\lambda_2 + \omega - 3\mu - \sigma - 2\gamma - \beta)S(t_0) + \sigma\frac{E(t_0)S(t_0)}{I(t_0)} \\
&+ \frac{(\mu+\gamma)^2 + (2\mu+\sigma+\gamma)(\mu+\sigma) + \lambda_1 - \lambda_2(2\mu+\sigma+\gamma)}{\beta}N\frac{E(t_0)}{I(t_0)}
\end{aligned} \tag{35}
$$

from (34). The controller tuning parameter λ_i for $i \in \{0, 1, 2\}$ are related to the roots $(-r_j)$, for $j \in \{1, 2, 3\}$, of the closed-loop characteristic polynomial $P(s)$, see *Remark 1 (i)*, by:

$$\lambda_0 = r_1 r_2 r_3 \;;\; \lambda_1 = r_1 r_2 + r_1 r_3 + r_2 r_3 \;;\; \lambda_2 = r_1 + r_2 + r_3 \tag{36}$$

The assignment of r_j for $j \in \{1, 2, 3\}$ such that the constraints (a) and (b) are fulfilled implies that:

$$\begin{aligned}&\lambda_2+\omega-3\mu-\sigma-2\gamma-\beta\geq 0\\&(\mu+\gamma)^2+(2\mu+\sigma+\gamma)(\mu+\sigma)+\lambda_1-\lambda_2(2\mu+\sigma+\gamma)\geq 0\\&\mu\sigma\beta+\lambda_0-\lambda_1(\mu+\gamma)+\lambda_2(\mu+\gamma)^2-(\mu+\gamma)^3=\mu\sigma\beta\geq 0\end{aligned} \quad (37)$$

Then, $\dot{R}(t_0)\geq 0$ by taking into account (37) in (35). The facts that $R(t)\geq 0 \quad \forall t\in[0,\ t_0)$, $R(t_0)=0$ and $\dot{R}(t_0)\geq 0$ imply that $R(t)\geq 0 \quad \forall t\in[0,\ t_f]$ via complete induction.

On the other hand, from (17) and (18), it follows:

$$\begin{aligned}u(t)=&\frac{\mu\sigma\beta-(\mu+\gamma)^3+\lambda_0-\lambda_1(\mu+\gamma)+\lambda_2(\mu+\gamma)^2}{\mu\sigma\beta}+\frac{\omega}{\mu N}R(t)\\&-\frac{(3\mu+\sigma+2\gamma-\omega-\lambda_2)}{\mu N}S(t)+\frac{\sigma}{\mu N}\frac{E(t)S(t)}{I(t)}-\frac{\beta}{\mu N^2}I(t)S(t)\\&+\frac{(\mu+\gamma)^2+(2\mu+\sigma+\gamma)(\mu+\sigma)+\lambda_1-\lambda_2(2\mu+\sigma+\gamma)}{\mu\beta}\frac{E(t)}{I(t)}\end{aligned} \quad (38)$$

$\forall t\in[0,\ t_f]$ by taking into account that $S(t)+E(t)+I(t)+R(t)=N$. Moreover:

$$\begin{aligned}u(t)\geq&\frac{\mu\sigma\beta-(\mu+\gamma)^3+\lambda_0-\lambda_1(\mu+\gamma)+\lambda_2(\mu+\gamma)^2}{\mu\sigma\beta}\\&+\frac{(\mu+\gamma)^2+(2\mu+\sigma+\gamma)(\mu+\sigma)+\lambda_1-\lambda_2(2\mu+\sigma+\gamma)}{\mu\beta}\frac{E(t)}{I(t)}\\&+\frac{\lambda_2+\omega-3\mu-\sigma-2\gamma-\beta}{\mu N}S(t) \qquad \forall t\in[0,\ t_f]\end{aligned} \quad (39)$$

where the facts that $0<\delta\leq I(t)\leq N$, $E(t)\geq 0$, $S(t)\geq 0$ and $R(t)\geq 0 \quad \forall t\in[0,\ t_f]$ have been used. If the roots of the polynomial $P(s)$ satisfy the conditions in (a) and (b), it follows from (39) that:

$$\begin{aligned}u(t)\geq\ &1+\frac{\lambda_2+\omega-3\mu-\sigma-2\gamma-\beta}{\mu N}S(t)\\&+\frac{(\mu+\gamma)^2+(2\mu+\sigma+\gamma)(\mu+\sigma)+\lambda_1-\lambda_2(2\mu+\sigma+\gamma)}{\mu\beta}\frac{E(t)}{I(t)}\geq 1\end{aligned} \quad (40)$$

$\forall t\in[0,\ t_f]$ by taking into account the third equation in (37) and the non-negativity of $S(t)$, $E(t)$ and $I(t)$ $\forall t\in[0,\ t_f]$. Finally, it follows that $u(t)\geq 0 \quad \forall t\in\mathbb{R}_0^+$ from (18) and (40).

4 SIMULATION RESULTS

An example based on an outbreak of influenza in a British boarding school in early 1978 (Keeling and Rohani, 2008) is used to illustrate the theoretical results presented. Such an epidemic can be described by the SEIR mathematical model (1)-(4) with $\mu^{-1}=70\ \text{years}=25550\ \text{days}$, $\beta=1.66$ per day, $\sigma^{-1}=\gamma^{-1}=2.2$ days and $\omega^{-1}=15$ days. A total population of $N=1000$ boys is considered with the initial conditions $S(0)=800$ boys, $E(0)=100$ boys, $I(0)=60$ boys and $R(0)=40$ boys. Two sets of simulation results are presented to compare the evolution of the SEIR mathematical model populations in two different situations, namely: when no vaccination control actions are applied and if a control action based on the feedback input-output linearization approach is applied.

4.1 Epidemic Evolution without Vaccination

The time evolution of the respective populations is displayed in Figure 1. The model tends to its endemic equilibrium point as time tends to infinity. There are susceptible, infected and infectious populations at such an equilibrium point. As a consequence, a vaccination control action has to be applied in order to eradicate the epidemics.

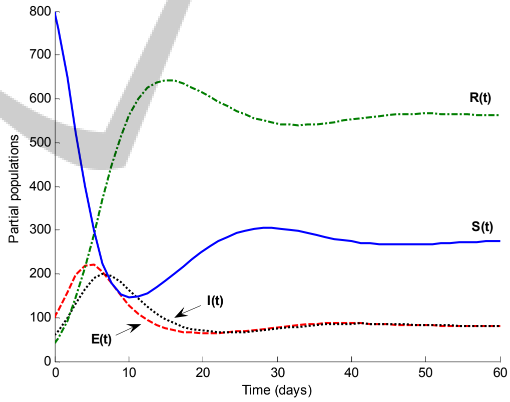

Figure 1: Time evolution of the individual populations without vaccination.

4.2 Epidemic Evolution with a Feedback Control Law

The control law given by (18)-(19) is applied with $\delta=0.001$ and the free-design controller parameters λ_i, for $i\in\{0,\ 1,\ 2\}$, being chosen so that the roots of the characteristic polynomial $P(s)$ associated with the closed-loop dynamics (12) are $-r_1=-\gamma$, $-r_2=-(\mu+\gamma)$ and $-r_3=-(2\mu+\gamma)$. Such values for λ_i are obtained from (36). The time evolution of the respective populations is displayed in Figure 2 and the vaccination function in Figure 3.

SCITEPRESS SCIENCE AND TECHNOLOGY PUBLICATIONS

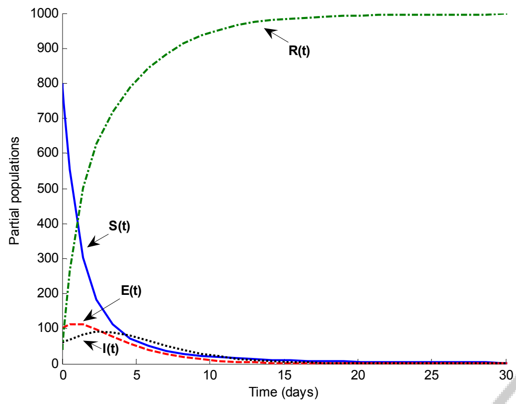

Figure 2: Time evolution of the individual populations with the vaccination control action.

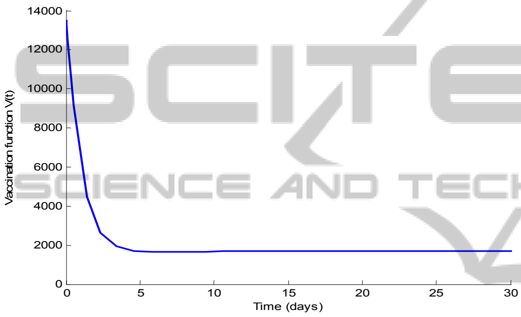

Figure 3: Time evolution of the vaccination function.

The vaccination control action achieves the control objectives as it is seen in Figure 2. The infection is eradicated from the population since both infectious and infected populations converge rapidly to zero. Also, the susceptible population converges to zero while the removed-by-immunity population tracks asymptotically the whole population as time tends to infinity. Such a result is coherent with the result proved in *Theorem 3*. Moreover, the positivity of the system is maintained for all time as it can be seen from such figures. Such a property is satisfied although all constraints of the assumption (b) of *Theorem 3* are not fulfilled by the system parameters and the chosen control parameters. However, such a result is coherent since such constraints are sufficient but not necessary to prove the positivity of the system. The switched off time instant for the vaccination is $t_f \approx 30$ days .

The time evolution of the respective partial populations under the application of the developed control strategy is similar to that obtained under the use of other vaccination strategies proposed in other papers by our research group, for instance, in (De la Sen and Alonso-Quesada, 2010). The purpose of the paper is to present an alternative method to obtain a vaccination control law from linearization techniques in the SEIR epidemic model.

5 CONCLUSIONS

A vaccination control strategy based on feedback input-output linearization techniques has been proposed to fight against the propagation of epidemic diseases. A SEIR model with known parameters is used to describe the propagation of the disease. The stability and the positivity properties of the closed-loop system as well as the eradication of the epidemics have been proved. *Such a strategy has a main drawback, namely, the control law needs the knowledge of the true values of the susceptible, infected and infectious populations at all time instants which may not be available in certain real situations. Future researches are going to deal with alternative approaches useful to overcome such a drawback. For instance, an observer may be added to estimate online all the partial populations*. Also, the application of the current approach and similar non-linear techniques to other disease propagation models can be considered.

ACKNOWLEDGEMENTS

The authors thank to the Spanish Ministry of Education by its support of this work through Grant DPI2009-07197 and to the Basque Government by its support through Grants IT378-10, SAIOTEK SPE07UN04 and SAIOTEK SPE09UN12.

REFERENCES

De la Sen, M., Agarwal, R. P., Ibeas, A., Alonso-Quesada, S., 2010. On a generalized time-varying SEIR epidemic model with mixed point and distributed time-varying delays and combined regular and impulsive vaccination controls. *Advances in Difference Equations 2010, Article ID 281612,* 42 pages, doi:10.1155/2010/281612.

De la Sen, M., Agarwal, R. P., Ibeas, A., Alonso-Quesada, S., 2011. On the existence of equilibrium points, boundedness, oscillating behavior and positivity of a SVEIRS epidemic model under constant and impulsive vaccination. *Advances in Difference Equations 2011, Article ID 748608,* 32 pages, doi:10.1155/2011/748608.

De la Sen, M., Alonso-Quesada, S., 2010. On vaccination control tools for a general SEIR-epidemic model. In *18th Mediterranean Conference on Control &*

Automation (MED'10), pp. 1322-1328, doi:10.1109/MED.2010.5547865.

Fulton, W., Harris, J. D., 1991. Representation Theory, *Springer*, New York.

Isidori, A., 1995. Nonlinear Control Systems, *Springer-Verlag*, London.

Jumpen, W., Orankitjaroen, S., Boonkrong, P., Wiwatanapataphee, B., 2011. SEIQR-SIS epidemic network model and its stability. *International Journal of Mathematics and Computers in Simulation 5, pp. 326-333*.

Keeling, M. J., Rohani, P., 2008. *Modeling Infectious Diseases in Humans and Animals*, Princeton University Press, Princeton and Oxford.

Li, M. Y., Graef, J. R., Wang, L., Karsai, J., 1999. Global dynamics of a SEIR model with varying total population size. *Mathematical Biosciences* 160, pp. 191-213.

Makinde, O. D., 2007. Adomian decomposition approach to a SIR epidemic model with constant vaccination strategy. *Applied Mathematics and Computation* 184, pp. 842-848.

Mollison, D., 2003. *Epidemic Models: Their Structure and Relation to Data*, Publications of the Newton Institute, Cambridge University Press.

Mukhopadhyay, B., Bhattacharyya, R., 2007. Existence of epidemic waves in a disease transmission model with two-habitat population. *International Journal of Systems Science* 38, pp. 699-707.

Safi, M. A., Gumel, A. B., 2011. Mathematical analysis of a disease transmission model with quarantine, isolation and an imperfect vaccine. *Computers and Mathematics with Applications* 61, pp. 3044-3070.

Song, X. Y., Jiang, Y., Wei, H. M., 2009. Analysis of a saturation incidence SVEIRS epidemic model with pulse and two time delays. *Applied Mathematics and Computation* 214, pp. 381-390.

Xu, R., Ma, Z., Wang, Z., 2010. Global stability of a delayed SIRS epidemic model with saturation incidence and temporary immunity. *Computers and Mathematics with Applications* 59, pp. 3211-3221.

Zhang, T. L., Liu, J. L., Teng, Z. D., 2009. Dynamic behaviour for a nonautonomous SIRS epidemic model with distributed delays. *Applied Mathematics and Computation* 214, pp. 624-631.

SCITEPRESS SCIENCE AND TECHNOLOGY PUBLICATIONS