COMPUTING VALID INEQUALITIES FOR GENERAL INTEGER PROGRAMS USING AN EXTENSION OF MAXIMAL DUAL FEASIBLE FUNCTIONS TO NEGATIVE ARGUMENTS

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Abstract: Dual feasible functions (DFFs) were used with much success to compute bounds for several combinatorial optimization problems and to derive valid inequalities for some linear integer programs. A major limitation of these functions is that their domain remains restricted to the set of positive arguments. To tackle more general linear integer problems, the extension of DFFs to negative arguments is essential. In this paper, we show how these functions can be generalized to this case. We explore the properties required for DFFs with negative arguments to be maximal, we analyze additional properties of these DFFs, we prove that many classical maximal DFFs cannot be extended in this way, and we present some non-trivial examples.

1 INTRODUCTION

Dual feasible functions (DFFs) were introduced in (Johnson, 1973), and used since then to compute bounds for different combinatorial optimization problems and valid inequalities for integer linear programs (see for example (Nemhauser and Wolsey, 1998), (Fekete and Schepers, 2001) and (Clautiaux et al., 2010)). To ensure the quality of the bounds, one has to resort to maximal DFFs. The criteria for a DFF to be maximal were described first by Carlier and Néron in (Carlier and Néron, 2007). Recently, in (Rietz et al., 2011), some of the strongest maximal DFFs of the literature were analyzed with respect to their worst cases in the computation of lower bounds.

In (Clautiaux et al., 2010), the authors showed that DFFs could be used to compute valid inequalities for integer programs. However, all the DFFs developed until now apply exclusively to positive data. This fact constitutes a clear restriction for their use in the computation of valid inequalities for general integer programs. The extension of DFFs to negative arguments is not trivial. It raises different issues that are addressed in this paper.

Example 1. *The function $f_{FS,1}$ was defined in (Fekete and Schepers, 2001) for $0 \le x \le 1$ and $k \in \mathbb{N}\setminus\{0\}$ as*

$$f_{FS,1}(x) = \begin{cases} x & \text{if } (k+1)x \in \mathbb{Z} \\ \lfloor (k+1)x \rfloor / k & \text{otherwise} \end{cases},$$

but it cannot be extended as a maximal DFF to $x \in \mathbb{R}$ with the same formula. Let $0 < \varepsilon < \frac{1}{k+1}$. Then $f_{FS,1}(\frac{k+2}{k+1} - \varepsilon) = \frac{k+1}{k} > \frac{k+2}{k+1} = f_{FS,1}(\frac{k+2}{k+1})$, and hence $f_{FS,1}$ would not be monotonous.

The paper is organized as follows. The definition and the characteristics of maximal DFFs with a domain that is the whole set of real numbers are introduced in the next section. Additional properties of these functions and some tools to construct maximal DFFs follow in Section 3. Several non-trivial examples of general DFFs (with positive and negative arguments) are presented in Section 4. In Section 5, we show through an example how these functions apply to general integer linear programs.

2 DEFINITIONS AND ESSENTIAL PROPERTIES

The notion of (maximal/extremal) dual feasible function can be extended to domain and range $\mathbb{R}$. The defining conditions of these functions remain nearly the same as for the DFFs restricted to positive

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arguments. These conditions are stated in the sequel.

Definition 1. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a* dual feasible function (DFF)*, if for all* $n \in \mathbb{N}$ *and all* $x_1, \ldots, x_n \in \mathbb{R}$ *with* $\sum_{i=1}^{n} x_i \leq 1$*, we have* $\sum_{i=1}^{n} f(x_i) \leq 1$.

Definition 2. *A DFF* $f : \mathbb{R} \to \mathbb{R}$ *is a* maximal dual feasible function (MDFF)*, if there is no other DFF* $g : \mathbb{R} \to \mathbb{R}$ *with* $g(x) \geq f(x)$ *for all* $x \in \mathbb{R}$.

Definition 3. *A MDFF* $f : \mathbb{R} \to \mathbb{R}$ *is* extremal*, if any MDFFs* $g, h : \mathbb{R} \to \mathbb{R}$ *with* $2f(x) = g(x) + h(x)$ *for all* $x \in \mathbb{R}$ *are necessarily identical to* f.

Note that the identity function f_{id} is clearly a DFF.

Any DFF $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

- $\sum_{i=1}^{n} x_i \leq 0 \Longrightarrow \sum_{i=1}^{n} f(x_i) \leq 0$, especially $f(x) \leq 0$ for all $x \leq 0$
- $0 < x \leq 1 \Longrightarrow f(x) \leq 1/\lfloor 1/x \rfloor$
- if $f(x_1) > 0$ for a certain $x_1 \in \mathbb{R}$, then $f(x) < 0$ for all $x < 0$.

In a general integer linear program with the decision variables $x_1, \ldots, x_n \in \mathbb{N}$ and a set of coefficients $a_1, \ldots, a_n \in \mathbb{R}$, if the inequality $\sum_{i=1}^{n} a_i x_i \leq 1$ is required, then the following inequality obtained by applying a DFF $f : \mathbb{R} \to \mathbb{R}$

$$\sum_{i=1}^{n} x_i \times f(a_i) \leq 1$$

is valid because

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} \sum_{j=1}^{x_i} a_i \leq 1,$$

and hence $1 \geq \sum_{i=1}^{n} \sum_{j=1}^{x_i} f(a_i)$.

In the following proposition, we show that MDFFs with domain $\mathbb{R}$ are different from those with domain $[0,1]$.

Proposition 1. *For every* $c \in [0,1]$*, the function* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) := cx$ *for all* $x \in \mathbb{R}$ *is a MDFF.*

Proof. For this proof, we resort to the definitions 1 and 2. Let $n \in \mathbb{N} \setminus \{0\}$ and $x_1, \ldots, x_n \in \mathbb{R}$ with $\sum_{i=1}^{n} x_i \leq 1$ be given. Then, $\sum_{i=1}^{n} f(x_i) = c \times \sum_{i=1}^{n} x_i \leq c$, and hence f is a DFF. Suppose that there is a DFF $g : \mathbb{R} \to \mathbb{R}$ with $g(x) \geq cx$, for all $x \in \mathbb{R}$, and $g(x_0) > cx_0$ for a certain $x_0 \in \mathbb{R}$. Since $g(-x_0) \geq f(-x_0)$, it follows that $g(x_0) + g(-x_0) > cx_0 - cx_0 = 0$. That is a contradiction. Since f is not dominated by another DFF g, the assertion follows. □

The following theorem characterizes the MDFFs. It is inspired on the theorem by Carlier and Néron (Carlier and Néron, 2007), but here the domain and range are $\mathbb{R}$ and not only an interval.

Theorem 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a given function.*

(a) If f *satisfies the following conditions, then* f *is a MDFF:*

1. $f(0) = 0$*;*
2. f *is superadditive, i.e. for all* $x_1, x_2 \in \mathbb{R}$*, it holds that*
$$f(x_1 + x_2) \geq f(x_1) + f(x_2); \tag{1}$$
3. *there is an* $\varepsilon > 0$*, such that* $f(x) \geq 0$ *for all* $x \in (0, \varepsilon)$*;*
4. *for all* $x \in \mathbb{R}$*, it holds that*
$$f(x) + f(1-x) = 1. \tag{2}$$

(b) If f *is a MDFF, then the above properties (1.)–(3.) hold for* f*, but not necessarily (4.);*

(c) If f *satisfies the above conditions (1.)–(3.), then* f *is monotonously increasing.*

Proof. The proof is made in the following order: first, we prove (c), then (b), and finally (a) is proved.

(c) If f satisfies the first three conditions, then for any $x > 0$ it follows that $n := \lfloor x/\varepsilon \rfloor + 1 \in \mathbb{N} \setminus \{0\}$ and $0 < x/n < \varepsilon$. Hence, we have $f(x/n) \geq 0$ and $f(x) \geq n \times f(x/n) \geq 0$. Therefore, the monotonicity follows immediately from $f(x_2) \geq f(x_1) + f(x_2 - x_1)$ for any $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$. The remaining proof is partially similar to Theorem 1 of (Carlier and Néron, 2007).

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a MDFF. We prove the properties (1.)–(3.). One has $f(0) \leq 0$ due to the condition for DFFs. On the other hand, $f(x) < 0$ for a certain $x \geq 0$ is impossible, because f is maximal and setting $f(x)$ to zero cannot violate the condition for DFFs. Assume that $f(x_1 + x_2) < f(x_1) + f(x_2)$ for certain $x_1, x_2 \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) := \begin{cases} f(x) & \text{if } x \neq x_1 + x_2 \\ f(x_1) + f(x_2) & \text{otherwise} \end{cases}.$$

Since f is a MDFF, g must violate the defining condition for a DFF. Replacing $g(x_1 + x_2)$ by $g(x_1) + g(x_2)$ and $x_1 + x_2$ by two ones x_1 and x_2 leads to a violation if $x_1, x_2 \neq 0$, because of the definition of g. That is a contradiction.

(a) The converse direction is to prove that if f satistfies the conditions (1.)–(4.), then f is a MDFF.

For any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$ with $\sum_{i=1}^{n} x_i \leq 1$, the superadditivity condition (2.) yields $\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i)$. Let $x_0 := 1 - \sum_{i=1}^{n} x_i \geq 0$. Therefore, we have $f(x_0) \geq 0$. Because of $f(1-x_0) + f(x_0) = 1$, it follows that f is a DFF. Let $g : \mathbb{R} \to \mathbb{R}$ be a DFF with $g(x) > f(x)$ for a certain $x \in \mathbb{R}$. Since g is a DFF, one has $g(1-x) + g(x) \leq 1$. It follows that $g(1-x) \leq 1 - g(x) < 1 - f(x) = f(1-x)$ due to (2), hence g does not dominate f. Therefore, f is a MDFF. □

The third condition is necessary for the assertion (a) as it can be shown through a counter example. The following function $f : \mathbb{R} \to \mathbb{R}$ obeys only the 1st, 2nd and 4th condition of the theorem and it is not a DFF (see Figure 1):

$$f(x) := \begin{cases} 3x-2 & \text{if } x < 0 \\ -x & \text{if } 0 \leq x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 2-x & \text{if } 1/2 < x \leq 1 \\ 3x & \text{otherwise} \end{cases} \qquad (3)$$

The first condition is obviously fulfilled. The fourth is also checked easily. If $x < 0$, then $1 - x > 1$ and $f(x) + f(1-x) = 3x - 2 + 3(1-x) = 1$. If $0 \leq x < 1/2$, then $f(x) + f(1-x) = -x + 2 - (1-x) = 1$. To check the superadditivity, assume that $x_1 \leq x_2$. If $x_2 < 0$ or $x_1 > 1$, then the proof is trivial. If $x_2 > 1$ and $0 \leq x_1 + x_2 < 1/2$, then $f(x_1+x_2) - f(x_1) - f(x_2) = -(x_1+x_2) - (3x_1 - 2) - 3x_2 = 2 - 4(x_1+x_2) > 0$. The other cases are left to the reader.

Figure 1: The need for monotonicity.

The following proposition simplifies the proof of a given real function to be a MDFF by Theorem 1.

Proposition 2. *If the function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) for all $x \leq 1/2$, then (2) holds for all $x \in \mathbb{R}$. If additionally the inequality (1) holds for all $x_1, x_2 \in \mathbb{R}$ with ($x_1 + x_2 \leq 2/3$ and) $x_1 \leq x_2 \leq \frac{1-x_1}{2}$, then the inequality (1) is true for all $x_1, x_2 \in \mathbb{R}$.*

Proof. If $x > 1/2$, then $z := 1 - x < 1/2$, and hence $f(z) + f(1-z) = 1$ due to (2). That implies $f(x) + f(1-x) = 1$. This symmetry will be assumed for the entire remaining proof.

The condition $x_1 \leq x_2 \leq \frac{1-x_1}{2}$ implies $x_1 + x_2 \leq 2/3$ and $x_1 \leq 1/3$, because $x_1 \leq \frac{1-x_1}{2}$ leads to $3x_1 \leq 1$ and therefore $x_1 + x_2 \leq \frac{1+x_1}{2} \leq \frac{1+1/3}{2} = \frac{2}{3}$. Obviously, the inequality (1) is valid if and only if it is true after swapping x_1 against x_2. Therefore, $x_1 \leq x_2$ can be enforced without loss of generality. Now we prove that the inequality (1) holds for all $x_1, x_2 \in \mathbb{R}$, if it is true for all $x_1, x_2 \in \mathbb{R}$ with $x_1 + x_2 \leq 2/3$. If $x_1 + x_2 > 2/3$, then $x_2 > 1/3$ due to $x_1 \leq x_2$. Hence $1 - x_2 < 2/3$ and $f(x_1) + f(1 - x_2 - x_1) \leq f(1 - x_2)$ according to the inequality (1). The symmetry (2) yields $f(x_1) + 1 - f(x_1 + x_2) \leq 1 - f(x_2)$, and hence $f(x_1) + f(x_2) \leq f(x_1 + x_2)$, as needed. Therefore, $x_1 + x_2 \leq 2/3$ can be assumed in the rest of the proof, and hence $x_1 \leq \frac{1}{3} \leq \frac{1-x_1}{2}$. If $2x_2 > 1 - x_1$, then let $x_3 := 1 - x_1 - x_2 < \frac{1-x_1}{2}$. Due to the previous parts of the proof and the prerequisites, the superadditivity rule (1) can be used, implying $f(x_1) + f(x_3) \leq f(x_1 + x_3)$. The symmetry rule (2) yields $f(x_1) + 1 - f(1 - x_3) \leq 1 - f(1 - x_1 - x_3)$, and hence $f(x_1) + f(1 - x_1 - x_3) = f(x_1) + f(x_2) \leq f(1 - x_3) = f(x_1 + x_2)$. □

3 ADDITIONAL PROPERTIES OF MDFFS

In this section, some further tools to (dis)prove that a given function is a MDFF are provided.

Proposition 3. *If $f : \mathbb{R} \to \mathbb{R}$ is a MDFF, then for all $\bar{x} \in \mathbb{R}$ the limits $\lim_{x \uparrow \bar{x}} f(x)$ and $\lim_{x \downarrow \bar{x}} f(x)$ exist and $\lim_{x \uparrow 0} f(x) = \inf_{\bar{x} \in \mathbb{R}} \{\lim_{x \uparrow \bar{x}} f(x) - \lim_{x \downarrow \bar{x}} f(x)\}$.*

Proof. f is monotonously increasing and defined for all real arguments. To verify the existence of the left and right limits at a certain $\bar{x} \in \mathbb{R}$, choose any sequences (x_n) and (y_n) of real numbers with $x_0 < \dots < x_n < \dots < \bar{x} < \dots < y_n < \dots < y_0$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \bar{x}$. Then $f(x_0) \leq \dots \leq f(x_n) \leq \dots \leq f(\bar{x}) \leq \dots \leq f(y_n) \leq \dots \leq f(y_0)$. Any monotonous and bounded sequence converges. Therefore, the claimed limits exist.

The superadditivity rule (1) implies $f(-2) \leq f(\bar{x} - 1) - f(\bar{x} + 1) \leq \lim_{x \uparrow \bar{x}} f(x) - \lim_{x \downarrow \bar{x}} f(x)$ for every $\bar{x} \in \mathbb{R}$, and hence $a := \inf_{\bar{x} \in \mathbb{R}} \{\lim_{x \uparrow \bar{x}} f(x) - \lim_{x \downarrow \bar{x}} f(x)\}$ is finite. Because of $f(x) \geq 0$ for all $x \geq 0$, it follows that $\lim_{x \uparrow 0} f(x) \geq a$. For any $\varepsilon < 0$, there is an $\bar{x} \in \mathbb{R}$ with $\lim_{x \downarrow \bar{x}} f(x) - \lim_{x \uparrow \bar{x}} f(x) > \varepsilon - a$. The superadditivity (1) implies $f(\varepsilon) \leq f(\bar{x} + \frac{\varepsilon}{2}) - f(\bar{x} - \frac{\varepsilon}{2})$. The monotonicity of f implies $f(\bar{x} + \frac{\varepsilon}{2}) \leq \lim_{x \uparrow \bar{x}} f(x)$ and $f(\bar{x} - \frac{\varepsilon}{2}) \geq \lim_{x \downarrow \bar{x}} f(x)$, and hence $\varepsilon - a < f(\bar{x} - \frac{\varepsilon}{2}) - f(\bar{x} + \frac{\varepsilon}{2}) \leq$

$-f(\varepsilon)$, *i.e.* $f(\varepsilon) < a - \varepsilon$. Since this holds for all $\varepsilon < 0$, it follows that $\lim_{x \uparrow 0} f(x) \leq a$. Together with $\lim_{x \uparrow 0} f(x) \geq a$, the assertion follows. □

The next proposition shows in contrast to the case of domain and range $[0,1]$ that the set of differentiable MDFFs is much stronger restricted.

Proposition 4. *If the function $f : \mathbb{R} \to \mathbb{R}$ possesses the properties (1.) and (2.) of Theorem 1, and if it is continuously differentiable at the points 0 and $a \in \mathbb{R}$, then $f'(a) = f'(0)$.*

Proof. Let $h \in \mathbb{R}$, $h > 0$. The superadditivity of f yields $f(a+h) \geq f(a) + f(h)$ and $f(a) \geq f(a+h) + f(-h)$, and hence $f(h) \leq f(a+h) - f(a) \leq -f(-h)$. Since $f(0) = 0$ and $h > 0$, this can be rewritten as $\frac{f(h)-f(0)}{h} \leq \frac{f(a+h)-f(a)}{h} \leq \frac{f(0)-f(-h)}{h} = \frac{f(-h)-f(0)}{-h}$. Using the limit $h \downarrow 0$ yields due to the assumed continuous differentiability of f the inequality chain $f'(0) \leq f'(a) \leq f'(0)$, and hence $f'(a) = f'(0)$. □

Corollary 1. *Any continuously differentiable MDFF $f : \mathbb{R} \to \mathbb{R}$ has the form $f(x) = cx$ with $c \in [0,1]$.*

Proof. The derivative is constant, and hence $f(x) = cx + d$ with certain constants $c, d \in \mathbb{R}$. Since $f(0) = 0$, it follows that $f(x) = cx$. Definition 1 yields $f(1) \leq 1$, hence $c \leq 1$. Since $f(x) \leq 0$ for $x < 0$, one has $c \geq 0$. □

Proposition 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a superadditive function. If there is an $a \in \mathbb{R} \setminus \{0\}$ with $f(a) + f(-a) = 0$, then the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := f(x) - x \times f(a)/a$ (for all $x \in \mathbb{R}$) is periodic with period a.*

Proof. The superadditivity of f implies $f(x+a) \geq f(x) + f(a)$ and $f(x) \geq f(x+a) + f(-a)$ for all $x \in \mathbb{R}$. Hence, $f(x+a) \leq f(x) - f(-a) = f(x) + f(a)$ because of $f(a) = -f(-a)$, and finally $f(x+a) = f(x) + f(a)$ for all $x \in \mathbb{R}$. That yields $g(x+a) - g(x) = f(x+a) - f(x) - (x+a) \times f(a)/a + x \times f(a)/a = f(a) - a \times f(a)/a = 0$. □

An example of this kind of MDFFs is the Burdett and Johnson function $f_{BJ,1}$ (see Proposition 12).

If a function is given, which satisfies most of the demands, but is not symmetric, then sometimes a MDFF can be constructed from it like it was done in the Theorem 1 of (Clautiaux et al., 2010), but not generally.

Proposition 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(1) > 0$ and satisfying the conditions (1.)–(3.) of our Theorem 1. Define the function $g : \mathbb{R} \to \mathbb{R}$ as*

$$g(x) := \begin{cases} \frac{f(x)}{f(1)} & \text{if } x < \frac{1}{2} \\ 1/2 & \text{if } x = 1/2 \\ 1 - g(1-x) & \text{otherwise} \end{cases}.$$

If $g(x) + g(y) \leq g(x+y)$ holds for all $x < 0$ and $y \in [\frac{1}{2}, \frac{1-x}{2}]$, then g is a MDFF, but not generally.

Proof. The function g satisfies obviously the conditions (1.), (3.) and (4.) of Theorem 1 by construction. We show the superadditivity of g under the additional constraint. According to Proposition 2, choose any $x, y \in \mathbb{R}$ with $x \leq y \leq \frac{1-x}{2}$. Five cases have to be distinguished:

1. $x, y, x+y < 1/2$: the superadditivity of f yields $g(x) + g(y) \leq g(x+y)$, because $f(1) > 0$;
2. $x, y < 1/2 = x+y$: the superadditivity of f implies $f(1) \geq 2 \times f(1/2)$ and $f(x) + f(y) \leq f(1/2) \leq \frac{1}{2} \times f(1)$, and hence $g(x) + g(y) \leq \frac{1}{2} = g(x+y)$;
3. $x, y < 1/2 < x+y$: the superadditivity of f leads to $f(x) + f(y) + f(1-x-y) \leq f(1)$, and hence $g(x) + g(y) + g(1-x-y) \leq 1$ and by symmetry $g(x) + g(y) \leq g(x+y)$;
4. $x = 0$: this case is trivial because of $g(0) = 0$;
5. $x < 0$, $\frac{1}{2} \leq y \leq \frac{1-x}{2}$: this case is explicitly given in the prerequisites.

There are no other cases, because $x > 0$ leads to $\frac{1-x}{2} < \frac{1}{2}$. Therefore, g is superadditive due to Proposition 2.

A counter-example to the superadditivity of g without the additional constraint arises from $f(x) := \lfloor 3x \rfloor$ for all $x \in \mathbb{R}$. This function satisfies all the conditions (1.)–(3.) of Theorem 1, but the resulting function g is not superadditive. We get $g(-1/3) = -1/3$, $g(7/12) = 1 - g(5/12) = 1 - \lfloor \frac{5}{4} \rfloor / 3 = 2/3$ and $g(1/6) = g(1/4) = 0$, and hence $g(-1/3) + g(1/2) = 1/6 > g(1/6)$ and $g(-1/3) + g(7/12) = 1/3 > g(1/4)$. □

Define $\mathrm{frac}(x) := x - \lfloor x \rfloor$ as an abbreviation for the non-integer part of any real expression x. The following proposition helps in proving superadditivity.

Proposition 7. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : [0,1) \to \mathbb{R}$ be superadditive functions such that for all $x, y, z \in \mathbb{R}$ with $0 < y \leq z < 1$ and $y + z \geq 1$ it holds that*

$$f(x+1) - f(x) \geq g(y) + g(z) - g(y+z-1). \quad (4)$$

Then the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := f(\lfloor x \rfloor) + g(\mathrm{frac}(x))$ is superadditive.

Proof. Choose any $x, y \in \mathbb{R}$. To verify $h(x+y) \geq h(x) + h(y)$, the non-integer parts of x, y need to be considered. If $\text{frac}(x) + \text{frac}(y) < 1$, then $\text{frac}(x+y) = \text{frac}(x) + \text{frac}(y)$ and $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, and hence $h(x+y) - h(x) - h(y) = f(\lfloor x \rfloor + \lfloor y \rfloor) - f(\lfloor x \rfloor) - f(\lfloor y \rfloor) + g(\text{frac}(x) + \text{frac}(y)) - g(\text{frac}(x)) - g(\text{frac}(y)) \geq 0$, because of the superadditivity of f and g.

The other case is $\text{frac}(x) + \text{frac}(y) \geq 1$ leading to $\text{frac}(x+y) = \text{frac}(x) + \text{frac}(y) - 1$ and $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1$. The prerequisite (4) brings $f(\lfloor x \rfloor + \lfloor y \rfloor + 1) \geq f(\lfloor x \rfloor + \lfloor y \rfloor) + g(\text{frac}(x)) + g(\text{frac}(y)) - g(\text{frac}(x) + \text{frac}(y) - 1)$, and hence $h(x+y) - h(x) - h(y) = f(\lfloor x \rfloor + \lfloor y \rfloor + 1) - f(\lfloor x \rfloor) - f(\lfloor y \rfloor) + g(\text{frac}(x) + \text{frac}(y) - 1) - g(\text{frac}(x)) - g(\text{frac}(y)) \geq f(\lfloor x \rfloor + \lfloor y \rfloor) - f(\lfloor x \rfloor) - f(\lfloor y \rfloor) \geq 0$. □

Proposition 7 becomes more useful in conjunction with the following proposition about composed functions.

Proposition 8. *The composition $f(g(\cdot))$ of superadditive functions $f, g : \mathbb{R} \to \mathbb{R}$ is superadditive, if the inner function g is additive or if the outer function f is monotonously increasing, but not generally.*

Proof. If g is additive, then $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$. The superadditivity of f implies in this case $f(g(x+y)) = f(g(x) + g(y)) \geq f(g(x)) + f(g(y))$. If f is monotonously increasing, then one gets $g(x+y) \geq g(x) + g(y)$ and $f(g(x+y)) \geq f(g(x) + g(y)) \geq f(g(x)) + f(g(y))$. If f is not monotonously increasing and g is not additive, then the composition needs not to be superadditive, as the following counter-example shows. Let f be the function (3) and g be the Burdett and Johnson function $f_{BJ,1}$ (see Proposition 12) with parameter $C = 9/2$. One gets $g(1/9) = 0$ and $g(2/9) = 1/4$, and hence $f(g(1/9)) = f(0) = 0$ and $f(g(2/9)) = f(1/4) = -1/4 < 2 \times f(g(1/9))$. □

4 EXAMPLES

In this section, we present and analyze non-trivial examples of MDFF whose domain and range is the set of real numbers $\mathbb{R}$.

Proposition 9. *Let $a, b \in \mathbb{R}$ with $0 \leq a \leq 1$ and $a \leq b$. The following function $f : \mathbb{R} \to \mathbb{R}$ satisfies all the conditions (1.)–(4.) of Theorem 1, and hence it is a MDFF:*

$$f(x) = \begin{cases} (1+b)x & \text{if } x \leq 0 \\ (1-a)x & \text{if } 0 \leq x \leq 1/4 \\ (1+a)x - \frac{a}{2} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ (1-a)x + a & \text{if } \frac{3}{4} \leq x \leq 1 \\ (1+b)x - b & \text{for } x \geq 1 \end{cases}$$

Proof. The function f is piecewise linear and continuous. In particular, we have $f(0) = 0$, $f(\frac{1}{4}) = \frac{1-a}{4}$, $f(\frac{3}{4}) = \frac{3+a}{4}$ and $f(1) = 1$. The conditions (1.), (3.) and (4.) can be checked easily. Only the superadditivity condition (2.) needs a large case distinction. For this purpose, choose any $x, y \in \mathbb{R}$ with $x \leq y$ (without loss of generality) and $x + y \leq 2/3$ according to Proposition 2. Define

$$d(x,y) := f(x+y) - f(x) - f(y).$$

We have to prove that $d(x,y) \geq 0$. This function is also piecewise linear and continuous.

1. If $x + y \leq 0$, then $x \leq 0$ due to $x \leq y$, and hence $d(x,y) = (1+b)(x+y-x) - f(y) = (1+b)y - f(y)$. Except for $\frac{3}{4} < y < 1$, the desired inequality $f(y) \leq (1+b)y$ is obvious. Since d is piecewise linear and continuous, we get $d(x,y) \geq 0$ for the excluded case too.
2. If $\frac{1}{4} \leq x + y \leq \frac{3}{4}$, then $x \leq \frac{3}{8}$, $y \geq \frac{1}{8}$, and the following subcases arise:
   (a) $x \leq 0$ and $\frac{1}{4} \leq y \leq \frac{3}{4}$ yields $d(x,y) = (1+a) \times (x+y) - \frac{a}{2} - (1+b)x - (1+a)y + \frac{a}{2} = (a-b)x \geq 0$.
   (b) $y \geq 1$ yields $x < 0$ and $d(x,y) = (1+a) \times (x+y) - \frac{a}{2} - (1+b)x - (1+b)y + b = (a-b) \times (x+y) + b - \frac{a}{2} \geq \frac{3a-3b}{4} + b - \frac{a}{2} = \frac{a+b}{4} \geq 0$.
   (c) $x \leq 0 \wedge \frac{3}{4} < y < 1$ needs not to be analyzed, because d is continuous and piecewise linear.
   (d) $0 < x \leq y \leq \frac{1}{4}$ gives $d(x,y) = (1+a) \times (x+y) - \frac{a}{2} - (1-a) \times (x+y) = 2a \times (x+y) - \frac{a}{2} \geq \frac{2a}{4} - \frac{a}{2} = 0$.
   (e) $0 < x \leq \frac{1}{4} < y < \frac{3}{4}$ brings $d(x,y) = (1+a) \times (x+y) - \frac{a}{2} - (1-a)x - (1+a)y + \frac{a}{2} = (1+a)x - (1-a)x = 2ax \geq 0$.
   (f) $\frac{1}{4} < x \leq y \leq \frac{3}{8}$ leads to $d(x,y) = (1+a) \times (x+y) - \frac{a}{2} - (1+a) \times (x+y) + a = \frac{a}{2} \geq 0$.

3. The remaining cases $0 < x+y < \frac{1}{4}$ or $\frac{3}{4} < x+y < 1$ need no further analysis, because setting $z := x+y$ yields $d(x,y) = f(z) - f(x) - f(z-x)$, and this expression is a continuous and piecewise linear function in both arguments x and z.

□

The next function has a simple structure similar to the function $f_{CCM,1}$ by Carlier, Clautiaux and Moukrim (see e.g. (Clautiaux et al., 2010)), but it is still different. Moreover, we will see that $f_{CCM,1}$ cannot be generalized to be a MDFF with domain $\mathbb{R}$.

Proposition 10. *Let $b \in \mathbb{R}$, $b \geq 1$. The following function $f : \mathbb{R} \to \mathbb{R}$ is a MDFF:*

$$f(x) = \begin{cases} b \times \lfloor 2x \rfloor & \text{if } x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 1 - b \times \lfloor 2 - 2x \rfloor & \text{if } x > 1/2 \end{cases} \qquad (5)$$

Proof. f satisfies obviously the conditions (1.), (3.) and (4.) of Theorem 1. Only the superadditivity (2.) needs a more careful check. Choose for this purpose any $x, y \in \mathbb{R}$ with $x \leq y \leq \frac{1-x}{2}$ according to Proposition 2, and hence $x \leq 1/3$ and therefore $f(x) \leq 0$. A case distinction follows.

1. If $y < 1/2$ and $x+y < 1/2$, then $f(x) + f(y) = b \times (\lfloor 2x \rfloor + \lfloor 2y \rfloor) \leq b \times \lfloor 2x+2y \rfloor = f(x+y)$.
2. If $y < 1/2 \leq x+y$, then $f(x) \leq 0$, $f(y) \leq 0$ and $f(x+y) \geq 1/2 > f(x) + f(y)$.
3. If $y = 1/2$ and $x+y < 1/2$, then $f(x+y) - f(x) = b \times (\lfloor 2x+2y \rfloor - \lfloor 2x \rfloor) = b \times (\lfloor 2x+1 \rfloor - \lfloor 2x \rfloor) = b \times 1 \geq 1 > f(y)$.
4. If $y = 1/2 \leq x+y$, then $f(x+y) \geq 1/2 = f(y) \geq f(x) + f(y)$.
5. If $y > 1/2 > x+y$, then $f(x+y) - f(x) - f(y) = b \times (\lfloor 2x+2y \rfloor - \lfloor 2x \rfloor + \lfloor 2-2y \rfloor) - 1 \geq b \times (\lfloor 2x+2y+1-2y \rfloor - \lfloor 2x \rfloor) - 1 = b - 1 \geq 0$.
6. If $x < 0$, then $x+y \leq x + \frac{1-x}{2} = \frac{1+x}{2} < \frac{1}{2}$, and hence the case $x+y = 1/2 < y$ is impossible.
7. If $y > 1/2$ and $x+y > 1/2$, then $f(x+y) - f(x) - f(y) = 1 - b \times \lfloor 2-2x-2y \rfloor - b \times \lfloor 2x \rfloor - 1 + b \times \lfloor 2-2y \rfloor \geq b \times (\lfloor 2-2y \rfloor - \lfloor 2-2x-2y+2x \rfloor) \geq 0$.

□

A generalization of this function runs into difficulties, as the following proposition shows.

Proposition 11. *Let $a, b, c, d \in \mathbb{R}$ and the MDFF $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:*

$$f(x) = \begin{cases} a + b \times \lfloor cx + d \rfloor & \text{if } x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 1 - f(1-x) & \text{otherwise} \end{cases} .$$

Then f is necessarily the function (5) with $b \geq 1$.

Proof. The superadditivity implies $f(x/2) \geq x/2$ for all $x \in \mathbb{N}$, because $f(1/2) = 1/2$. Therefore, we have $bc \neq 0$. Hence, the function f possesses gaps of at least the size $|b|$. According to Proposition 3, it follows that $\lim_{x \uparrow 0} f(x) \leq -|b| < 0 = f(0)$. Therefore, we have $c > 0$ and $d \in \mathbb{Z}$, causing $\lfloor cx + d \rfloor = \lfloor cx \rfloor + d$. If $d \neq 0$, then replace a by $a + bd$ and after that d by zero. That does not change f. Therefore, we may assume $d = 0$ for the rest of the proof. We get $0 = f(0) = a + b \times \lfloor c \times 0 \rfloor = a$. The assumption $0 < c < 2$ leads to $f(\frac{1}{2} - \frac{1}{c}) - f(\frac{-1}{c}) = b \times \lfloor \frac{c}{2} - 1 \rfloor - b \times \lfloor -1 \rfloor = b \times (-1) - b \times (-1) = 0$ in contradiction to $\frac{1}{2} = f(\frac{1}{2}) \leq f(\frac{1}{2} - \frac{1}{c}) - f(\frac{-1}{c})$ according to the superadditivity rule. Therefore, we have that $c \geq 2$.

Suppose now that $c > 2$. Let $\varepsilon := \frac{1 - \mathrm{frac}(c)}{2c}$. Then $f(1 + \varepsilon - \frac{1}{c}) = 1 - f(\frac{1}{c} - \varepsilon) = 1$, because $0 < \varepsilon < \frac{1}{c} < \frac{1}{2}$. Since $f(\frac{1 - \lfloor c \rfloor}{c}) = b \times (1 - \lfloor c \rfloor)$, the superadditivity rule implies $1 + b \times (1 - \lfloor c \rfloor) \leq f(1 + \varepsilon - \frac{1}{c} + \frac{1 - \lfloor c \rfloor}{c}) = f(1 + \varepsilon - \frac{\lfloor c \rfloor}{c}) = f(\frac{\mathrm{frac}(c)}{c} + \varepsilon) = f(\frac{1 + \mathrm{frac}(c)}{2c}) = 0$, and hence $b \geq \frac{1}{\lfloor c \rfloor - 1}$. On the other hand, $\frac{1}{c} < \frac{1}{2}$ and $f(\frac{1}{c}) = b$ yield due to $\lfloor c \rfloor \times \frac{1}{c} \leq 1$ and the superadditivity the contradiction $1 \geq f(\lfloor c \rfloor \times \frac{1}{c}) \geq \lfloor c \rfloor \times f(\frac{1}{c}) = \lfloor c \rfloor \times b \geq \lfloor c \rfloor \times \frac{1}{\lfloor c \rfloor - 1} > 1$. Hence, $c > 2$ is impossible, such that finally $c = 2$ follows.

Due to the superadditivity rule, $f(-\frac{1}{2}) = -b$, $f(\frac{1}{4}) = 0$ and $f(\frac{3}{4}) = 1$ yield $f(-\frac{1}{2}) + f(\frac{3}{4}) \leq f(\frac{1}{4})$, and hence $b \geq 1$. □

If more general functions are allowed like $g : \mathbb{R} \to \mathbb{R}$ with parameters $b, c, d \in \mathbb{R}$ and a function $f : [0, 1) \to \mathbb{R}$ according to

$$g(x) := \begin{cases} b \times \lfloor cx + d \rfloor + f(\mathrm{frac}(cx + d)), & x < 1/2, \\ 1/2, & x = 1/2, \\ 1 - g(1-x), & \text{otherwise,} \end{cases}$$

then, there are more possibilities. Since $\lfloor cx + d \rfloor = \lfloor cx + \mathrm{frac}(d) \rfloor + \lfloor d \rfloor$ and $\mathrm{frac}(cx + d) = \mathrm{frac}(cx + \mathrm{frac}(d))$, we can assume $0 \leq d < 1$. Otherwise, the additional constant $b \times \lfloor d \rfloor$ shall become a part of f, such that d can be replaced by $\mathrm{frac}(d)$, leading to the same function g. Some necessary conditions for g to be a MDFF are the following:

- $f(d) = 0$, because of $0 \leq d < 1$ and $g(0) = 0$;
- $bc > 0$, because g must be monotonously increasing and $g(0) = 0$ and $g(x) < 0$ for $x < 0$. If we had $bc = 0$ then g would be constant or periodic for $x < 1/2$, and $bc < 0$ would yield $\lim_{x \to -\infty} g(x) = +\infty$;

- f must be monotonous, namely non-increasing if $b < 0$ and non-decreasing if $b > 0$, because of the needed monotonicity of g;
- f must be bounded, namely $|f(0) - \lim_{x \uparrow 1} f(x)| \leq |b|$, otherwise g would not be monotonous;
- if $d = 0$, then f must be superadditive in the interval $[0, \min\{1, \frac{c}{2}\})$ if $c > 0$ and in the entire domain $[0, 1)$ if $c < 0$, because of the needed superadditivity of g.

An example of this type of functions g is presented in the following proposition.

Proposition 12. *Let $C \in \mathbb{R}$, $C \geq 1$. The function $f_{BJ,1} : \mathbb{R} \to \mathbb{R}$ due to Burdett and Johnson (Burdett and Johnson, 1977) with $f_{BJ,1}(x) = (\lfloor Cx \rfloor + \max\{0, \frac{\mathrm{frac}(Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)}\}) / \lfloor C \rfloor$ is a MDFF.*

Proof. The conditions (1.) and (3.) of Theorem 1 are obviously satisfied. To prove the other ones, choose any $x, y \in \mathbb{R}$. We have $C - Cx = \lfloor C \rfloor + \mathrm{frac}(C) - \lfloor Cx \rfloor - \mathrm{frac}(Cx)$ and

- either $\mathrm{frac}(Cx) > \mathrm{frac}(C)$, and hence

$$\begin{aligned} & \lfloor C \rfloor \times (f_{BJ,1}(x) + f_{BJ,1}(1-x)) \\ = & \lfloor Cx \rfloor + \lfloor C - Cx \rfloor \\ & + \max\{0, \frac{\mathrm{frac}(Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)}\} \\ & + \max\{0, \frac{\mathrm{frac}(C - Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)}\} \\ = & \lfloor C \rfloor - 1 + \frac{\mathrm{frac}(Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \\ & + \frac{\mathrm{frac}(C) + 1 - \mathrm{frac}(Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \\ = & \lfloor C \rfloor - 1 + \frac{1 - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} = \lfloor C \rfloor, \end{aligned}$$

- or $\mathrm{frac}(Cx) \leq \mathrm{frac}(C)$ and therefore $\lfloor C \rfloor \times (f_{BJ,1}(x) + f_{BJ,1}(1-x)) = \lfloor Cx \rfloor + \lfloor C - Cx \rfloor + 0 + 0 = \lfloor Cx \rfloor + \lfloor C \rfloor - \lfloor Cx \rfloor = \lfloor C \rfloor$,

such that the symmetry (2) is verified. The superadditivity $f_{BJ,1}(x+y) \geq f_{BJ,1}(x) + f_{BJ,1}(y)$ is obviously valid for $\mathrm{frac}(Cx) \leq \mathrm{frac}(C)$ or $\mathrm{frac}(Cy) \leq \mathrm{frac}(C)$. Therefore, assume $\mathrm{frac}(Cx) > \mathrm{frac}(C)$ and $\mathrm{frac}(Cy) > \mathrm{frac}(C)$. We have $Cx + Cy = \lfloor Cx \rfloor + \lfloor Cy \rfloor + \mathrm{frac}(Cx) + \mathrm{frac}(Cy)$ and $d := \lfloor C \rfloor \times (f_{BJ,1}(x+y) - f_{BJ,1}(x) - f_{BJ,1}(y)) = \lfloor \mathrm{frac}(Cx) + \mathrm{frac}(Cy) \rfloor + \max\{0, \frac{\mathrm{frac}(\mathrm{frac}(Cx) + \mathrm{frac}(Cy)) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)}\} - \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - 2\mathrm{frac}(C)}{1 - \mathrm{frac}(C)}$. Three cases arise:

1. $\mathrm{frac}(Cx) + \mathrm{frac}(Cy) < 1$ yields $d = \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} - \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - 2\mathrm{frac}(C)}{1 - \mathrm{frac}(C)} = \frac{\mathrm{frac}(C)}{1 - \mathrm{frac}(C)} > 0$;
2. $1 \leq \mathrm{frac}(Cx) + \mathrm{frac}(Cy) \leq 1 + \mathrm{frac}(C)$ brings $d = 1 - \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - 2\mathrm{frac}(C)}{1 - \mathrm{frac}(C)} = \frac{1 + \mathrm{frac}(C) - \mathrm{frac}(Cx) - \mathrm{frac}(Cy)}{1 - \mathrm{frac}(C)} \geq 0$;
3. $\mathrm{frac}(Cx) + \mathrm{frac}(Cy) > 1 + \mathrm{frac}(C)$ gives $d = 1 + \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - 1 - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} - \frac{\mathrm{frac}(Cx) + \mathrm{frac}(Cy) - 2\mathrm{frac}(C)}{1 - \mathrm{frac}(C)} = 1 + \frac{\mathrm{frac}(C) - 1}{1 - \mathrm{frac}(C)} = 0$.

In all cases the superadditivity is also valid, such that all conditions of Theorem 1 are satisfied. □

The survey (Clautiaux et al., 2010) already stated that $f_{BJ,1}$, restricted to the domain $[0, 1]$, is a MDFF, but without a complete proof. This function $f_{BJ,1}$ can be seen as a use of Propositions 7 and 8. The next function $f_{LL,1}$ due to Letchford and Lodi is built similarly, cf. (Clautiaux et al., 2010). On the contrary, the improved function $f_{LL,2}$ of (Clautiaux et al., 2010) cannot be extended to a MDFF with domain $\mathbb{R}$.

Proposition 13. *Let $C \in \mathbb{R} \setminus \mathbb{N}$, $C > 1$ and $k \in \mathbb{N}$, $k \geq \lceil \frac{1}{\mathrm{frac}(C)} \rceil$. The following function $f_{LL,1} : \mathbb{R} \to \mathbb{R}$ satisfies the conditions (1.)–(3.) of Theorem 1, but is not symmetric and cannot be improved by Proposition 6 in spite of $f_{LL,1}(1) = 1$:*

$$f_{LL,1}(x) := \frac{\lfloor Cx \rfloor + \max\{0, \frac{\lceil (k-1) \times \frac{\mathrm{frac}(Cx) - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil}{k}\}}{\lfloor C \rfloor}$$

Proof. One gets $f_{LL,1}(0) = (0 + \max\{0, \lceil (k-1) \times \frac{-\mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil / k\}) / \lfloor C \rfloor = 0$ and $f_{LL,1}(1) = (\lfloor C \rfloor + \max\{0, 0\}) / \lfloor C \rfloor = 1$. Moreover, for all $x > 0$ it holds obviously that $f_{LL,1}(x) \geq 0$, because $\lfloor Cx \rfloor \geq 0$ and $\lfloor C \rfloor > 0$. To prove the superadditivity, Propositions 7 and 8 are used. We set $f(x) := x$ and $g(x) := \max\{0, \lceil (k-1) \times \frac{x - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil\} / k$ in Proposition 7. We verify that for all $x, y, z \in \mathbb{R}$ with $0 \leq y \leq z < 1$ and $y + z > 1$ the inequality $f(x+1) - f(x) \geq g(y) + g(z) - g(y+z-1)$ holds. That is equivalent to $k \geq \max\{0, \lceil (k-1) \times \frac{y - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil\} + \max\{0, \lceil (k-1) \times \frac{z - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil\} - \max\{0, \lceil (k-1) \times \frac{y + z - 1 - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil\}$. Since $z < 1$, it follows that $\lceil (k-1) \times \frac{z - \mathrm{frac}(C)}{1 - \mathrm{frac}(C)} \rceil \leq$

$k-1$, and hence the desired inequality is obviously fulfilled, if $y \leq \mathsf{frac}(C)$. Therefore, assume $y > \mathsf{frac}(C)$, such that the inequality becomes $k \geq \lceil (k-1) \times \frac{y-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil + \lceil (k-1) \times \frac{z-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil - \max\{0, \lceil (k-1) \times \frac{y+z-1-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil\}$. Since for all $x \in \mathbb{R}$ it holds that $x \leq \lceil x \rceil < x+1$, one gets $\lceil (k-1) \times \frac{y-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil + \lceil (k-1) \times \frac{z-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil - \max\{0, \lceil (k-1) \times \frac{y+z-1-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil\} < 2 + (k-1) \times \frac{y+z-2\times\mathsf{frac}(C)}{1-\mathsf{frac}(C)} - (k-1) \times \frac{y+z-1-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} = 2 + \frac{k-1}{1-\mathsf{frac}(C)} \times (-2 \times \mathsf{frac}(C) + 1 + \mathsf{frac}(C)) = k+1$. The left part of this inequality is integer, implying that it is not above k. Therefore, the chosen functions f and g satisfy the prerequisite (4) of Proposition 7, such that the function $x \mapsto \lfloor x \rfloor + \max\{0, \lceil (k-1) \times \frac{\mathsf{frac}(x)-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil / k\}$ is superadditive. This function can be composed with the linear function $x \mapsto Cx$ according to Proposition 8. Finally, dividing the entire expression by $\lfloor C \rfloor \geq 1$ has no influence on the superadditivity. Therefore, $f_{LL,1}$ is superadditive.

It remains to show that the additional constraint of Proposition 6 is violated for some feasible parameter choices (and hence $f_{LL,1}$ is not symmetric). Choose any $C \in (1,2)$ and any enough large odd $k \in \mathbb{N}$. Let $x := \frac{-1}{C}$ and $y := \frac{1}{2}$. That yields $f_{LL,1}(x) = \lfloor -1 \rfloor / \lfloor C \rfloor = -1$ and $f_{LL,1}(x+y) = f_{LL,1}(\frac{C-2}{2C}) = \lfloor \frac{C}{2} - 1 \rfloor + \max\{0, \lceil (k-1) \times \frac{\mathsf{frac}(C/2-1)-\mathsf{frac}(C)}{1-\mathsf{frac}(C)} \rceil / k\} = -1 + \max\{0, \lceil (k-1) \times \frac{C/2-C+1}{1-C+1} \rceil / k\} = -1 + \max\{0, \lceil \frac{k-1}{2} \rceil / k\} = -1 + \frac{k-1}{2k}$, because k is odd, and finally $f_{LL,1}(x+y) = \frac{-1}{2} - \frac{1}{2k} < -1 + \frac{1}{2}$. □

5 USING MDFFS TO COMPUTE VALID INEQUALITIES

In this section, we demonstrate how the MDFFs can be used to generate valid inequalities to solve integer linear programs, and we illustrate their use through a simple example. Let be given an instance

$$\begin{aligned} \max \quad & \mathbf{c}^\top \mathbf{x} \\ s.t. \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{N}^n, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ whith $m,n \in \mathbb{N} \setminus \{0\}$. We can take every non-negative linear combination of the constraints to generate another valid inequality, *i.e.* choose any $\mathbf{u} \in \mathbb{R}^m_+ \setminus \{\mathbf{o}\}$ to obtain one inequality $\mathbf{u}^\top \mathbf{Ax} \leq \mathbf{b}^\top \mathbf{u}$. If $f : \mathbb{R} \to \mathbb{R}$ is any monotonously increasing superadditive function, e.g. a MDFF, then the monotonicity of f yields $f(\mathbf{b}^\top \mathbf{u}) \geq f(\mathbf{u}^\top \mathbf{Ax})$, and this is not below $\sum_{j=1}^{n} f(\sum_{i=1}^{m} a_{ij} u_i) \times x_j$ due to the superadditivity of f and the condition $\mathbf{x} \in \mathbb{N}^n$.

Example 2. *Consider the following integer linear program (with negative coefficients):*

$$\begin{aligned} \max \quad & z := 10x_1 - 3x_2 \\ s.t. \quad & 7x_1 - 2x_2 \leq 9 \\ & x_1, x_2 \in \mathbb{N} \end{aligned}$$

The solution $\mathbf{x} := (3,6)^\top$ is feasible and yields $z = 12$. In the sequel, we show that this solution is optimal by showing that the following inequality

$$10x_1 - 3x_2 \leq 12$$

is in fact a valid inequality. Furthermore, we show that this inequality can be derived using a MDFF with domain and range $\mathbb{R}$.

Choose any $u > 0$ and a MDFF f to get the valid inequality $f(7u) \times x_1 + f(-2u) \times x_2 \leq f(9u)$. We try several functions f, namely according to

- *Proposition 9: to get the desired inequality, we set $u := 1/9$, yielding $\frac{3}{4} \leq 7u \leq 1$, and hence*

$$((1-a) \times \frac{7}{9} + a) \times x_1 - \frac{2}{9} \times (1+b) \times x_2 \leq 1 \quad (6)$$

  with $0 \leq a \leq 1$ and $b \geq a$. The inequality (6) becomes the sharpest for the smallest possible b, i.e. for $b = a$. That yields

$$(\frac{7}{9} + \frac{2}{9}a) \times x_1 - (\frac{2}{9} + \frac{2}{9}a) \times x_2 \leq 1.$$

  Choosing $a := 1/14$ yields $\frac{50}{63}x_1 - \frac{15}{63}x_2 \leq 1$ or equivalently $10x_1 - 3x_2 \leq \frac{63}{5}$. Since the left hand side is integer, it follows that $z = 12$ is optimal.
- *Proposition 10: here we have to distinguish several cases with respect to $u > 0$, but this function fails to give the needed strong valid inequality. For instance $\frac{3}{14} < u \leq \frac{2}{9}$ yields the valid inequality $2x_1 - x_2 \leq 2$, but it remains too weak.*
- *Proposition 12: We may use any $C \geq 1$ and $u > 0$, yielding $(\lfloor 7Cu \rfloor + \max\{0; \frac{\mathsf{frac}(7Cu)-\mathsf{frac}(C)}{1-\mathsf{frac}(C)}\}) \times x_1 + (\lfloor -2Cu \rfloor + \max\{0; \frac{\mathsf{frac}(-2Cu)-\mathsf{frac}(C)}{1-\mathsf{frac}(C)}\}) \times x_2 \leq \lfloor 9Cu \rfloor + \max\{0; \frac{\mathsf{frac}(9Cu)-\mathsf{frac}(C)}{1-\mathsf{frac}(C)}\}$, for instance $C := 10/7$ and $u := 1$, yielding $10x_1 - 3x_2 \leq 12\frac{3}{4}$. The choice $C := \frac{13}{7}$ and $u := \frac{10}{13}$ leads directly to $10x_1 - 3x_2 \leq 12$, as desired.*

Note that the valid inequality $10x_1 - 3x_2 \leq 12$ *is a Chvátal-Gomory-inequality (cf. (Nemhauser and Wolsey, 1998)), and it can be obtained by using* $u := 10/7$ *and rounding down to the next integer.*

6 CONCLUSIONS

In this paper, we generalized the notion of (maximal) dual feasible functions to functions of which the domain comprises the entire set of real numbers. This extension is important to allow the use of DFFs for deriving valid inequalities for any general integer linear program. This generalization is also non-trivial. Indeed, the well-known symmetry condition, which was necessary for a DFF with domain $[0,1]$ to be maximal, does not hold for all MDFFs with domain $\mathbb{R}$. Furthermore, the influence of the conditions that characterize these functions becomes more restrictive, *i.e.* many well known classical MDFFs cannot be generalized to domain $\mathbb{R}$. On the contrary, besides the MDFF $f_{BJ,1}$, some other non-trivial MDFFs were defined. Some examples were proposed and discussed in this paper. Finally, we illustrated through the use of an example how valid inequalities could be derived using these new MDFFs.

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