MEAN FIELD MONTE CARLO STUDIES OF ASSOCIATIVE MEMORY
Understanding the Dynamics of a Many-pattern Model

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Abstract: Dynamics of a Hebbian model of associative memory is studied using Mean field Monte-Carlo method. Under the assumption of infinite system, we have derived single-spin equations, using the generating functional method from statistical mechanics, for the purpose of simulations. This approach circumvents the strong finite-size effects of the usual calculations on this system. We have tried to understand the retrieval of a stored pattern in presence of another condensed pattern undergoing reinforcement, positive or negative. We find that the retrieval is faster and the retrieval quality is better for the case of positive reinforcement.

1 INTRODUCTION

In the theoretical study of the neural network representation of associative memory, models composed of spin-like elements play an important role. In most simple cases, two state (Ising-Spin) variables are used to model the basic processing elements i.e. neurons (Hopfield, 1982). The dynamics of this system is described by *update rules* that specify the behavior of the state of the neuron (spin) as governed by the net synaptic input (local field at the spin) from the other neurons in the network. These interactions between the neurons are obtained from the employed learning rule and are represented in the synaptic matrix. To analyze the retrieval performance such as speed and quality of retrieval of the network, it is necessary to use dynamical techniques from non equilibrium statistical physics (as opposed to methods from equilibrium statistical physics). Another feature characteristic of biological systems which neccessitates the use of dynamical methods is asymmetry in the couplings i.e. $J_{ij} \neq J_{ji}$where J_{ij} is the interaction strength between i^{th} and j^{th} neurons (Amit, 1992).

In the direct numerical simulations done within the above scheme, N^2 couplings have to be stored for a connected model where N is the total number of neurons. This makes the task of extrapolation from a finite number of spins to the Large N (thermodynamic) limit a highly non-trivial one (Kohring and Schreckenberg, 1991). Thus, to understand the behaviour of macroscopic quantities, like suitably defined *order parameters*, of large networks, it is relevant to use fully connected models for which a dynamical mean field theory can be formulated for an infinite system. This technique was implemented by (Eissfeller and Opper, 1992) who used the generating functional technique to develop a numerical method for analyzing the dynamics of a system consisting of interactions in the Sherrington-Kirkpatrick (SK) model for spin glasses (Sherrington and Kirkpatrick, 1975). In their work, the generating functional method was employed to derive stochastic single-spin dynamical rules in the large-N limit and this self-consistent single spin dynamics is simulated using a Monte Carlo Procedure to calculate disorder averaged quantities. The same method was gainfully used by (Singh and Dasgupta, 2003) to study the dynamics of pattern retrieval in a Hopfield-like model of associative memory with one stored pattern. While discussing the possible relevance of this one pattern model for the Hopfield-like models, it was argued the pattern stored in the network represents the condensed pattern the network is trying to recall. The effect of the other uncondensed pattern is mimicked by the SK model like coupling term in the synaptic matrix. It is natural to ask as to what happens to the recall process when there are other condensed patterns. This could correspond to the physical situation when

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other patterns are *recollected (with a positive acquisition strength) or* forgotten (with a negative acqisition strength). In an attempt to answer this question we generalize the method for the two patterns (further generalization for more than two patterns is straightforward).

The present work focuses on a neural network model of associative memory with two binary patterns. The synaptic matrix consists of two parts – one corresponding to the Hebbian learning (Hebb, 2002) of the patterns with respective acquisition strengths and the other corresponding to the coupling matrix of SK model. We oncentrate on the retrieval dynamics of the one of the stored patterns. The SK part in the synaptic matrix may effectively mimick the interference from a macroscopic number (of the order of number of neurons) 'other' uncondensed memories when the connection between this simple model and the Hopfield like models is sought. In this framework, we can understand the effect of interference due to specific condensed patterns (those having a finite macroscopic overlap with the pattern under consideration) as opposed to just the interference from a large number of uncondensed (not having any macroscopic overlap with the initial pattern) patterns. We present the generating functional treatment used to derive the self-consistent stochastic dynamical rules and the results of the Monte-Carlo Simulation of the subsequent single-spin dynamics.

The paper is organized as follows: Section 2 describes, in detail, the model under consideration. In Section 3, we discuss the derivation of the Single Spin Equations in the Large N limit. Section 4 details the method used to simulate the single spin equation. We conclude with results and discussions in Section 5.

2 ASYMMETRIC SK MODEL WITH MULTIPLE PATTERNS

2.1 The Neurons and their Interaction

The model network is composed of N two-state neurons (Ising Spins) $\sigma_i = \pm 1$. Every neuron σ_i is interacting with all other neurons σ_j by couplings J_{ij}:

$$J_{ij} = J_{ij}^{\mathrm{SK}} + \frac{J_1}{N}\xi_i^1\xi_j^1 + \frac{J_2}{N}\xi_i^2\xi_j^2 + \ldots, \quad i \neq j, \quad J_{ii} = 0, \tag{1}$$

The first term represents the couplings in the SK Model with Random Asymmetric Interactions. This can be taken as the two (multi) pattern analogue of the *tabula non rasa* scenario proposed by Toulouse, Dehaene, and Changeux (Toulouse et al., 1986). The terms that follow it represent Hebbian Learning of memory patterns $\xi_i^1, \xi_i^2, \ldots$ with $J_1, J_2, \ldots$ are the respective acquisition strengths for the patterns. The couplings J_{ij}^{SK} are independent Gaussian random variables for all $i < j$ and are drawn from the distribution:

$$P\left(J_{ij}^{\mathrm{SK}}\right) = \sqrt{\frac{1}{2\pi/N}}\exp\left\{-\frac{\left(J_{ij}^{\mathrm{SK}}\right)^2}{2/N}\right\}, \quad i < j. \tag{2}$$

The information about the symmetry of the coupling matrix is expressed in terms of η, the average symmetry parameter:

$$\left[J_{ij}^{\mathrm{SK}} J_{ji}^{\mathrm{SK}}\right] = \eta/N, \tag{3}$$

The Brackets, here, denote the (ensemble) average over the distribution of couplings. The values $\eta = 1$ and $\eta = -1$ correspond to symmetric couplings and fully antisymmetric couplings respectively. The case $\eta = 0$ denotes totally uncorrelated couplings. Couplings with the average symmetry η can be constructed via[]

$$J_{ij}^{\mathrm{SK}} = \left[\frac{1+\eta}{2}\right]^{1/2} J_{ij}^{\mathrm{s}} + \left[\frac{1-\eta}{2}\right]^{1/2} J_{ij}^{\mathrm{as}}, \tag{4}$$

Here, $J_{ij}^{\mathrm{s}} (= J_{ji}^{\mathrm{s}})$ and $J_{ij}^{\mathrm{as}} (= -J_{ji}^{\mathrm{as}})$, the symmetric and the antisymmetric components of the SK interaction, are independent Gaussian random variables, for all $i < j$, and are drawn from the same distribution as that in Eq. (2). However for all the results reported here we have considered only the symmetrical couplings. The effect of asymmetry will be taken up elsewhere.

2.2 The Dynamical Rule

In this work, we consider the noise-free (Zero Temperature) dynamics of the system with a synchronous update of all the spins as a response to the local field $h_i(t)$ acting on each spin σ_i. At any elementary time step t, the update rule is given by:

$$\sigma_i(t+1) = \mathrm{sgn}\left(h_i(t)\right), \quad i = 1, \ldots, N, \tag{5}$$

Where

$$\begin{aligned} h_i(t) &= \textstyle\sum_{j\neq i} J_{ij}\sigma_j(t), \\ &= \left[\tfrac{1+\eta}{2}\right]^{1/2} \textstyle\sum_{j\neq i} J_{ij}^{\mathrm{s}}\sigma_j(t) + \\ &\left[\tfrac{1-\eta}{2}\right]^{1/2} \textstyle\sum_{j\neq i} J_{ij}^{\mathrm{as}}\sigma_j(t) + \tfrac{J_1}{N}\xi_i^1\xi_j^1 + \\ &\tfrac{J_2}{N}\xi_i^2\xi_j^2 + \ldots. \end{aligned} \tag{6}$$

The first term plays the role of the noise due to the interference from a large number (Finite fraction of the number of neurons) of patterns stored in the network.

The second one is the antisymmetric term from the synaptic matrix. The terms following that correspond to the pattern under retrieval and the interference term due to the other condensed pattern.

2.3 Assessing the Retrieval Properties

In this paper, we aim to assess the retrieval properties of a network of associative memory with more than one condensed memories. We shall work in the framework of an initial value problem, in which the initial ($t = 0$) neural network configuration, $\{\sigma_i(0)\}$, has an overlap m_k with the stored pattern ξ_i^k. The evolution of the system to a state that has overlap m_l with a stored pattern ξ_i^l that is sufficiently close to unity is referred to as the successful of the latter pattern. In the presence of interference from an additional condensed pattern, the following quantities are of particular concern:

1. Retrieval quality, i.e., the degree of closeness of the final state to the relevant pattern under retrieval.
2. Convergence time, i.e., time taken by the neural network to converge to a final state close to the corresponding stored pattern.

The simulation is done on self-consistent single-spin dynamics that is obtained using the Dynamical Mean Field Theory described in the following section.

3 DYNAMICAL MEAN FIELD THEORY

Under the assumption that each spin is coupled to all the other spins we shall show that the internal field, $h_i(t)$, which depends explicitly on the states of all the spins, can be replaced by an effective 'mean field' which depends only on some macroscopic order parameters. This effective internal field is a time-dependent random process and is different from the averaged internal field $[h_i(t)]$. The random processes for effective field $h_i(t)$ can be constructed conveniently using the Dynamical Generating Functional technique. This effective field can be used to generate stochastic spin trajectories in a monte-carlo simulation.

We are interested in the statistical properties of a large, but finite number N_T of spin trajectories over t_f time steps, at the sites $i = 1, \ldots, N_T$, out of a system in which the total number of spins, N, may be very large. To derive these properties, we consider the dynamical generating function $\langle Z(\mathbf{l})\rangle_J$ for the local fields $h_i(t)$, $i = 1, \ldots, N_T$; $t = 1, \ldots, t_f$ in a system with just 2 patterns:

$$\langle Z(\mathbf{l})\rangle_J = \langle Tr_{\sigma(t)} \int \prod_{i=1}^{N} \prod_{t=1}^{t_f} \{h_i(t)\, \Theta(\sigma_i(t+1)\, h_i(t)) \times \delta(h_i(t) - [\tfrac{1+\eta}{2}]^{1/2} \sum_{j\neq i} J_{ij}^{\mathrm{s}} \sigma_j(t) + \left[\tfrac{1-\eta}{2}\right]^{1/2} \sum_{j\neq i} J_{ij}^{\mathrm{as}} \sigma_j(t) + \tfrac{J_1}{N}\, \xi_i^1\, \xi_j^1 + \tfrac{J_2}{N}\, \xi_i^2\, \xi_j^2)\} \times \exp(\sum_{t=0}^{t_f} \sum_{i=1}^{N_T} l_i(t)\, h_i(t))\rangle_J. \qquad (7)$$

Here, $\langle \cdots \rangle_J$ represents the average over the all possible random couplings and Tr_σ denotes the sum over all $2^{N t_f}$ possible states of the spin-system, $\sigma_i(t) = \pm 1$. $\theta(x)$ and $\delta(x)$ are the unit step function and the dirac delta function respectively. These functions ensure that only those 'spin paths' $\sigma_i(t)$ which are consistent with the equations of motion (5) and (6)contribute to $\langle Z(\mathbf{l})\rangle_J$.

The calculation of $\langle Z(\mathbf{l})\rangle_J$ follows, closely, the derivation given in (Eissfeller and Opper, 1994) (for the asynchronous case with $J_0 = 0$), which in turn is a generalization of the derivation given by (Henkel and Opper, 1991) for the synchronous dynamics of a neural network. Analysis showed us that in the large-N limit, the generating function can be completely factorised into independent components for the N_T spins:

$$\langle Z(\mathbf{l})\rangle_J \propto \prod_{i=1}^{N_T} \left\langle Tr_{\sigma_i(t)} \int \prod_t \{\mathrm{dh_i(t)}\, \Theta(\sigma_\mathrm{i}(\mathrm{t}+1)\, h_i(t))\} \exp\{\mathrm{i} \sum_\mathrm{t} \mathrm{l_i(t)\, h_i(t)}\} \prod_t \delta\left(h_i(t) - J_1 m_1(t)\xi_i^2 - J_2 m_2(t)\xi_i^2 - \phi_i(t) - \eta \sum_s K(t,s)\sigma_i(s)\right)\right\rangle_\phi . \qquad (8)$$

In this form of the generating function, we see that the dynamics of the spin system is described by the *uncorrelated system of dynamical equations*:

$$\sigma_i(t+1) = \mathrm{sign}\,(h_i(t))\,, \qquad (9)$$

where

$$h_i(t) = J_1 m_1(t)\xi_i^1 + J_2 m_2(t)\xi_i^2 + \phi_i(t) + \eta \sum_{s<t} K(t,s)\sigma_i(s)\,. \qquad (10)$$

We have effectively replaced the time-independent random couplings to other spins by a Gaussian random variables $\phi_i(t)$, with zero mean and covariance $\langle \phi_i(t)\phi_i(s)\rangle_\phi = C(t,s)$, introduced independently for each site i. In the above equation (Eq. 10),the first two terms in the above 'effective' local field come from the mean field theory and are responsible pattern retrieval, the third term is a Gaussian noise, while the fourth term represents a retarded self-interaction.

The order parameters can be rewritten in terms of the Gaussian averages:

$$C(t,s) = \langle \phi(t)\phi(s)\rangle_\phi = \langle \sigma(t)\sigma(s)\rangle_\phi\,, \qquad (11)$$

$$K(t,s) = -\mathrm{i}\left\langle \hat{\mathrm{h}}(\mathrm{s})\sigma(\mathrm{t})\right\rangle_\phi = \left\langle \frac{\partial}{\partial\phi(\mathrm{s})}\sigma(\mathrm{t}) \right\rangle_\phi \qquad (12)$$

From Eq. (12) we can physically interpret $K(t,s)$ as a response function. In order to evaluate it, we express the above average of the partial derivative in terms of the correlation function $\langle\sigma(t)\phi(s)\rangle$ using a discrete version of Novikov's theorem (Hanggi, 1978).

$$\langle\sigma(t)\phi(s)\rangle = \sum_{\tau=0}^{t} K(t,\tau)C(\tau,s)\,. \qquad (13)$$

4 RESULTS: RETRIEVAL PROPERTIES

4.1 Method: The Monte Carlo Simulations

We have used the effective single-spin equations, (9) and (10) to calculate the exact averages in the $N \to \infty$ limit. To carry this out, spin variables have been expressed as explicit functions of the Gaussian Fields and integrations (weighted by the multivariate Gaussian) have been performed by a Monte-Carlo process. At each time step, the necessary averages over the system (of $N \to \infty$ spins) have been calculated by summing over a system of large number (N_T) of single-spin trajectories.

To reduce the numerical errors of the Monte-Carlo integration, we have used $N_t = 2.5 \times 10^6$. Simulations over time scales of $t_f = 100$ time steps can safely be carried out neglecting the error propagation effect from imperfections in the Gaussian random distribution used.

We have investigated the effect of an additional condensed pattern on the retrieval properties (including time and quality of retrieval) of a system. The additional condensed pattern may have a positive coupling strength (as in the case of *retrieval* of the memory) or negative (corresponding to the process of *forgetting* of a memory).

4.2 The Evolution of Overlap with the Stored Pattern

Figures 1 and 2 show the time evolution of the overlap of the current state of the system with the respective stored pattern. While Fig. 1 refers to a system which is simultaneously in the process of *recollecting* another memory (with coupling strength ($J_1 = 0.6$), Fig. 2 corresponds to a the negative recollection ($J_1 = -0.6$) or (*forgetting* of the second memory. The different curves represent different strength of the initial overlap of the system with the two memory patterns.

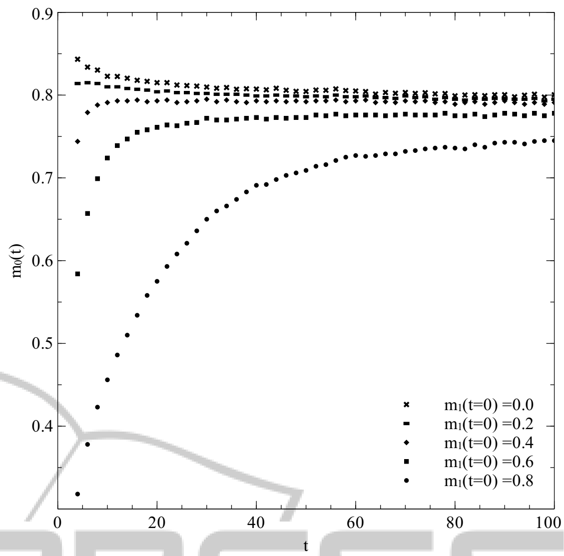

Figure 1: Value of overlap of pattern at time t with one condensed pattern (with coupling strength $J_0 = 1.5$) in presence of another pattern which has Positive coupling strength ($J_1 = 0.6$).

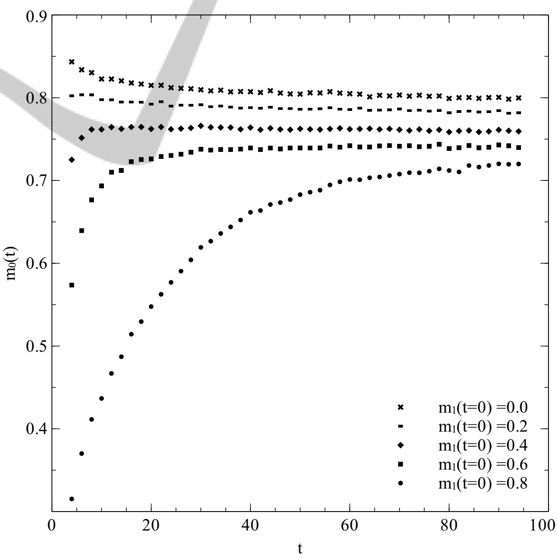

Figure 2: Value of overlap of pattern at time t with one condensed pattern (with coupling strength $J_0 = 1.5$) in presence of another pattern which has negative coupling strength ($J_1 = -0.6$).

The following function fits the above results very well:

$$m(t) = m_\infty + \text{const} \times t^{-a} \exp(-t/\tau)\,. \qquad (14)$$

Here, m_∞ is a measure of the quality of retrieval and τ and a describe the speed of retrieval of the memory.

4.3 Speed of Retrieval

Figure 3 shows ratio of time of retrieval for additional coupling negative ($J_1 = -0.6$) to additional coupling positive ($J_1 = 0.6$) as it varies with initial overlap of the system with the additional stored pattern $m_1(t = 0)$. For small values of starting overlap with the additional memory, the case with recolection is faster than the one with forgetting of an additional stored pattern.

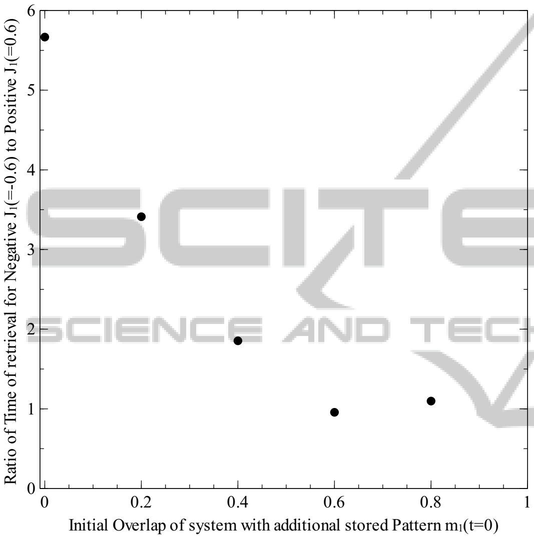

Figure 3: Variation of Ratio of time of retrieval for additional coupling negative ($J_1 = -0.6$) to additional coupling positive ($J_1 = 0.6$) w.r.t. initial overlap of the system with the additional stored pattern $m_1(t = 0)$.

4.4 The Quality of Retrieval

Figure 4 shows the difference between the quality of retrieval of a stored pattern for the cases of positve and negative coupling of additional pattern. In the model under consideration, it is clear that the presence of an additional memory which is being recollected leads to better retrieval quality.

5 CONCLUSIONS

We have studied the dynamics of associative memory using mean field Monte-Carlo method. In the case of an infinite system, we have derived and simulated single-spin dynamical equations. We find that the retrieval is faster and the retrieval quality is better in case another stored memory is being recollected as compared to the case in which another stored memory is being forgotten.

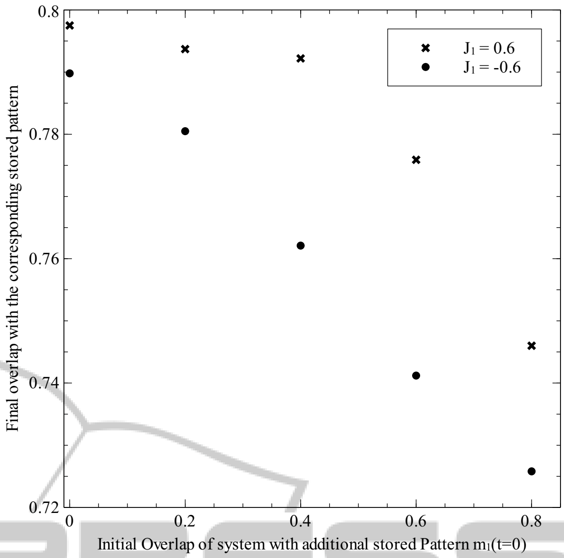

Figure 4: Quality of retrieval of the final stored pattern w.r.t. initial overlap of the system with the additional stored pattern $m_1(t = 0)$.

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