ON THE EXTENSION OF THE MEDIAN CENTER AND THE MIN-MAX CENTER TO FUZZY DEMAND POINTS

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Abstract: A common research topic has been the search of an optimal center, according to some objective function that considers the distance between the potential solutions and a given set of points. For crisp data, closed form expressions obtained are the median center, for the Manhattan distance, and the min-max center, for the Chebyshev distance. In this paper, we prove that these closed form expressions can be extended to fuzzy sets by modeling data points with fuzzy numbers, obtaining centers that, through their membership function, model the "appropriateness" of the final location.

1 INTRODUCTION

Finding an optimal center in space became a common process in planning, because it allows to affect a set of demands to one or several locations that offer dedicated facilities. For instance, a center collecting wastes, a vehicle depot for logistic purpose or a hospital complex, all require a relevant metric to minimize cost or maximize access to them. Mathematicians, economists and geographers developed methods which locate these centers according to either equity (minimax) or versus efficiency (minisum) objectives, following the work in k-facilities location problems on networks (Hakimi, 1964), that respectively correspond to the k-median and the k-center. Indeed, there exist many mathematical problems and formalisms for optimal location problems (Hansen et al., 1987). More recently, we can see a larger scope of the domain and sets where these issues appear (Chan, 2005). Other books complete the state-of-the-art (Drezner and Hamacher, 2004; Griffith et al., 1998; Nickel and Puerto, 2005) or focus on applications in transportation (Labbé et al., 1995; Thomas, 2002) or health care (Brandeau et al., 2004).

Methodologies for optimal location can be applied on continuous space, finite space or networks (graphs or roads for instance). If $k = 1$, then the aim is to find a single center. The choice of the metric p is also significant because it involves, on the one hand, the method to set the distance separating the demands to the center, and on the other, how to combine these distances according to a given objective function. Thus, there exist many ways to calculate a center for many points of demand, even when reducing complexity by considering a continuous space, a unique center and the Minkowski distance of L_p norms. The first parameter, p, defines the norm of the distance separating the demand points to the center: rectilinear ($p = 1$), Euclidean ($p = 2$) or Chebyshev's ($p \to \infty$). The second parameter, p', relates to the calculation of the center itself. The sum of the distances is minimized when $p' = 1$, the sum of the squared distances when $p' = 2$ and the maximum of the distances when ($p' \to \infty$).

Among all the possibilities crossing p and p' of the L_p norms, only three cases can be computed in closed form: the median center, which minimizes the sum of the rectilinear distances ($p = p' = 1$), the centroid or barycenter, which minimizes the sum of the squared Euclidean distances ($p = p' = 2$) and the min-max center, which minimizes the maximum of the maximum distances ($p \to \infty$ and $p' \to \infty$).

Scientists and planners use to consider the final location to be accurate and crisp, or, at least, as a finite set of possible predefined locations. However, there might be uncertainty on the estimated distances, due to uncertainty carried by the demand location it-

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self. This is particularly true when considering urban sprawl, as it can generate non negligible variations on the location of the town's center, which in place might affect the location of the optimal center. There is also the case when subjective or vague information is used to define the demand location. The result, then, cannot possibly be a crisp point, and solutions that assume crisp data when non is available, might be at risk being far from optimal.

By modeling the demand points as bi-dimensional fuzzy sets we prove in this paper that the results obtained for crisp environments can be easily extended to the fuzzy ones, attaining homologous closed form expressions. As the solutions depend only on arithmetic operations of fuzzy numbers, thus obtaining fuzzy numbers as its coordinates, the approach followed in this work deviates from the path trailed by many fuzzy location papers, in which constraints are fuzzy, but the solution is not (Darzentas, 1987; Canós et al., 1999; Chen, 2001; Moreno Pérez et al., 2004).

Fuzzy solutions also give some leeway to planners which might be forced to select the final location of the center away from the place with the highest membership value, but that can the measure the impact of their decision and, thus, asses its "appropriateness".

This paper is structured in the following way. In Section 2, we introduce the closed form expressions for centers usually used in the literature. Then, on Section 3, the basic concepts of fuzzy sets and fuzzy numbers used through our paper are defined. Section 4 covers the demonstrations used to prove that the closed form expressions found for some centers in crisp environments can be extended to fuzzy points. A small numerical example, joined by some figures in which the results can be easily seen, is developed in Section 5. Finally, Section 6 presents the conclusions as well as the future work based on our results.

2 THE MEDIAN CENTER AND THE MIN-MAX CENTER

A recurrent problem in geography is the need to find the center of a set of demand points that minimizes a given objective function. Without taking into consideration the road network that links these points, i.e., in an open space, there are two simple, but also widely used methods to solve this problem, the median center and the min-max center.

Definition 1. *For a set $P = \{p^{(i)}\}$ of n points in $\mathbb{R}^2$, i.e., $p^{(i)} = \{p^{(i,x)}, p^{(i,y)}\}$, the median center $m = \{m^{(x)}, m^{(y)}\}$ is found by the median of their coordinates in x and y:*

$$m^{(x)} = median\left(p^{(i,x)}\right) \tag{1}$$

$$m^{(y)} = median\left(p^{(i,y)}\right). \tag{2}$$

Definition 2. *For a set $P = \{p^{(i)}\}$ of n points in $\mathbb{R}^2$, i.e., $p^{(i)} = \{p^{(i,x)}, p^{(i,y)}\}$, the min-max center $z = \{c^{(x)}, c^{(y)}\}$ is found by the average of the extremes in x and y:*

$$z^{(x)} = \frac{\max_{i=1,\dots,n}\left(p^{(i,x)}\right) + \min_{i=1,\dots,n}\left(p^{(i,x)}\right)}{2} \tag{3}$$

$$z^{(y)} = \frac{\max_{i=1,\dots,n}\left(p^{(i,x)}\right) + \min_{i=1,\dots,n}\left(p^{(i,x)}\right)}{2}. \tag{4}$$

The median center is affected by changes in the middle points, but changes in extreme points affect only the min-max center. The selection of the appropriate method to find the center depends on which points are most likely to change (Ciligot-Travain and Josselin, 2009).

3 FUZZY SETS AND FUZZY NUMBERS

When it is difficult to say that an object clearly belongs to a class, classical set theory loses its usefulness. The fuzzy sets theory (Zadeh, 1965) overcomes this problem by assigning degrees of membership of elements to a set. In this section we will recall the concepts of the fuzzy set theory that will be used in this paper.

3.1 Basic Definitions

Definition 3. *A fuzzy subset $\underset{\sim}{A}$ is a set whose elements do not follow the law of the excluded middle that rules over Boolean logic, i.e., their membership function is mapped as:*

$$\mu_{\underset{\sim}{A}} : X \to [0,1]. \tag{5}$$

In general, a fuzzy subset $\underset{\sim}{A}$ can be represented by a set of pairs composed of the elements x of the universal set X, and a grade of membership $\mu_{\underset{\sim}{A}}(x)$:

$$\underset{\sim}{A} = \left\{\left(x, \mu_{\underset{\sim}{A}}(x)\right) \mid x \in X, \mu_{\underset{\sim}{A}}(x) \in [0,1]\right\}. \tag{6}$$

Definition 4. *An α-cut of a fuzzy subset $\underset{\sim}{A}$ is defined by:*

$$A_\alpha = \{x \in X : \mu_{\underset{\sim}{A}}(x) \geq \alpha\}, \tag{7}$$

i.e., the subset of all elements that belong to $\underset{\sim}{A}$ at least in a degree α.

Definition 5. *A fuzzy subset $\underset{\sim}{A}$ is convex, if and only if:*

$$\lambda x_1 + (1 - \lambda x_2) \in A_\alpha \forall x_1, x_2 \in A_\alpha,\ \alpha, \lambda \in [0,1], \quad (8)$$

i.e., all the points in $[x_1, x_2]$ must belong to A_α, for any α.

Definition 6. *A fuzzy subset $\underset{\sim}{A}$ is normal, if and only if:*

$$\max_{x \in X} \left(\mu_{\underset{\sim}{A}}(x) \right) = 1. \quad (9)$$

Definition 7. *The core of a fuzzy subset $\underset{\sim}{A}$ is defined as:*

$$N_{\underset{\sim}{A}} = \left\{ x : \mu_{\underset{\sim}{A}}(x) = 1 \right\}. \quad (10)$$

Definition 8. *A fuzzy number $\underset{\sim}{A}$ is a normal, convex fuzzy subset with domain in $\mathbb{R}$ for which:*

1. $\bar{x} := N_{\underset{\sim}{A}}$, $\mathrm{card}(\bar{x}) = 1$, and
2. $\mu_{\underset{\sim}{A}}$ is at least piecewise continuous.

The mean value $\bar{x}$ (Zimmermann, 2005), also called maximum of presumption (Kaufmann and Gupta, 1985), identifies a fuzzy number in such a way that the proposition "about 9" can be modeled with a fuzzy number whose maximum of presumption is $x = 9$. As Zimmermann explains, for computational simplicity there is a tendency to call "fuzzy number" any normal, convex fuzzy subset whose membership function is, at least, piecewise continuous, without taking into consideration the uniqueness of the maximum of presumption. Thus, this definition will include "fuzzy intervals", fuzzy numbers in which $\bar{x}$ covers an interval[1], and particularly trapezoidal fuzzy numbers (TrFN).

Definition 9. *A TrFN is defined by the membership function:*

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 1 - \frac{x_2 - x}{x_2 - x_1}, & \text{if } x_1 \le x < x_2 \\ 1, & \text{if } x_2 \le x \le x_3 \\ 1 - \frac{x - x_3}{x_4 - x_3}, & \text{if } x_3 < x \le x_4 \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

This kind of fuzzy interval represents the case when the maximum of presumption, the modal value, can not be identified in a single point, but only in an interval between x_2 and x_3, decreasing linearly to zero at the worst case deviations x_1 and x_4. The TrFN is represented by a 4-tuple whose first and fourth elements correspond to the extremes from where the membership function begins to grow, and whose second and third components are the limits of the interval where the maximum certainty lies, i.e., $\underset{\sim}{A} = (x_1, x_2, x_3, x_4)$.

[1] As a matter of fact, they are also called "flat fuzzy numbers" (Dubois and Prade, 1979).

Definition 10. *The image of a TrFN is defined as:*

$$Im\left(\underset{\sim}{A}\right) = (-a_4, -a_3, -a_2, -a_1).$$

Definition 11. *The addition and subtraction of two TrFN $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are defined as:*

$$\underset{\sim}{A} \oplus \underset{\sim}{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4) \quad (12)$$

$$\underset{\sim}{A} \ominus \underset{\sim}{B} = \underset{\sim}{A} \oplus Im(\underset{\sim}{B}). \quad (13)$$

3.2 Miscelaneous Definitions

Comparing fuzzy numbers is a task that can only be achieved via defuzzification, i.e., by calculating its expected value. For its simplicity, we have selected the graded mean integrated representation (GMIR) of a TrFN (Chen and Hsieh, 1999) as the method used in this paper to defuzzify and compare TrFN.

Definition 12 (Chen and Hsieh, 1999)**.** *The GMIR of non-normal TrFN is:*

$$E\left(\underset{\sim}{M}\right) = \frac{\int_0^{\max\left(\mu_{\underset{\sim}{M}}\right)} \frac{\mu}{2} \left(L_{\underset{\sim}{M}}^{-1}(\mu) + R_{\underset{\sim}{M}}^{-1}(\mu) \right) d\mu}{\int_0^{\max\left(\mu_{\underset{\sim}{M}}\right)} \mu \, d\mu}. \quad (14)$$

Remark 1. *For a normal TrFN as defined in* (11)*, the GMIR is:*

$$E\left(\underset{\sim}{A}\right) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}. \quad (15)$$

Remark 2. *The GMIR is linear, i.e., $E(\underset{\sim}{A} \oplus \underset{\sim}{B}) = E(\underset{\sim}{A}) + E(\underset{\sim}{B})$ and $E(\alpha \cdot \underset{\sim}{A}) = \alpha \cdot E(\underset{\sim}{A})$.*

To calculate the distance between two TrFN, we must first define the absolute value of a TrFN. We will rely on the work of (Chen and Wang, 2009) for this.

Definition 13 (Chen and Wang, 2009)**.** *The absolute value of a TrFN is defined as:*

$$\left|\underset{\sim}{A}\right| = \begin{cases} \underset{\sim}{A}, & \text{if } E\left(\underset{\sim}{A}\right) > 0 \\ 0, & \text{if } E\left(\underset{\sim}{A}\right) = 0 \\ Im\left(\underset{\sim}{A}\right), & \text{if } E\left(\underset{\sim}{A}\right) < 0. \end{cases} \quad (16)$$

Proposition 1. *For a TrFN $\underset{\sim}{A}$, $E(|\underset{\sim}{A}|) = |E(\underset{\sim}{A})|$.*

Proof. For $E(\underset{\sim}{A}) \ge 0$ the proof is trivial. For $E(\underset{\sim}{A}) < 0$ we have:

$$\begin{aligned} E\left(\left|\underset{\sim}{A}\right|\right) &= E\left(Im\left(\underset{\sim}{A}\right)\right) \\ &= \frac{-a_4 - 2a_3 - 2a_2 - a_1}{6} \\ &= -E\left(\underset{\sim}{A}\right) \\ &= \left|E\left(\underset{\sim}{A}\right)\right|. \end{aligned}$$

□

Definition 14. *The fuzzy Minkoswki family of distances between two fuzzy n-dimensional vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ composed of TrFN:*

$$\underset{\sim}{d}_p\left(\underset{\sim}{A},\underset{\sim}{B}\right)=\left(\sum_{i=1}^{n}\left(\left|\underset{\sim}{A}_i\ominus\underset{\sim}{B}_i\right|\right)^p\right)^{\frac{1}{p}}. \tag{17}$$

Remark 3. *As with the crisp Minkowski family of distances, the fuzzy Manhattan distance is defined for $p=1$, the fuzzy Euclidean distance is defined for $p=2$, and the fuzzy Chebyshev distance is defined for $p=\infty$.*

Remark 4. *In our proofs, we will use the form:*

$$\underset{\sim}{d}^p\left(\underset{\sim}{A},\underset{\sim}{B}\right)=\sum_{i=1}^{n}\left(\left|\underset{\sim}{A}_i\ominus\underset{\sim}{B}_i\right|\right)^p, \tag{18}$$

except for $p=\infty$ in which:

$$\underset{\sim}{d}^\infty\left(\underset{\sim}{A},\underset{\sim}{B}\right)=\underset{\left|\underset{\sim}{A}_i\ominus\underset{\sim}{B}_i\right|}{\arg}\ \max_{i=1}^{n}E\left(\left|\underset{\sim}{A}_i\ominus\underset{\sim}{B}_i\right|\right). \tag{19}$$

4 FUZZY MEDIAN CENTER AND FUZZY MIN-MAX CENTER

We will prove that for a set of fuzzy points, the fuzzy median center and the fuzzy min-max center are extensions of their respective counterparts in crisp settings, i.e., that they can be obtained by the median or the average of the maximum X and Y coordinates of the fuzzy points, respectively.

Proposition 2. *For two TrFN $\underset{\sim}{p^{(1)}}$ and $\underset{\sim}{p^{(2)}}$, such that $E(\underset{\sim}{p^{(1)}})<E(\underset{\sim}{p^{(2)}})$, $\underset{\underset{\sim}{c}}{\arg\min}E(\sum_{i\in\{1,2\}}\underset{\sim}{d}^1(\underset{\sim}{p^{(i)}},\underset{\sim}{c}))=\{\underset{\sim}{c}:E(\underset{\sim}{c})\in[E(\underset{\sim}{p^{(1)}}),E(\underset{\sim}{p^{(2)}})]\}$.*

Proof. Let $\underset{\sim}{p^{(i)}}=(p_1^{(i)},p_2^{(i)},p_3^{(i)},p_4^{(i)})$ and $\underset{\sim}{c}=(c_1,c_2,c_3,c_4)$, hence:

$$\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)=\left|\underset{\sim}{p^{(i)}}\ominus\underset{\sim}{c}\right|.$$

By properties of the GMIR:

$$E\left(\underset{\sim}{p^{(i)}}\ominus\underset{\sim}{c}\right)=E\left(\underset{\sim}{p^{(i)}}\right)-E\left(\underset{\sim}{c}\right).$$

If $E(\underset{\sim}{c})\leq E(\underset{\sim}{p^{(1)}})$ and by (16), then:

$$\underset{\sim}{d}^1\left(\underset{\sim}{p^{(1)}},\underset{\sim}{c}\right)=\underset{\sim}{p^{(1)}}\ominus\underset{\sim}{c}, \tag{20}$$

$$\underset{\sim}{d}^1\left(\underset{\sim}{p^{(2)}},\underset{\sim}{c}\right)=\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{c}. \tag{21}$$

By (20) and (21):

$$\underset{\underset{\sim}{c}}{\arg\min}\,E\left(\sum_{i\in\{1,2\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right)=\underset{\sim}{p^{(1)}},$$

as $\underset{\sim}{p^{(1)}}\ominus\underset{\sim}{c}=(0,0,0,0)$ and $\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{c}=\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{p^{(1)}}$. For any $\{\underset{\sim}{c}:E(\underset{\sim}{c})<E(\underset{\sim}{p^{(1)}})\}$, $E(\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{c})>0$ and $E(\underset{\sim}{c}\ominus\underset{\sim}{p^{(1)}})>E(\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{p^{(1)}})$.

Equivalently, if $E(\underset{\sim}{p^{(2)}})\leq E(\underset{\sim}{c})$ by (16), then:

$$\underset{\sim}{d}^1\left(\underset{\sim}{p^{(1)}},\underset{\sim}{c}\right)=\underset{\sim}{c}\ominus\underset{\sim}{p^{(1)}}, \tag{22}$$

$$\underset{\sim}{d}^1\left(\underset{\sim}{p^{(2)}},\underset{\sim}{c}\right)=\underset{\sim}{c}\ominus\underset{\sim}{p^{(2)}}. \tag{23}$$

By (22) and (23),

$$\underset{\underset{\sim}{c}}{\arg\min}\,E\left(\sum_{i\in\{1,2\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right)=\underset{\sim}{p^{(2)}},$$

as $\underset{\sim}{c}\ominus\underset{\sim}{p^{(2)}}=(0,0,0,0)$ and $\underset{\sim}{c}\ominus\underset{\sim}{p^{(1)}}=\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{p^{(1)}}$. For any $\{\underset{\sim}{c}:E(\underset{\sim}{p^{(2)}})<E(\underset{\sim}{c})\}$, $E(\underset{\sim}{c}\ominus\underset{\sim}{p^{(2)}})>0$ and $E(\underset{\sim}{c}\ominus\underset{\sim}{p^{(1)}})>E(\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{p^{(1)}})$.

Given that $E(\underset{\sim}{p^{(1)}})<E(\underset{\sim}{c})$ and by (16), then:

$$\begin{aligned}\underset{\sim}{d}^1\left(\underset{\sim}{p^{(1)}},\underset{\sim}{c}\right)&=\underset{\sim}{c}\ominus\underset{\sim}{p^{(1)}}\\&=\left(c_1-p_4^{(1)},c_2-p_3^{(1)},c_3-p_2^{(1)},c_1-p_4^{(1)}\right).\end{aligned} \tag{24}$$

Given that $E(\underset{\sim}{c})<E(\underset{\sim}{p^{(2)}})$ and by (16), then:

$$\begin{aligned}\underset{\sim}{d}^1\left(\underset{\sim}{p^{(2)}},\underset{\sim}{c}\right)&=\underset{\sim}{p^{(2)}}\ominus\underset{\sim}{c}\\&=\left(p_1^{(2)}-c_4,p_2^{(2)}-c_3,p_3^{(2)}-c_2,p_4^{(2)}-c_1\right).\end{aligned} \tag{25}$$

From (24) and (25):

$$\begin{aligned}\Sigma_{i\in\{1,2\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)&=\left(c_1-p_4^{(1)},c_2-p_3^{(1)},c_3-p_2^{(1)},c_4-p_1^{(1)}\right)\oplus\\&\quad\left(p_1^{(2)}-c_4,p_2^{(2)}-c_3,p_3^{(2)}-c_2,p_4^{(2)}-c_1\right)\\&=\left(p_1^{(2)}-p_1^{(1)}+c_1-c_4,p_2^{(2)}-p_2^{(1)}+c_2-c_3,\right.\\&\quad\left.p_3^{(2)}-p_3^{(1)}+c_3-c_2,p_4^{(2)}-p_4^{(1)}+c_4-c_1\right).\end{aligned} \tag{26}$$

Applying GMIR to (26) :

$$E\left(\Sigma_{i\in\{1,2\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right)=E\left(\underset{\sim}{p^{(2)}}-\underset{\sim}{p^{(1)}}\right). \tag{27}$$

Being that (27) is independent from $\underset{\sim}{c}$:

$$\underset{\underset{\sim}{c}}{\arg\min}E\left(\Sigma_{i\in\{1,2\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right)=\left\{\underset{\sim}{c}:E\left(\underset{\sim}{c}\right)\in\left[E\left(\underset{\sim}{p^{(1)}}\right),E\left(\underset{\sim}{p^{(2)}}\right)\right]\right\}.$$

□

The result obtained in Proposition 2 shows than any fuzzy point $\underset{\sim}{c}$ between two fuzzy points $\underset{\sim}{p^{(1)}}$ and $\underset{\sim}{p^{(2)}}$ gives an equally good solution to the problem of the minimization of distances. An arbitrary, but frequently found solution to the crisp version of this problem, is using the average of both points:

$$\underset{\sim}{c} = \frac{\underset{\sim}{p^{(1)}} \oplus \underset{\sim}{p^{(2)}}}{2}. \tag{28}$$

In the following proposition we will see what happens for a set of n fuzzy points, but first, let us define the notion of order statistic for fuzzy numbers.

Definition 15. *For a set $P = \{\underset{\sim}{p^{(i)}}\}$, $\forall i = 1,\ldots,n$, of TrFN, the $k-$th order statistic $\underset{\sim}{p^{([k])}}$ is defined as the $k-$th point for which $E(\underset{\sim}{p^{([k])}}) \leq E(\underset{\sim}{p^{([k+1])}})$.*

Proposition 3 (Fuzzy median center in $\mathbb{R}$)**.** *For a set $P = \{\underset{\sim}{p^{(i)}}\}$, $i = 1,\ldots,n$, of TrFN, $\underset{\sim}{c}^*$ is the point for which $\arg\min_{\underset{\sim}{c}} E(\sum_{i=1}^{n} \underset{\sim}{d}^1(\underset{\sim}{p^{([i])}},\underset{\sim}{c})) = \{\underset{\sim}{c} : E(\underset{\sim}{c}) \in [E(\underset{\sim}{p^{([\frac{n}{2}])}}), E(\underset{\sim}{p^{([\frac{n}{2}+1])}})]\}$, if n is even, but if it is odd $\arg\min_{\underset{\sim}{c}} E(\sum_{i=1}^{n}(\underset{\sim}{d}^1(\underset{\sim}{p^{([i])}},\underset{\sim}{c}))) = \underset{\sim}{p^{([\frac{n+1}{2}])}}$.*

Proof. Given that the $k-$th order statistic of the set P is $\underset{\sim}{p^{([k])}}$, we can apply iteratively the result in Proposition 2. In first place, it is known that:

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i\in\{1,n\}} \underset{\sim}{d}\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right)\right) = \left\{\underset{\sim}{c} : E\left(\underset{\sim}{c}\right) \in \left[E\left(\underset{\sim}{p^{([1])}}\right), E\left(\underset{\sim}{p^{([n])}}\right)\right]\right\}.$$

From Definition 15, it is also known that:

$$\left[E\left(\underset{\sim}{p^{([2])}}\right), E\left(\underset{\sim}{p^{([n-1])}}\right)\right] \in \left[E\left(\underset{\sim}{p^{([1])}}\right), E\left(\underset{\sim}{p^{([n])}}\right)\right],$$

so the solution is now:

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i\in\{1,2,n-1,n\}} \underset{\sim}{d}^1\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right)\right) = \left\{\underset{\sim}{c} : E\left(\underset{\sim}{c}\right) \in \left[E\left(\underset{\sim}{p^{([2])}}\right), E\left(\underset{\sim}{p^{([n-1])}}\right)\right]\right\}.$$

If we keep applying iteratively this logic, and n is even, we get that

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i=1}^{n} \underset{\sim}{d}^1\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right)\right) = \left\{\underset{\sim}{c} : E\left(\underset{\sim}{c}\right) \in \left[E\left(\underset{\sim}{p^{([\frac{n}{2}])}}\right), E\left(\underset{\sim}{p^{([\frac{n}{2}+1])}}\right)\right]\right\}.$$

If n is odd, we will have three points in the next-to-last iteration, $\left\{\underset{\sim}{p^{([\frac{n-1}{2}])}}, \underset{\sim}{p^{([\frac{n+1}{2}])}}, \underset{\sim}{p^{([\frac{n+3}{2}])}}\right\}$. We can present the problem as:

$$\arg\min_{\underset{\sim}{c}} E\sum_{i=1}^{n}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right)\right) =$$

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i=\frac{n-1}{2}}^{\frac{n-3}{2}} \underset{\sim}{d}^1\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right)\right)$$

$$= \arg\min_{\underset{\sim}{c}} E\left(\sum_{i=\{\frac{n-1}{2},\frac{n-3}{2}\}} \underset{\sim}{d}^1\left(\underset{\sim}{p^{([i])}},\underset{\sim}{c}\right) + \underset{\sim}{d}^1\left(\underset{\sim}{p^{([\frac{n+1}{2}])}},\underset{\sim}{c}\right)\right).$$

We know that:

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i=\{\frac{n-1}{2},\frac{n-3}{2}\}}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right) = \left\{\underset{\sim}{c} : E\left(\underset{\sim}{c}\right) \in \left[E\left(\underset{\sim}{p^{([\frac{n-1}{2}])}}\right), E\left(\underset{\sim}{p^{([\frac{n-3}{2}])}}\right)\right]\right\}.$$

Therefore, it is clear that:

$$\arg\min_{\underset{\sim}{c}} E\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{([\frac{n+1}{2}])}},\underset{\sim}{c}\right)\right) = \underset{\sim}{p^{([\frac{n+1}{2}])}}.$$

So, given that $\underset{\sim}{c} = \underset{\sim}{p^{([\frac{n+1}{2}])}}$ and that $E(\underset{\sim}{p^{([\frac{n+1}{2}])}}) \in [E(\underset{\sim}{p^{([\frac{n-1}{2}])}}), E(\underset{\sim}{p^{([\frac{n-3}{2}])}})]$, for n even:

$$\arg\min_{\underset{\sim}{c}} E\left(\sum_{i=1}^{n}\left(\underset{\sim}{d}^1\left(\underset{\sim}{p^{(i)}},\underset{\sim}{c}\right)\right)\right) = \underset{\sim}{p^{([\frac{n+1}{2}])}}.$$

□

Applying (28) to the result of Proposition 3 we get the definition of the median for a set of TrFN.

Definition 16. *The median of a set $P = \{\underset{\sim}{p^{(i)}}\}$, $\forall i = 1,\ldots,n$, of TrFN is defined as:*

$$median\,(P) = \begin{cases} \dfrac{\underset{\sim}{p^{([\frac{n}{2}])}} \oplus \underset{\sim}{p^{([\frac{n}{2}+1])}}}{2}, & \text{if } n \text{ is odd,} \\ \underset{\sim}{p^{([\frac{n+1}{2}])}}, & \text{if } n \text{ is even.} \end{cases}$$

In an $\mathbb{R}^2$ space, the solution is equivalent, as we will see in the following proposition.

Proposition 4. *For a set* $P = \left\{ \underset{\sim}{P^{(i)}} : \underset{\sim}{P^{(i)}} = \left\{ \underset{\sim}{p^{(i,j)}} \right\}, \forall i = 1, \ldots, n,\ j \in \{x, y\} \right\}$, *where* $\underset{\sim}{p^{(i,j)}}$ *is a TrFN,* $\underset{\underset{\sim}{C}}{\arg\min} E\left(\sum_{i=1}^{n} \underset{\sim}{d}^{1}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}\right)\right) = \left\{ median\left(\underset{\sim}{p^{(i,x)}}\right), median\left(\underset{\sim}{p^{(i,y)}}\right) \right\}$.

Proof. Due to the linearity of the GMIR:

$$E\left(\sum_{i=1}^{n}\left(\underset{\sim}{d}^{1}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}\right)\right)\right) = \sum_{i=1}^{n}\sum_{j\in\{x,y\}}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{c^{j}}\right)\right|$$
$$= \sum_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,x)}}\right) - E\left(\underset{\sim}{c^{(x)}}\right)\right| + \sum_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,y)}}\right) - E\left(\underset{\sim}{c^{(y)}}\right)\right|. \quad (29)$$

As both terms in (29) are independent from each other:

$$\min_{c^{(j)}}\left(\sum_{j\in\{x,y\}}\sum_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,x)}}\right) - E\left(\underset{\sim}{c^{(x)}}\right)\right|\right) = \sum_{j\in\{x,y\}}\min_{c^{(j)}}\sum_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,x)}}\right) - E\left(\underset{\sim}{c^{(x)}}\right)\right|.$$

The optimization problem is then reduced to applying independently for each $j \in \{x, y\}$ the result of Proposition 3 with Definition 16. Thus:

$$\underset{\underset{\sim}{C}}{\arg\min} E\left(\sum_{i=1}^{n}\underset{\sim}{d}^{1}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}\right)\right) = \left\{\text{median}\left(\underset{\sim}{p^{(i,x)}}\right), \text{median}\left(\underset{\sim}{p^{(i,y)}}\right)\right\}. \quad (30)$$

□

Finally, we will address the subject of the fuzzy min-max center, found using (19).

Proposition 5. *For a set* $P = \left\{\underset{\sim}{p^{(i)}}\right\}, \forall i = 1, \ldots, n,$ *of TrFN,* $\max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}\right|) = \frac{1}{2}\cdot E(p^{([n])} - p^{([1])})$.

Proof. Let $E(\underset{\sim}{c}) = \frac{1}{2}(E(p^{([1])}) + E(\underset{\sim}{p^{([n])}}))$. Due to the linearity of the GMIR and Proposition 1, $\max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}\right|) = \max_{i=1}^{n}\left|E(\underset{\sim}{p^{(i)}}) - E(\underset{\sim}{c})\right|$.

So:

$$-\frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(p^{([1])}\right)}{2} \leq E\left(\underset{\sim}{p^{([i])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2} \leq \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(p^{([1])}\right)}{2}$$

then:

$$\left|E\left(\underset{\sim}{p^{([i])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right| \leq \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(p^{([1])}\right)}{2}$$

$$\max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{([i])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right| \leq \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(p^{([1])}\right)}{2}.$$

In fact:

$$\left|E\left(\underset{\sim}{p^{([n])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right| = \left|\frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right|$$
$$= \left|-\frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right|$$
$$= \left|E\left(\underset{\sim}{p^{([1])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right|$$
$$\geq \left|E\left(\underset{\sim}{p^{([i])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right|.$$

So:

$$\max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{([i])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}\right| = \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2},$$

i.e.:

$$\max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{([i])}}\right) - E\left(\underset{\sim}{c}\right)\right| = \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2}.$$

□

Proposition 6. *For a TrFN* $\underset{\sim}{c}'$*, such that for every TrFN* $\underset{\sim}{p}$ $\max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{p}\right|) \geq \max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}'\right|))$*, then* $E(\underset{\sim}{c}) = (\underset{\sim}{c}')$.

Proof. Let $E(\underset{\sim}{c}) = \frac{1}{2}(E(p^{([1])}) + E(\underset{\sim}{p^{([n])}}))$. Taking $\underset{\sim}{p} = \underset{\sim}{c}$:

$$\max_{i=1}^{n}\left(E\left(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}'\right|\right)\right) \leq \max_{i=1}^{n}\left(E\left(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}\right|\right)\right) = \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2},$$

so:

$$E\left(\left|\underset{\sim}{p^{([1])}} \ominus \underset{\sim}{c'}\right|\right) \le \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2}$$

$$E\left(\left|\underset{\sim}{p^{([n])}} \ominus \underset{\sim}{c'}\right|\right) \le \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2}$$

and:

$$E\left(\underset{\sim}{c'}\right) \le \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2} + E\left(\underset{\sim}{p^{([1])}}\right) = \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}$$

$$E\left(\underset{\sim}{c'}\right) \ge E\left(\underset{\sim}{p^{([n])}}\right) - \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2} = \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}.$$

So:

$$E\left(\underset{\sim}{c'}\right) = \frac{E\left(\underset{\sim}{p^{([n])}}\right) + E\left(\underset{\sim}{p^{([1])}}\right)}{2}$$

□

Proposition 7. *For a set* $P = \left\{\underset{\sim}{p^{(i)}}\right\}$, $\forall i = 1, \ldots, n$, *of TrFN and a TrFN* $\underset{\sim}{p}$, $\max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{p}\right|) = \max_{i=1}^{n}(E(\left|\underset{\sim}{p^{(i)}} \ominus \underset{\sim}{c}\right|)$.

Proof. Let $E(\underset{\sim}{c}) = \frac{1}{2}(E(p^{([1])}) + E(\underset{\sim}{p^{([n])}}))$. If $E(\underset{\sim}{p}) \le E(\underset{\sim}{c})$, $E(\underset{\sim}{p^{([n])}}) - E(\underset{\sim}{p}) \ge E(\underset{\sim}{p^{([n])}}) - E(\underset{\sim}{c})$. So:

$$\max_{i=1}^{n}\left|E(\underset{\sim}{p^{([i])}}) - E(\underset{\sim}{p})\right| \ge \left|E(\underset{\sim}{p^{([n])}}) - E(\underset{\sim}{p})\right| \ge \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2} = \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{([i])}}\right) - E\left(\underset{\sim}{c}\right)\right|.$$

If $E(\underset{\sim}{p}) \ge E(\underset{\sim}{c})$, $E(\underset{\sim}{p}) - E(\underset{\sim}{p^{([1])}}) \ge E(\underset{\sim}{c}) - E(\underset{\sim}{p^{([1])}})$. So:

$$\max_{i=1}^{n}\left|E(\underset{\sim}{p^{([i])}}) - E(\underset{\sim}{p})\right| \ge \left|E(\underset{\sim}{p}) - E(\underset{\sim}{p^{([1])}})\right| \ge \frac{E\left(\underset{\sim}{p^{([n])}}\right) - E\left(\underset{\sim}{p^{([1])}}\right)}{2} = \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{([i])}}\right) - E\left(\underset{\sim}{c}\right)\right|.$$

Again, due to the linearity of the GMIR, $\max_{i=1}^{n}\left|E(\underset{\sim}{p^{([i])}}) - E(\underset{\sim}{c})\right| = \max_{i=1}^{n} E\left|\underset{\sim}{p^{([i])}} \ominus \underset{\sim}{c}\right|$ □

Proposition 8. *For a set* $P = \{\underset{\sim}{P^{(i)}} : \underset{\sim}{P^{(i)}} = \{\underset{\sim}{p^{(i,j)}}\}\}, \forall i \in 1, \ldots, n, j \in \{x, y\}$, *where* $\underset{\sim}{p^{(i,j)}}$ *is a TrFN, and the fuzzy center* $\underset{\sim}{C} = \{\underset{\sim}{c^{(j)}}\}$, $\max_{i=1}^{n}(\underset{\sim}{d^{\infty}}(\underset{\sim}{P^{(i)}}, \underset{\sim}{P})) \ge \max_{i=1}^{n}(\underset{\sim}{d^{\infty}}(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}))$.

Proof. Let $E(\underset{\sim}{c^{(j)}}) = \frac{1}{2}E(\underset{\sim}{p^{([1],j)}} \oplus \underset{\sim}{p^{([n],j)}})$ and a the fuzzy point $\underset{\sim}{P} = \{\underset{\sim}{p^{(j)}}\}$. Then:

$$\max_{i=1}^{n} \underset{\sim}{d^{\infty}}\left(d^{\infty}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{P}\right)\right) = \max_{i=1}^{n} \max_{j \in \{x,y\}} E\left(\left|\underset{\sim}{p^{(i,j)}} \ominus \underset{\sim}{p^{(j)}}\right|\right) = \max_{j \in \{x,y\}} \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{p^{(j)}}\right)\right|.$$

From Proposition 7, we will recall that:

$$\max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{p^{(j)}}\right)\right| \ge \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{c^{(j)}}\right)\right|,$$

so:

$$\max_{j \in \{x,y\}} \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{p^{(j)}}\right)\right| \ge \max_{j \in \{x,y\}} \max_{i=1}^{n}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{c^{(j)}}\right)\right| = \max_{i=1}^{n} \underset{\sim}{d^{\infty}}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}\right).$$

□

Proposition 9. *For a set* $P = \left\{\underset{\sim}{P^{(i)}} : \underset{\sim}{P^{(i)}} = \left\{\underset{\sim}{p^{(i,j)}}\right\}, \forall i \in 1, \ldots, n, j \in \{x, y\}\right\}$, *where* $\underset{\sim}{p^{(i,j)}}$ *is a TrFN, the fuzzy min-max center* $\underset{\sim}{C^{*}} = \left\{\underset{\sim}{c^{(j)}}\right\}$ *is* $\arg\min_{\underset{\sim}{C}} \max_{i=1}^{n} \underset{\sim}{d^{\infty}}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{C}\right) = \{\underset{\sim}{c} : E(c^{(j)}) = \frac{1}{2}E(\underset{\sim}{p^{([1],j)}} \oplus \underset{\sim}{p^{([n],j)}})\}$.

Proof. Let the fuzzy point $\underset{\sim}{P} = \{\underset{\sim}{p^{(j)}}\}$, then:

$$\max_{i=1}^{n} E\left(\underset{\sim}{d^{\infty}}\left(\underset{\sim}{P^{(i)}}, \underset{\sim}{P}\right)\right) = \max_{i=1}^{n} \max_{j \in \{x,y\}} E\left(\left|\underset{\sim}{p^{(i,j)}} \ominus \underset{\sim}{p^{(j)}}\right|\right) = \max_{i=1}^{n} \max_{j \in \{x,y\}}\left|E\left(\underset{\sim}{p^{(i,j)}}\right) - E\left(\underset{\sim}{p^{(j)}}\right)\right|.$$

By the result of Proposition 8:

$$E\left(\underset{\sim}{c}^{(j)}\right) = \frac{E\left(\underset{\sim}{p}^{(i,j)}\right) + E\left(\underset{\sim}{p}^{(j)}\right)}{2}.$$

Then:

$$\max_{i=1}^{n} \underset{\sim}{d}^{\infty}\left(\underset{\sim}{P}^{(i)}, \underset{\sim}{P}\right) \geq \max_{i=1}^{n} \underset{\sim}{d}^{\infty}\left(\underset{\sim}{P}^{(i)}, \underset{\sim}{C}\right).$$

$\square$

Given that the solution of the fuzzy min-max center is a set of fuzzy points, we will extend the result for crisp values with the following definition.

Definition 17 (Fuzzy min-max center in $\mathbb{R}^2$). *For a set $P = \left\{\underset{\sim}{P}^{(i)} : \underset{\sim}{P}^{(i)} = \left\{\underset{\sim}{p}^{(i,j)}\right\}, \forall i \in 1, \ldots, n,\ j \in \{x, y\}\right\}$, where $\underset{\sim}{p}^{(i,j)}$ is a TrFN, the fuzzy min-max center $\underset{\sim}{C} = \left\{\underset{\sim}{c}^{(j)}\right\}$ is defined as:*

$$\underset{\sim}{c}^{(j)} := \frac{\underset{\sim}{p}^{([1],j)} \oplus \underset{\sim}{p}^{([n],j)}}{2}. \tag{31}$$

5 NUMERICAL EXAMPLE

In the following numerical example we will see how the three centers are found and how much they differ from each other. Lets suppose there are three fuzzy demand points:

$$\begin{aligned}
\underset{\sim}{P}^{(1)} &= \left\{\underset{\sim}{p}^{(1,x)}, \underset{\sim}{p}^{(1,y)}\right\} \\
\underset{\sim}{p}^{(1,x)} &= (18, 35, 37, 40) \\
\underset{\sim}{p}^{(1,y)} &= (31, 49, 49, 68)
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{P}^{(2)} &= \left\{\underset{\sim}{p}^{(2,x)}, \underset{\sim}{p}^{(2,y)}\right\} \\
\underset{\sim}{p}^{(2,x)} &= (58, 75, 75, 94) \\
\underset{\sim}{p}^{(2,y)} &= (87, 103, 105, 121)
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{P}^{(3)} &= \left\{\underset{\sim}{p}^{(3,x)}, \underset{\sim}{p}^{(3,y)}\right\} \\
\underset{\sim}{p}^{(3,x)} &= (73, 83, 86, 107) \\
\underset{\sim}{p}^{(3,y)} &= (10, 20, 21, 29)
\end{aligned}$$

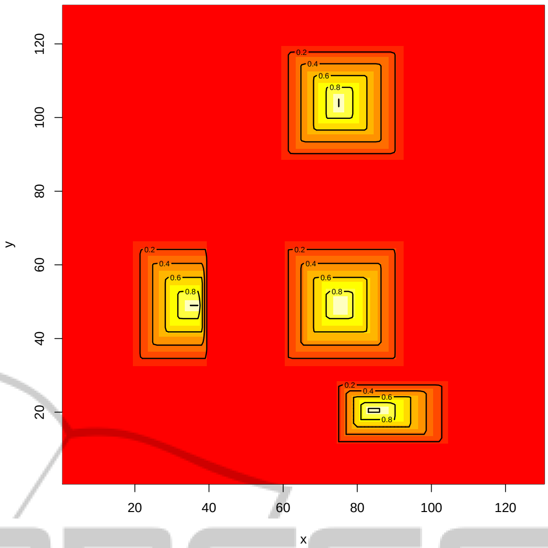

Figure 1: Fuzzy median center.

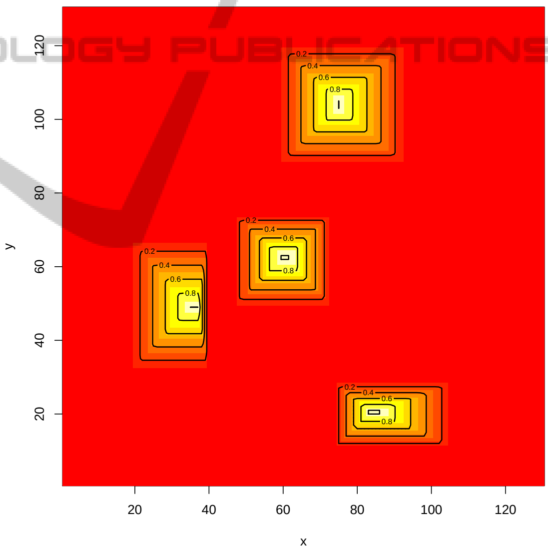

Figure 2: Fuzzy min-max center.

The expected values for these three points would be:

$$\begin{aligned}
E\left(\underset{\sim}{p}^{(1,x)}\right) &= 33.667 \\
E\left(\underset{\sim}{p}^{(1,y)}\right) &= 49.167
\end{aligned}$$

$$\begin{aligned}
E\left(\underset{\sim}{p}^{(2,x)}\right) &= 75.333 \\
E\left(\underset{\sim}{p}^{(2,y)}\right) &= 104
\end{aligned}$$

$$\begin{aligned}
E\left(\underset{\sim}{p}^{(3,x)}\right) &= 86.333 \\
E\left(\underset{\sim}{p}^{(3,y)}\right) &= 20.167
\end{aligned}$$

For these points, the fuzzy median center (see Figure 1) would be:

$$\begin{aligned}
\underset{\sim}{M} &= \left\{\underset{\sim}{m^{(x)}}, \underset{\sim}{m^{(y)}}\right\} \\
\underset{\sim}{m^{(x)}} &= \underset{\sim}{\text{median}}_{i=1,\dots,3}\left(\underset{\sim}{p^{(i,x)}}\right) \\
&= (58, 75, 75, 94) \\
\underset{\sim}{m^{(y)}} &= \underset{\sim}{\text{median}}_{i=1,\dots,3}\left(\underset{\sim}{p^{(i,y)}}\right) \\
&= (31, 49, 49, 68)\,.
\end{aligned}$$

And the min-max center (see Figure 2) would be:

$$\begin{aligned}
\underset{\sim}{Z} &= \left\{\underset{\sim}{z^{(x)}}, \underset{\sim}{z^{(y)}}\right\} \\
\underset{\sim}{z^{(x)}} &= \tfrac{1}{2}\textstyle\sum_{i\in\{1,3\}} \underset{\sim}{p^{([i],x)}} \\
&= (49.667, 64.333, 66, 80.333) \\
\underset{\sim}{z^{(y)}} &= \tfrac{1}{2}\textstyle\sum_{i\in\{1,3\}} \underset{\sim}{p^{([i],y)}} \\
&= (42.667, 57.333, 58.333, 72.667)\,.
\end{aligned}$$

As we can see from the figures, the option of using fuzzy numbers to model the demand points is much more closer to what in reality geographers and planners face. The results obtained will give them flexibility in the final location of the center, according to constraints not easily modeled otherwise.

6 CONCLUSIONS

In this paper we have shown that the results found for the solution of the median center and the min-max center can be extended to fuzzy environments, where both the demand points and the center are modeled with fuzzy numbers. The use of fuzzy numbers is due to the need to reflect the uncertainty about available information on demand. Not only the data might be vague or subjective, but it could also involve disagreements or lack of confidence in the methodology used in its collection. Therefore, it is necessary to have a solution that, while simply obtained, incorporates this uncertainty.

Fuzzy solutions can also give flexibility to planners on the final location of the center, according to constraints that are not easily modeled. The selected center will have a membership value that reflects its "appropriateness" according to the data.

Future work deriving from this methodology will follow solving the fuzzy *1*-median problem as well as the barycenter, when the solution is modeled with fuzzy numbers. We would like to use the results found for multicriteria analysis, creating a fuzzy Pareto front by intersecting the solutions found for different values of p.

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