SATISFIABILITY DEGREE THEORY FOR TEMPORAL LOGIC*

Jian Luo, Guiming Luo and Mo Xia
Key Laboratory for Information System Security, Ministry of Education, Tsinghua National Laboratory for Information Science and Technology, School of Software, Tsinghua University, Beijing 100084, China

Keywords: Satisfiability degree, Propositional logic, Temporal logic, Reasoning.

Abstract: The truth value of propositional logic is not capable of representing the real word full of complexity and diversity. The requirements of the proposition satisfiability are reviewed in this paper. Every state is labeled with a vector, which is defined by the proposition satisfiability degree. The satisfiability degree for temporal logic is proposed based on the vector of satisfiability degree. It is used to interpret the truth degree of the temporal logic instead of true or false. A sound and precise reasoning system for temporal logic is established and the computation is given. One example of a leadership election is included to show that uncertain information can be quantized by the satisfiability degree.

1 INTRODUCTION

The idea of temporal logic (Mattolini and Nesi, 2000) is that a formula is not statically true or false in a time model. Instead, the models of temporal logical contain several states and a formula can be true in some states and false in others. The formulas may change their truth values as the system evolves from state to state, but the truth values of the formulas are true or false. Sometimes, a state partially satisfies a formula, so it is not absolutely true or false, and the semantic of the temporal logic, which is based the classical Boolean logic, cannot interpret this case. Thus, the world requires new ways to express uncertainty. Many studies have used non-classical logic, such as fuzzy logic (Bergmann, 2008), probabilistic logic (Raedt and Kersting, 2003), modal logic, etc.

Satisfiability degree, a new precise logic presentation, was proposed in (Luo and Yin, 2009). It describes the extent to which a proposition is true based on the truth table by finding out the proportion of satisfiable interpretations. Unlike fuzzy logic and probabilistic logic, satisfiability degree does not need membership function or distribution function and it is determined by the proposition itself. Moreover, satisfiability degree extends the concepts of satisfaction and contradictory propositions in Boolean logic and truth values of propositions are precisely interpreted as their satisfiability degrees.

*This work is supported by the Funds NSFC 60973049, 60635020, and TNList cross-discipline foundations.

Sometimes, given the premise is true, we want to deduce the truth degree of a considered conclusion. The conditional satisfiability degree was proposed in (Luo and Yin, 2009), to quantitatively represent the deductive reasoning, which is based on if the satisfiability degree of premise is given, we deduce the satisfiability degree of the conclusion.

There are many algorithms to compute the satisfiability degree, such as the backtracking algorithm (Yin et al., 2009), the satisfiability degree computation based on CNF (Hu et al., 2009), the algorithm based on binary decision diagrams (Luo and Yin, 2009) and the propositional matrix search algorithm (Luo and Luo, 2010). Once a propositional formula is given, its satisfiability degree can be precisely computed using those algorithms. Thus, this paper only focuses on the performance and properties of satisfiability degree based on the temporal logic.

Because the temporal logic is based on propositional logic and temporal connectives, the truth value of a temporal formula can be precise interpreted by satisfiability degree. Thus, if there are only several models are available for the concerned formula, we can choose the model with maximum satisfiability degree. Sometimes, a model checker may not find counterexamples but it does means the system cannot be applied to some domains, but satisfiability degree can provide us a quantitative analysis of model checking (Kang and Park, 2005).

Luo J., Luo G. and Xia M..
SATISFIABILITY DEGREE THEORY FOR TEMPORAL LOGIC.
DOI: 10.5220/0003672804970500
In *Proceedings of the International Conference on Evolutionary Computation Theory and Applications* (FCTA-2011), pages 497-500
ISBN: 978-989-8425-83-6

2 SATISFIABILITY DEGREE

Thus, we use satisfiability degree to define the truth values of propositions in states instead of "true" or false. Sometimes, it is difficult to choose some atomic descriptions, because some properties cannot be subdivided into. We define a new temporal model on n propositions set and use satisfiability degree to interpret the truth values of those propositions.

2.1 Temporal Model

Definition 1. Let Σ be a set of n propositions. A model M over Σ is a triple $M = (S, R, V)$ where:

- S is a set of states;
- $R \subset S \times S$ is a total transition relation, i.e., any state $s \in S$, there is a state $s' \in S$, s.t. $(s, s') \in R$;
- $V : \Sigma^n \times S \rightarrow [0,1]^n$ is a function that mapping the satisfiability degree of each proposition in s .

Thus, our model has a collection of state S, a relation R, saying how the system can move from state to state, and, associated with each state s, one has the n-dimension vector $V(s)$, components of which are satisfiability degree, denoted as $f(p,s)$:

$$V(s) = (f(p_1,s), f(p_2,s), ...f(p_n,s))^T \qquad (1)$$

Where $f(p_i,s)$ is the satisfiability degree of p_i in state s. We can express all the information about a model M using a directed graph and Figure 1 bellow shows us an example, where $S = \{s_0, s_1, s_2\}, \Sigma = \{p,q,r\}$ and $V(s_0) = (0.5, 0.6, 0.2), V(s_1) = (0.2, 0.3, 0.1)$ and $V(s_2) = (0.8, 0, 1)$.

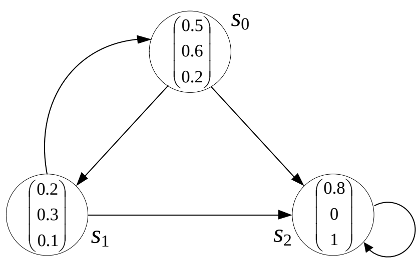

Figure 1: A temporal model based on satisfiability degree.

Note that, if all the propositions in set Σ are atomic propositions, then their satisfiability degree are 0 or 1. That is the classical temporal model referred to the paper (Huth and Ryan, 2005). Therefore, the models we define are more expressive than that of classical temporal models.

Definition 2. Let $M = (S, R, V)$ be a model. A path ρ in M is an infinite sequence $s_0 \cdot s_1 \cdot s_2 \ldots$ of states such that, for $\forall i \geq 0, (s_i, s_{i+1}) \in R$.

Consider the path $\rho = s_0 \cdot s_1 \cdot s_2 \ldots$ It represents a possible future of our system. We write ρ^i for the suffix starting at s_i, e.g.ρ^3 is $s_3 \cdot s_4 \ldots$

2.2 Satisfiability Degree for CTL*

The syntax of CTL* involves two classes of formulas, state formula and path formula.

- state formula, which are evaluated in states:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{A}[\alpha] \mid \mathbf{E}[\alpha] \qquad (2)$$

  Where $p \in \Sigma$ and α is any path formula; and
- path formula, which are evaluated along paths:

$$\alpha ::= \varphi \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha\, \mathbf{U}\, \alpha \mid \mathbf{G}\, \alpha \mid \mathbf{F}\, \alpha \mid \mathbf{X}\, \alpha \quad (3)$$

The semantics (Huth and Ryan, 2005) of CTL* was defined by a satisfaction relation, denoted by $\vDash$, which is characterized as the least relation on the paths. Since the interpretation of a CTL* formula varies over states, we denote the satisfiability degree of a CTL* formula φ at state s according to the model M as $f_s^M(\varphi)$.

Definition 3. For a model $M = (S, R, V)$, the satisfiability degree of a CTL* formula at state s is defined inductively as:

- $\forall\ \varphi \in \mathbf{P}, f_s^M(\varphi) = f(\varphi, s)$, where $\mathbf{P}$ is the set of Boolean formulas.
- $f_s^M(\neg\varphi) = 1 - f_s^M(\varphi)$
- $f_s^M(\varphi_1 \wedge \varphi_2) = \min(f_s^M(\varphi_1), f_s^M(\varphi_2)), \varphi_1 \wedge \varphi_2 \notin \mathbf{P}$
- $f_s^M(\mathbf{X}\ \alpha) = f_{s^*}^M(\alpha)$, where s^* is the next state.
- $f_s^M(\mathbf{F}\ \alpha) = \underset{\forall s^* \in \rho}{sup}\ f_{s^*}^M(\alpha)$, where ρ start with s and s^* is any state on ρ.
- $f_s^M(\mathbf{G}\ \alpha) = \inf\limits_{\forall s^* \in \rho} f_{s^*}^M(\alpha)$, where ρ start with s and s^* is any state on ρ.
- $f_s^M(\alpha_1 \mathbf{U} \alpha_2)$ $= \underset{\forall s_j \in \rho}{sup}\ (f_{s_0}^M(\alpha_2), \min\limits_{0 \leq i \leq j}(f_{s_i}^M(\alpha_1), f_{s_j}^M(\alpha_2)))$, where $\rho = s_0 \cdot s_1 \cdot s_2 \ldots$, and $s_0 = s$, starts with s.
- $f_s^M(\mathbf{A}[\alpha]) = \inf\limits_{\forall\rho} f_s^M(\alpha)$, where ρ starts with s.
- $f_s^M(\mathbf{E}[\alpha]) = \underset{\forall\rho}{sup}\, f_s^M(\alpha)$, where ρ starts with s.

The truth value of each propositional formula is determined by the state s with respect to the satisfiability degree vector $V(s)$. Suppose $M = (S, R, V)$ be a model, $s \in S$, and φ a CTL* formula. We have the follow conclusions: If $M, s \models \varphi$ if, and only if we have $f_s^M(\varphi) = 1$.

3 A REASONING SYSTEM

A signed temporal formula can is a tuple (φ, f) where f is the satisfiability degree of φ . The proposed reasoning system is a pair $\nu = (A, L)$ where A is a set of temporal logic axioms, and L is a collection of inference rules. An inference rule based on satisfiability degree is a pair $l = (l_{op}, l_f)$ where l_{op} is syntactical component that operates on temporal formulas, while l_f is a valuation component operating on satisfiability degree to compute the satisfiability degree of the conclusion depending on the satisfiability degrees of the premises. A rule l is usually written as:

$$\frac{\varphi_1, \varphi_2, \dots \varphi_n}{l_{op}(\varphi_1, \varphi_2 \dots \varphi_n)}, \frac{f_1, f_2, \dots f_n}{l_f(f_1, f_2 \dots f_n)} \quad (4)$$

This expression means that if the formulas $\varphi_1, \varphi_2 \dots \varphi_n$ have the satisfiability degree $f_1, f_2 \dots f_n$ respectively, then $l_{op}(\varphi_1, \varphi_2, \dot{\varphi}_n)$ is satisfiable at least to the satisfiability degree $l_f(f_1, f_2, \dots f_n)$.

The inference rules for Boolean operations:

$$L_\wedge : \frac{p,\ q}{p \wedge q}, \frac{f_1, f_2}{f_1 f_2} \quad (5)$$

$$L_\vee : \frac{p,\ q}{p \vee q}, \frac{f_1, f_2}{f_1 + f_2 - f_1 f_2} \quad (6)$$

$$L_\rightarrow : \frac{p,\ q}{p \rightarrow q}, \frac{f_1, f_2}{1 - f_1 + f_1 f_2} \quad (7)$$

And for any CTL* formula φ, we have

$$L_\neg : \frac{\varphi}{\neg\varphi}, \frac{f}{1 - f} \quad (8)$$

The theorems proved by the natural deduction can also be proved by our reasoning system, because we have the following theorem. Its proof is omitted.

Theorem 1. For propositional formulas φ and ϕ :

(1) $\varphi \vDash \phi$ if and only if $f(\phi \mid \varphi) = 1$;

(2) $\varphi \vdash \phi$ if and only if $f(\phi \mid \varphi) = 1$.

In addition, there are four kinds of generalization rules for temporal logic.

$$L_{AG} : \frac{\varphi}{\mathbf{AG}\ \varphi}, \frac{f}{f} \quad (9)$$

$$L_{GF} : \frac{\mathbf{G}\ \varphi}{\mathbf{F}\ \varphi}, \frac{f}{f} \quad (10)$$

$$L_X : \frac{\varphi}{\mathbf{F}\ \varphi}, \frac{f}{f} \quad (11)$$

$$L_{AE} : \frac{\mathbf{A}\ \alpha}{\mathbf{E}\ \alpha}, \frac{f}{f} \quad (12)$$

Theorem 2. Inference rules (5)-(12) are sound.

Proof: the rule $L_\neg$ is trivial. Let $\omega_1, \omega_2 \in \Omega$ be interpretations such that $p(\omega_1) = 1$, and $q(\omega_2) = 1$, then $\omega = (\omega_1, \omega_2) \in \Omega \times \Omega, p \wedge q(\omega) = 1$. By the Definition of satisfiability degree, we have

$$f(p \wedge q) = \frac{|\Omega_p \times \Omega_q|}{|\Omega \times \Omega|} = f(p)f(q) = f_1 f_2 \quad (13)$$

The rules such as $L_\vee$ and $L_\rightarrow$ can be derive from $L_\wedge$ and $L_\neg$, since $p \wedge q = \neg(\neg p \wedge \neg q)$ and $p \rightarrow q = \neg p \vee q$. If a CTL formula φ has the minimum satisfiability degree f regardless of models, then $\mathbf{G}\ \varphi$ has the same satisfiability degree f, so L_G is sound. Since for any state formula, we have:

$$f_s^M(\mathbf{M}\ \varphi) \geq f_s^M(\mathbf{G}\ \varphi) \quad (14)$$

Thus L_{GF} is sound. L_X and L_{AE} are trivial sound. It is very difficult to establish a complete one for CTL*, the completeness of our deductive system for CTL* need to be proved in our future works.

4 APPLICATION

This section presents another application, called Leader Election. This problem provide a protocol to select a leader out of their midst. The winning candidate needs more than half the available votes cast. Assume everyone is a candidate as well as a voter, he has a unique identity, *id* for short, selected from {1, 2,.., N}, we have two methods: one, the candidate can randomly vote his guy, including himself, then we check the result, if someone has more than half votes, he wins, otherwise we enter the next round and do it again until someone is elected as the leader. Second method, we do the same thing on the first round, but if nobody is selected, we change the rule on next round, we choose only two candidates with the most votes, of course, if there exist equal votes, select the one with bigger *id*. Then, vote again. We cannot make the election round and round because the expense of election is high. We should emphasize that a voter may change his opinion or position such that the results of each round are different, thus the extent of successful election varies with states or time. We use two propositions to describe the selection process of the two methods:

- p : a leader is elected now.
- r : it needs the next election.

Let M_1 be the model of the first method described as Figure 2. The meanings of each state are:

- s_{10} : the initial sate that no leader is elected and it does need the next election.

- s_{11} : the last election can come to nothing such that it will need the next election.
- s_{12} : The last election can be successful, so it needs no election any more.

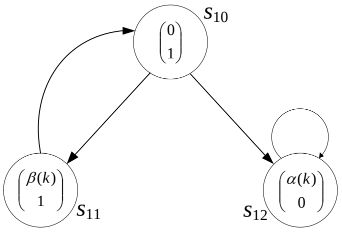

Figure 2: The model of the first method.

Note that $\alpha(k)$ is the satisfiability degree of a successful kth election, $\beta(k)$ for unsuccessful kth election, where

$$\alpha(k) + \beta(k) = 1 \tag{15}$$

Because, the second method can change its rules if the first election fails, so it is different from M_1. We use M_2 to model the second method, please see Figure 3. The meanings of each state are:

- s_{20}: the initial state that no leader is elected.
- s_{21}: the last election can come to nothing such that it will need the next election. Once the first election fails, it will go to state s_{21} and change its rules. Thus, it will go to itself rather than s_{20} like the model M_1.
- s_{22}: the last election can be successful, so it needs no election any more.

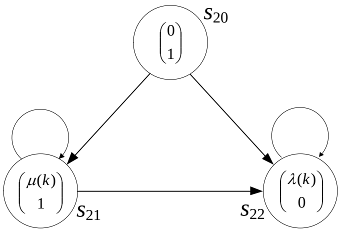

Figure 3: The model of the second method.

Note that $\lambda(k)$ is the satisfiability degree of a successful kth election, $\mu(k)$ for unsuccessful kth election, where

$$\lambda(k) + \mu(k) = 1 \tag{16}$$

Then, we concern the property that a leader is elected in only two rounds, which is written as:

$$\varphi = \mathbf{X}\ (p \wedge r = 0) \vee \mathbf{X}\,\mathbf{X}(p \wedge r = 0) \tag{17}$$

If $f_{s_{10}}^{M_1}(\varphi) < f_{s_{20}}^{M_2}(\varphi)$, M_2 is better than M_1, vice versa. However, if we use model checker, both M_1 and M_2 may not satisfy the formula, thus satisfiability degree can better differentiate the two protocols.

5 CONCLUSIONS

Satisfiability degree is a new method to precisely express the satisfiable extent of a formula. It is an inherent attribute of the proposition not depending on the interpretations. Since the propositional logic is the basis of temporal logic, satisfiability degree can be extended to the temporal logic such that the new temporal model can be constructed by satisfiability degree. That model is more expressive than the classical temporal model. On a model, the truth value of a temporal formula is precisely interpreted as its satisfiability degree. Based on satisfiability degree, a deductive system for CTL* is established; some interference rules and axiom schemes are given, and its soundness is proved. The example, Leader Election, shows that satisfiability degree can be used to do a quantitative analysis for uncertain model checking.

REFERENCES

Bergmann, M. (2008). *An introduction to many-valued logic: semantics, algebras, and derivation systems*. Cambridge University Press, Cambridge.

Hu, P., Luo, G. M., and Yin, X. Y. (2009). Computation of satisfiability degree based on cnf. In *Proceedings of the International Conference on Fuzzy Systems and Knowledge Discovery*. v6:142-146.

Huth, M. and Ryan, M. (2005). *Logic in Computer Science: Modeling and Reasoning about Systems*. China Machine Press, Beijing, 2nd edition.

Kang, H. and Park, I. (2005). Sat-based unbounded symbolic model chcking. In *IEEE Transactions on Computer-aided Design of Integrated circuits and systems*. vol.24(2):129-140.

Luo, G. M. and Yin, C. Y. (2009). An algorithm for calculating the satisfiability degee. In *proceedings of the 2009 sixth international Conference on Fuzzy System and Knowledge Discovery*. pp.322-326.

Luo, J. and Luo, G. M. (2010). Propositional matrix search algorithm for satisfiability degree computation. In *the 9th IEEE International Conference on Cognitive Informatics*. Beijing, China, pp.974-977.

Mattolini, P. B. R. and Nesi, P. (2000). Temporal logics for real time system specification. In *ACM Comput. Surveys*. vol.32, no.1, pp.12-42.

Raedt, L. D. and Kersting, K. (2003). Probabilistic logic learning. In *ACM SIGKDD Explorations Newsletter*. vol.5(1),31-48.

Yin, C. Y., Luo, G. M., and Hu, P. (2009). Backtracking search algorithm for satisfiability degree calculation. In *proceedings of the 2009 sixth international Conference on Fuzzy System and Knowledge Discovery*. Tianjin,China,2009,pp.3-7.