FURTHER REMARKS ON INVARIANCE PROPERTIES OF TIME-DELAY AND SWITCHING SYSTEMS

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Abstract: The present paper deals with correlation in the context of mRPI sets between discrete linear systems affected by time delay and switching systems. Existence and uniqueness of mRPI set for both systems are studied. One of the possible construction procedures of invariant approximations of mRPI set is also outlined. In order to keep this exposure as coherent as possible, all results are firstly consider separately for both cases. Special attention is put on the link between obtained results. An illustrative example is provided at the end.

1 INTRODUCTION

Time delay is often the essential property of the dynamic systems, primarily due to the transport and transfer phenomena (materials, energy, informations) (Sipahi et al., 2011), (Niculescu, 2001). Delay systems could be also affected by exogenous, additive disturbance input. For this problem, employing the invariant set theory could be of great help in analysis and synthesis as long as it provides useful information about limit behavior and the contractive properties (Lombardi et al., 2011).
In this study we consider delay systems with additive disturbance w_k and fixed delay $d \in \mathbb{Z}_+$ (d is positive integer), described by following linear delay difference equation in state-space:

$$\Sigma_d: \quad \mathbf{x}_{k+1} = A_0\mathbf{x}_k + A_d\mathbf{x}_{k-d} + \mathbf{w}_k. \tag{1}$$

Invariant sets (in particular positive invariant sets) have received increased attention in automatic control recently, especially in constrained and robust control. When the considered system is autonomous and linear with bounded additive disturbance, one of the issues is the characterization and the computation of the minimal robust positive invariant set (Kolmanovsky and Gilbert, 1998). This set can be observed as the set of states that can be reached from the origin under bounded disturbance signal (often referred as 0-reachable set (Blanchini and Miani, 2008)). From previous results in the field, it is well-known that by lifting dynamics to the space of sets and using contractive set-iterations is possible to construct invariant approximations of mRPI set very elegantly (Artstein and Rakovic, 2008). For this purpose we will mostly use polyhedral sets, since they have an advantage to follow shape of limit sets more precisely, in spite of their computational complexity.

Switching systems are a particular group of systems that could be described as finite number of independent dynamics, represented by its differential equation, combined by means of switching signal (Liberzon, 2003). At all instance of time, switching signal determines which of a finite dynamics is currently active. In this work we will particularly focus on the switching systems for which stability is not affected by admissible switching function (arbitrary switching case). Switching system considered in present study is given by subsequent linear difference equation in state-space:

$$\Sigma_s: \quad \mathbf{x}_{k+1} = A_i\mathbf{x}_k + \mathbf{w}_k, \tag{2}$$

where $i : \mathbb{Z}_+ \to \{0, d\}$ is switching signal.

In both cases, Σ_d and Σ_s, we assume that disturbance is uniformly distributed and takes values from compact and convex set W with 0 in its interior.

The main goal of our study is to point out a certain correlation, from the set theoretic point of view, between mRPI sets for systems affected by time delay and switching dynamics.

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The remaining paper is organized as follows: Section 2 includes some preliminary results and related background material. In Section 3 we introduce the main results of the paper. Section 4 provides illustrative examples while concluding remarks are outlined in Section 5.

Notation

Throughout present study sets of real numbers, non-negative real numbers, integers and non-negative integers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively. For a matrix $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ stands for the largest absolute value of its eigenvalues. Thus, the spectral norm $\sigma(A)$ is defined as $\sigma(A) := \sqrt{\rho(A^TA)}$. For two sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$ and vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$ set addition (Minkowski sum) is defined as $X \oplus Y := \{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in X, \mathbf{y} \in Y\}$ while set product is defined as $X \times Y := \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in X, \mathbf{y} \in Y\}$. For a given set X and a real matrix (or a scalar) M of compatible dimensions, we define $MX := \{Mx \mid x \in X\}$. Set obtained as the intersection of finite number of open and/or closed half-spaces is a polyhedral set while closed and bounded polyhedral set will be referred as polytope. A set $X \subset \mathbb{R}^n$ is 0-symmetric set if holds $X = -X$.
For two arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$, the p-norm distance d is defined as $d(\mathbf{x},\mathbf{y}) = (\Sigma_{i=1}^n |x_i - y_i|^p)^{1/p}$. For two non-empty sets X and Y, the Hausdorff distance is defined as $d_H(X,Y) := min_\alpha\{\alpha \mid X \subseteq Y \oplus \alpha L, Y \subseteq X \oplus \alpha L\}$, where L is a given symmetric, compact and convex set with 0 in its interior. For some $\varepsilon > 0$ we denote $\mathbb{B}_p^n(\varepsilon) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_p \le \varepsilon\}$.

2 PRELIMINARIES AND PREREQUISITES

Present paper relies on following standard results in the literature (Blanchini and Miani, 2008):

Lemma 1 (Banach fixed point theorem)**.** *Let $(X, d(\cdot,\cdot))$ be a complete metric space and let $f(\cdot): X \to X$ be a contractive function with contraction factor $\lambda \in [0,1)$ that is*

$$d(f(x), f(y)) \le \lambda d(x,y), \quad \lambda \in [0,1)$$

holds for all $x,y \in X$. Then there exists exactly one point $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$.

Lemma 2. *Let X,Y and Z be convex and compact sets with 0 in its interior, α,β are real parameters such that $\alpha \ge \beta > 0$ and M is a matrix of appropriate dimension. Then: $X \oplus Y = Y \oplus X$, $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$, $\alpha X \oplus \beta X = (\alpha+\beta)X$, $M(X \oplus Y) = MX \oplus MY$ and $X \subseteq Y \Leftrightarrow X \oplus Z \subseteq Y \oplus Z$.*

In the sequel we will use the following definition.

Definition 1 (Robust positively invariant (RPI) set)**.**

(*i*) *A set $\Phi^d \subset \mathbb{R}^n$ is a RPI set for the system Σ_d (1) with all initial conditions $\mathbf{x}_{-i} \in \Phi^d$, $i \in \mathbb{Z}_{[0,d]}$, if and only if $\mathbf{x}_k \in \Phi^d$, for $\forall k \in \mathbb{Z}_+$ and $\forall w \in W$ (alternatively in set-theoretic framework $A_0\Phi^d \oplus A_d\Phi^d \oplus W \subseteq \Phi^d$).*

(*ii*) *A set $\Phi^s \subset \mathbb{R}^n$ is a RPI set for the system Σ_s (2) if and only if $\mathbf{x}_k \in \Phi^s$ for $\forall k \in \mathbb{Z}_+$, $\forall w \in W$ and for all switchings i (alternatively in set-theoretic framework $\{A_0\Phi^s \oplus W\} \bigcup \{A_d\Phi^s \oplus W\} \subseteq \Phi^s$).*

(*iii*) *The minimal robust positively invariant set is defined as the RPI set contained in any closed RPI set. The mRPI set is unique, compact and contain the origin if W contains the origin.*

Definition 2. *Given a scalar $\varepsilon > 0$ and sets $\Omega \subset \mathbb{R}^n$ and $\Phi \subset \mathbb{R}^n$. The set $\Phi \subset \mathbb{R}^n$ is an outer ε-approximation of Ω if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}_p^n(\varepsilon)$.*

Definition 3. *A C-set is a convex and compact subset of $\mathbb{R}^n$ including the origin as an interior point.*

3 MAIN RESULTS

As we mentioned in the introduction, first we will focus on invariance properties of time-delay systems, and we will consider the switching case next. The correlation between obtained results and supplementary discussion are exposed at the end.

3.1 Time-delay Case

In the current subsection remarks on existence, uniqueness and construction of the invariant approximations of mRPI set for the system Σ_d (1) are presented.

Let consider dynamics Σ_d (1), expressed in space of sets by following set-valued map:

$$\Omega^d: \quad \Omega^d(X) = A_0X \oplus A_dX \oplus W, \quad X \subset \mathbb{R}^n \qquad (3)$$

whose range is a convex set if X is convex.

For future analysis we invoke the following assumption:

Assumption 1. *Suppose there exist a symmetric C-set L such that $A_0L \oplus A_dL \subseteq \lambda L$, where $\lambda \in [0,1)$.*

Remark 1. *Previous statement assume existence of the symmetric set L which is contractive with respect to the dynamics $\mathbf{x}_{k+1} = A_0\mathbf{x}_k + A_d\mathbf{x}_{k-d}$. Necessary and sufficient conditions such that assumption 1 hold is still an open problem. One of the possibilities to overcome this is using one of the recently presented sufficient conditions in (Lombardi et al., 2011),*

$\sigma(A_0)+\sigma(A_d)<1$, *or any other condition that meets assumption 1. Note that introduced assumption implies Hurwitz stability of the matrices* A_0 *and* A_d.

According to (Artstein and Rakovic, 2008) and based on the results from the Banach fixed point theorem and Assumption 1, the existence and uniqueness of the mRPI set for Σ_d (1) are obtained from the following theorem:

Theorem 1. *Suppose that assumption 1 holds and* L *and* λ *are used for the computation of the Hausdorff distance* d_H*. Then the set-valued map* Ω^d *(3) is contractive with respect to the Hausdorff distance. Moreover, there exists unique set,* Ω^d_∞*, which is the mRPI for the dynamics* Σ_d *(1).*

Proof. Let denote by X and Y two arbitrary C-sets and $d_H(X,Y)=\alpha$. By the definition of the Hausdorff distance we have:

$$X \subseteq Y \oplus \alpha L, \qquad X \subseteq Y \oplus \alpha L.$$

From previous assertion, using results from Lemma 2 and following statements,

$$A_d X \subseteq A_d Y \oplus \alpha A_d L$$
$$A_d Y \subseteq A_d X \oplus \alpha A_d L$$

we derive:

$$\Omega^d(X) \subseteq \Omega^d(Y) \oplus \alpha(A_0 L \oplus A_d L)$$
$$\Omega^d(Y) \subseteq \Omega^d(X) \oplus \alpha(A_0 L \oplus A_d L)$$

Recalling result from Lemma 3, $A_0 L \oplus A_d L \subseteq \lambda L$, we have:

$$\Omega^d(X) \subseteq \Omega^d(Y) \oplus \alpha\lambda L, \qquad \Omega^d(Y) \subseteq \Omega^d(X) \oplus \alpha\lambda L.$$

which is, by the definition of the Hausdorff distance, $d_H(\Omega^d(X),\Omega^d(Y))=\lambda\alpha$. Since $\alpha=d_H(X,Y)$, the contractivness of the set-valued map Ω^d (3) is guaranteed i.e. $d_H(\Omega^d(X),\Omega^d(Y)) \leq \lambda d_H(X,Y)$.

Because Ω^d (3) is a contraction, according to the Banach fixed point theorem, it has a unique globally asymptotically stable fixed point Ω^d_∞. □

For the simplicity of the exposure let denote by $\Omega^d_k=\Omega^d_k(W)$. In order to define 0-reachable set for dynamics with time delay, we use the follow set iteration:

$$\Omega^d_{k+1} = A_0\Omega^d_k \oplus A_d\Omega^d_k \oplus W. \tag{4}$$

where Ω^d_k is reachable set at forward step k, starting from $\{0\}$. We can notice here that $\Omega^d_k \subseteq \Omega^d_{k+1}$.

Now we will formulate analytic description of k^{th} sequence from the previous iteration. In order to simplify the comprehension of this step we introduce following set of indices $S=\{‘0‘,‘d‘,‘1‘\}$ in correspondence with map Ω^d (3), along with set product $S^k=S\times S\times\cdots\times S$, where $k\in\mathbb{Z}_+$ and $S^0=\{1\}$. Here is important to emphasize that ‘0‘, ‘d‘ and ‘1‘ are not values but indices, and ‘1‘ is an identity element with respect to multiplication e.g. $S^2=S\times S=\{‘00‘,‘0d‘,‘0‘,‘d0‘,‘dd‘,‘d‘,‘1‘\}$. For proposed notations, we can write reachable set at k^{th} forward step as:

$$\Omega^d_k = \bigoplus_{p\in S^k} \mathbb{A}_p W, \tag{5}$$

where $\mathbb{A}_1 := I_n$ and $\mathbb{A}_p$ stands for product of matrices with respect to index p. Since the origin is included in the relative interior of W it follows that it is also included in interior of Ω^d_k. Notice that $S^k \subset S^{k+1}$.

As it is already remarked in (Artstein and Rakovic, 2008), (Rakovic, 2008), mRPI set is given as limit value of the set iteration (4) when $k\to\infty$ i.e.

$$\Omega^d_\infty = \lim_{k\to\infty}\Big(\bigoplus_{p\in S^k}\mathbb{A}_p W\Big) \tag{6}$$

and it is the unique solution to the set-valued map (3)

$$\Omega^d_\infty = A_0\Omega^d_\infty \oplus A_d\Omega^d_\infty \oplus W. \tag{7}$$

This statement can be proved if we observe the limit value of the difference between two subsequent sequences Ω^d_k and Ω^d_{k+1}.

The constructive procedure relies on the results exposed in (Rakovic, 2008) and (Olaru et al., 2010). Invariant approximations of the mRPI set could be constructed from any invariant set for the dynamics Σ_d (1). If such set exists (Olaru et al., 2010), invariant approximations could be obtained by using that set in the contractive map Ω^d (3). This procedure is outlined as follows:

Theorem 2. *If there exists a family of invariant sets with respect to the dynamics* Σ_d *(1), then set iteration* $\Omega^d_k(\Phi^d)$*, for any set* Φ^d *from that family, tends to the mRPI set when* $k\to\infty$ *i.e.* $\lim_{k\to\infty}\Omega^d_k(\Phi^d)=\Omega^d_\infty$*. Moreover, for* $\forall k$ $\Omega^d_k(\Phi^d)$ *is an invariant set.*

Proof. Suppose Φ^d is an invariant set for time-delay system Σ_d (1).
Let first define following set-valued map:

$$R^d: \quad R^d(X) = A_0 X \oplus A_d X$$

along with corresponding set-iteration

$$R^d_{k+1}(X) = A_0 R^d_k(X) \oplus A_d R^d_k(X),$$

where $R^d_0(X)=X$ and X is an C-set. Since the assumption $\rho(A_0)<1$ and $\rho(A_d)<1$ hold, we can notice that $\lim_{k\to\infty} R^d_k(X)=0$.

Let consider map Ω^d (3) with respect to the set Φ^d i.e. $\Omega^d(\Phi^d)=A_0\Phi^d\oplus A_d\Phi^d\oplus W$. This map can be written as:

$$\Omega^d_{k+1}(\Phi^d) = R^d_{k+1}(\Phi^d) \oplus \Omega^d_k$$

where Ω_k^d is defined by equation (5). Since $\lim_{k\to\infty} R_k^d = 0$, limit value of the previous equation may be written as:

$$\lim_{k\to\infty} \Omega_{k+1}^d(\Phi^d) = \lim_{k\to\infty} \Omega_k^d = \Omega_\infty^d.$$

□

Foregoing results point out the existence and uniqueness of the mRPI set for dynamics Σ_d.

3.2 Switching Case

In this subsection results for the class of switching systems are presented in analogy with the time-delay case. In particular, we deal here with existence, uniqueness and approximative construction of mRPI set.

Throughout this study we assume that there exists common Lyapunov function for switching dynamics Σ_s which guaranties the asymptotic stability (Liberzon, 2003).

Assumption 2. *There exists a matrix $P \in \mathbb{R}^{n\times n}$ and $\lambda \in (0,1)$ such that*

$$A_i^T P A_i - P \leq -\lambda P, \quad P = P^T > 0 \tag{8}$$

for $\forall i$.

As in previous case, our observation of the problem is related to the set-theoretic framework. In this sense we introduce the following map:

$$\Omega^s: \quad \Omega^s(X) = \bigcup_i (A_i X \oplus W), \quad X \subseteq \mathbb{R}^n \tag{9}$$

where $i \in \{‘0‘, ‘d‘\}$. In spite of the time delay case, range of the map Ω^s is not a convex set in general, even if X is convex.

As a direct consequence of Assumption 1 we have the following Lemma:

Lemma 3. *Suppose that Assumption 1 holds. Then there exists symmetric C-set L such that*

$$\bigcup_i (A_i L) \subseteq \lambda L, \quad \lambda \in [0,1) \tag{10}$$

where $i \in \{‘0‘, ‘d‘\}$.

Proof. Since Assumption 1 holds, for $\forall c > 0$ we can define $L_c = \{x \in \mathbb{R}^n \mid x^T P x \leq c\}$. Set L_c is an invariant set for switching system Σ_s since it is a level surface of the common Lyapunov function. This set is also symmetric as consequence of the quadratic form of the common Lyapunov function. □

In the sequel the existence and uniqueness of the mRPI set for switching system Σ_s are obtained using Banach fixed point theorem and Lemma 3.

Theorem 3. *Suppose that Assumption 2 is satisfied and L and λ are used for the computation of the Hausdorff distance d_H. Then the set-valued map Ω^s (9) is contractive with respect to the Hausdorff distance, for any compact and convex sets X and Y. Moreover, there exists a unique set, Ω_∞^s, which is the mRPI set for the dynamics Σ_s (2).*

Proof. Let denote by X and Y two arbitrary C-sets and $d_H(X,Y) = \alpha$ such that:

$$X \subseteq Y \oplus \alpha L, \quad Y \subseteq X \oplus \alpha L.$$

By using relations from Lemma 2 we have:

$$\begin{aligned} A_0 X \oplus W &\subseteq A_0 Y \oplus W \oplus \alpha A_0 L \\ A_d X \oplus W &\subseteq A_d Y \oplus W \oplus \alpha A_d L, \end{aligned} \tag{11}$$

and

$$\begin{aligned} A_0 Y \oplus W &\subseteq A_0 X \oplus W \oplus \alpha A_0 L \\ A_d Y \oplus W &\subseteq A_d X \oplus W \oplus \alpha A_d L. \end{aligned} \tag{12}$$

Union of the left-hand sides of pair (11) and pair (12) are included in the union of of the right-hand sides of (11) and (12), respectively.

$$\bigcup_i (A_i X \oplus W) \subseteq \bigcup_i (A_i Y \oplus W) \oplus \bigcup_i \alpha A_i L,$$

$$\bigcup_i (A_i Y \oplus W) \subseteq \bigcup_i (A_i X \oplus W) \oplus \bigcup_i \alpha A_i L.$$

Recalling the set-valued map Ω^s (9) and Lemma 4, previous inclusions may be written as

$$\Omega^s(X) \subseteq \Omega^s(Y) \oplus \alpha\lambda L, \tag{13}$$

$$\Omega^s(Y) \subseteq \Omega^s(X) \oplus \alpha\lambda L. \tag{14}$$

Using the definition of Hausdorff distance, previous statements can be written as

$$d_H(\Omega^s(X), \Omega^s(Y)) = \alpha\lambda \tag{15}$$

which is indeed $d_H(\Omega^s(X), \Omega^s(Y)) \leq \lambda d_H(X,Y)$, since $d_H(X,Y) = \alpha$ and $\lambda \in [0,1)$.

Because the set-valued map Ω^s (9) is a contraction, according to the Banach fixed point theorem it has a unique globally asymptotically stable fixed point, Ω_∞^s (Artstein and Rakovic, 2008). □

Based on the set-valued map Ω^s (9), let define the following set-iteration for $X = \{0\}$:

$$\Omega_{k+1}^s = \bigcup_i (A_i \Omega_k^s \oplus W) \tag{16}$$

where $\Omega_k^s \subseteq \Omega_{k+1}^s$ and $\Omega_0^s = \{0\}$. Here by Ω_k^s is denoted reachable set from the origin at k^{th} forward step of iteration. Since the origin is in the relative interior of W it follows that it is also in the interior of Ω_k^s for $\forall k \in \mathbb{Z}_+$.

Minimal robust positive invariant set is given as the limit value of (16) when $k \to \infty$:

$$\Omega^s_\infty = \lim_{k\to\infty} \Omega^s_k. \qquad (17)$$

We can notice here that minimal robust positive invariant set Ω^s_∞ is not convex in general.

Constructive procedure reported here relies on results proposed in (Rakovic, 2008), (Olaru et al., 2010). For this purpose we invoke following set-iteration:

$$R^s_{k+1} = \bigcup_i (A_i R^s_k), \quad R^s_0 = \{\Phi^s\} \quad k \in \mathbb{Z}_+. \qquad (18)$$

where Φ^s is an initial invariant set with respect to the switching dynamics Σ_s (2). For more details on the computation of invariant approximations of mRPI sets we refer to the (Artstein and Rakovic, 2008), (Rakovic, 2008) and (Olaru et al., 2010).

Theorem 4. *Suppose that Assumption 1 holds. Thus, there exists invariant set Φ^s with respect to the dynamics Σ_s (2), with 0 in its interior. Then $\Omega^s_k \subseteq \Omega^s_{k+1} \subseteq \Omega^s_k \oplus R^s_k$ is satisfied for $\forall k \in \mathbb{Z}_+$. Moreover, set $\Omega^s_k \oplus R^s_k$ is an invariant outer approximation of the minimal robust positive invariant set Ω^s_∞ for $\forall k \in \mathbb{Z}_+$.*

Proof. For two arbitrary non-empty sets X and $Y \subset \mathbb{R}^n$, following properties hold (Rakovic et al., 2005):

$$\bigcup_i [A_i(X \oplus Y) \oplus W] \subseteq \bigcup_i (A_i X \oplus W) \oplus \bigcup_i (A_i Y) \qquad (19)$$

where $i \in \{‘0‘, ‘d‘\}$ and

$$X \subseteq Y \Rightarrow \bigcup_i (A_i X \oplus W) \subseteq \bigcup_i (A_i Y). \qquad (20)$$

Since we assumed the existence of the common Lyapunov quadratic function, then there exist invariant set Φ^s with respect to the switching dynamics.

Statement $\Omega^s_k \subset \Omega^s_{k+1} \subseteq \Omega^s_k \oplus R^s_k$ will be proved by the principle of mathematical induction.
Because Ω^s_k is 0-reachable set, it is evident that $\Omega^s_k \subseteq \Omega^s_{k+1}$ for $\forall k \in \mathbb{Z}_+$. For $\Omega^s_1 = W$ and $\Omega^s_0 \oplus R^s_0 = \Phi^s$ we have by definition $\Omega^s_1 \subset \Omega^s_0 \oplus R^s_0$. Now we assume that $\Omega^s_{k+1} \subseteq \Omega^s_k \oplus R^s_k$. Then, using properties (19) and (20) we have:

$$\Omega^s_{k+2} := \bigcup_i (A_i \Omega^s_{k+1} \oplus W) \subseteq \bigcup_i [A_i(\Omega^s_k \oplus R^s_k) \oplus W]$$

$$\subseteq \bigcup_i (A_i \Omega^s_k \oplus W) \oplus \bigcup_i (A_i R^s_k) = \Omega^s_{k+1} \oplus R^s_{k+1},$$

for $\forall k \in \mathbb{Z}_+$.
Limit value of $\Omega^s_k \oplus R^s_k$, when $k \to \infty$ is:

$$\lim_{k\to\infty} (\Omega^s_k \oplus R^s_k) = \lim_{k\to\infty} \Omega^s_k \oplus \lim_{k\to\infty} R^s_k.$$

Because $\lim_{k\to\infty} R^s_k = 0$, then $\lim_{k\to\infty} \Omega^s_k = \Omega^s_\infty$. □

Most of results reported in this subsection are already proposed in the literature in similar or different form (Rakovic et al., 2005).

3.3 Correlation between Time-delay and Switching Dynamics

In the previous subsections, results on the existence and uniqueness of the minimal robust positive invariant sets are presented using Banach fixed point theorem. In this subsection we propose new approach on analysis of time delay systems from the invariant set point of view, using corresponding switching dynamics. First result in that direction is stated in the following theorem:

Theorem 5. *Let consider matrices $A_0 \in \mathbb{R}^{n\times n}$,$A_d \in \mathbb{R}^{n\times n}$ and a C-set $W \leq \mathbb{R}^n$ that correspond to both dynamics, Σ_d (1) and Σ_s (2). If mRPI sets for both dynamics exist, then mRPI set for the switching dynamics Σ_s is always a subset of mRPI set for the time delay system Σ_d i.e. $\Omega^s_\infty \subseteq \Omega^d_\infty$.*

Proof. For any three sets X, Y and $Z \subset \mathbb{R}^n$, such that $X \subseteq Z$ and $Y \subseteq Z$, the relation $(X \cup Y) \subseteq Z$ holds.

We will prove Theorem 5 using the principal of mathematical induction.

Since $0 \in W$, we can notice that $\Omega^s_0 \subseteq \Omega^d_0$. Assume that $\Omega^s_{k+1} \subseteq \Omega^d_{k+1}$ where Ω^s_{k+1} and Ω^d_{k+1} are defined as:

$$\Omega^s_{k+1} = \bigcup_i (A_i \Omega^s_k \oplus W)$$
$$\Omega^d_{k+1} = A_0 \Omega^d_k \oplus A_d \Omega^d_k \oplus W$$

By definition we have:

$$\Omega^s_{k+2} = (A_0 \Omega^s_{k+1} \oplus W) \cup (A_d \Omega^s_{k+1} \oplus W)$$
$$\Omega^d_{k+2} = A_0 \Omega^d_{k+1} \oplus A_d \Omega^d_{k+1} \oplus W.$$

Since by assumption $\Omega^s_{k+1} \subseteq \Omega^d_{k+1}$ we can get following relations:

$$A_0 \Omega^s_{k+1} \oplus W \subseteq A_0 \Omega^d_{k+1} \oplus A_d \Omega^d_{k+1} \oplus W$$
$$A_d \Omega^s_{k+1} \oplus W \subseteq A_0 \Omega^d_{k+1} \oplus A_d \Omega^d_{k+1} \oplus W.$$

Using Property 1 in previous result we have:

$$\bigcup_i (A_i \Omega^s_{k+1} \oplus W) \subseteq A_0 \Omega^d_{k+1} \oplus A_d \Omega^d_{k+1} \oplus W,$$

that is $\Omega^s_{k+2} \subseteq \Omega^d_{k+2}$. Proof of the Theorem 5 follows from the principal of mathematical induction so we have:

$$\Omega^s_k \subseteq \Omega^d_k, \quad \forall k \in \mathbb{Z}_+.$$

□

Remark 2. *The assumption that the origin is contained in the interior of the convex disturbance set can be relaxed assuming that W has nonempty interior and that there exists a point $c \in W$ that is the* *analytical center of the convex body. The mRPI set corresponding to W now can be expressed as a translation of the mRPI set corresponding to $W \oplus \{-c\}$. For more details we refer to the (Olaru et al., 2010).*

Corollary 1. *A necessary condition for existence of the bounded mRPI set for the time delay system Σ_d (1) is the boundedness of the mRPI set for the switching dynamics Σ_s (2).*

Corollary states necessary condition for existence mRPI set for time-delay systems via existence of mRPI set for switching systems. What is more important, it has shown that these two different systems dynamics may be correlated from the stability point of view.

4 ILLUSTRATIVE EXAMPLE

In order to clarify exposed theory, an illustrative example is outlined in this section.

Consider the discrete time-delay system Σ_d (1) and switching dynamics Σ_s (2) represented by the triplet (A_0, A_d, W), where

$$A_0 = \begin{bmatrix} 0.2 & 0 \\ -0.15 & 0.3 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.3 & -0.15 \\ 0.2 & 0.25 \end{bmatrix}$$

and $W = \|x\|_1 \leq 1$, $x \in \mathbb{R}^2$. Initial invariant set for both systems is arbitrary chosen as $\Phi^d = \Phi^s = \|x\|_1 \leq 6$.

All reachable sets were obtained by a direct application of defined set-iterations. Since the Minkowski addition is computationally very expensive, we present results just for lower dimensional polytopes, i.e. iterations 0 to 5 (See Fig1).

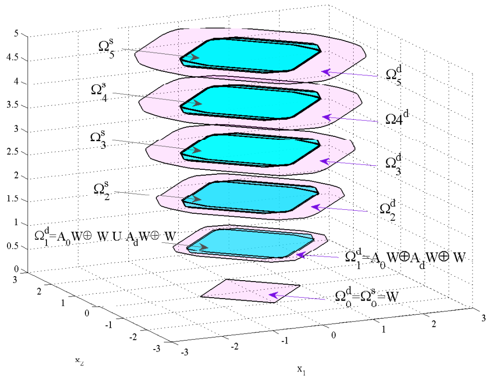

Figure 1: 0-reachable set for time delay system and switching dynamics - lower dimensional polytopes.

5 CONCLUSION REMARKS

This paper has reported discussion on minimal robust positive invariant set for time delay and switching systems and their correlation. We showed that the existence of mRPI set for switching system is a necessary condition for existence of mRPI set for corresponding time delay dynamics. What is more important, we set up connection between two classes of different dynamics, which gives us new theoretical approach in the analysis of some open questions such as necessary and sufficient conditions for existence of invariant sets for time-delay systems.

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