HIDDEN ATTRACTOR IN CHUA'S CIRCUITS

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Abstract: Notion of hidden attractor (basin does not contain neighborhoods of equilibria) is discussed. Effective analytical-numerical procedure for hidden attractors localization is considered. Existence of hidden attractor in Chua's circuits is demonstrated.

1 INTRODUCTION

The classical attractors of Lorenz (Lorenz, 1963), Rossler (Rossler, 1976), Chua (Chua & Lin, 1990), Chen (Chen & Ueta, 1999), and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it.

However there are attractors of another type: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria* (Leonov et. al., 2011). Here equilibria are not "*connected*" with attractor and creation of numerical procedure of integration of trajectories for the passage from equilibrium to periodic solution is impossible because the neighbourhood of equilibrium does not belong to such attractor. The simplest examples of systems with such hidden attractors are hidden oscillations in counterexamples to widely-known Aizerman's and Kalman's conjectures on absolute stability (see, e.g., (Leonov, 2010; Leonov et. al., 2010b)). Similar computational problems arise in investigation of semi-stable and nested limit cycles in 16th Hilbert problem (see, e.g., (Kuznetsov & Leonov, 2008; Leonov & Kuznetsov, 2010; Leonov et. al., 2011)).

Here a special analytical-numerical algorithm for localization of hidden attractors is considered. Example of hidden attractor localization in Chua's circuit, which is used for hidden chaotic communication (Zhiguo et al., 2008), is demonstrated.

Chua's circuit (see Fig. 1) can be described by differential equations in dimensionless coordinates:

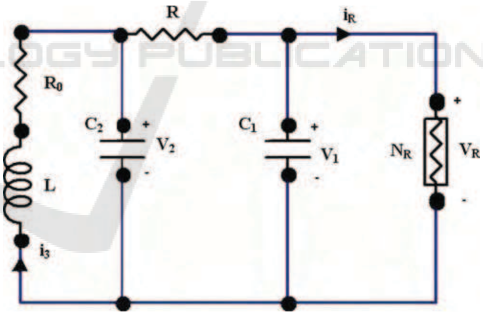

Figure 1: Classical Chu's circuit.

$$\begin{aligned}\dot{x} &= \alpha(y-x) - \alpha f(x),\\ \dot{y} &= x - y + z,\\ \dot{z} &= -(\beta y + \gamma z).\end{aligned} \tag{1}$$

Here the function

$$\begin{aligned} f(x) &= m_1 x + (m_0 - m_1)\mathrm{sat}(x) = \\ &= m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)\end{aligned} \tag{2}$$

characterizes a nonlinear element, of the system, called *Chua's diode*; $\alpha, \beta, \gamma, m_0, m_1$ are parameters of the system. In this system it was discovered the strange attractors (Chua, 1992; Chua, 1995) called then Chua's attractors. All known classical Chua's attractors are the attractors that are excited from unstable equilibria. and this makes it possible to compute such attractors with relative easy (see, e.g., attractors gallery in (Bilotta & Pantano, 2008).

The applied in this work algorithm shows the possibility of existence of hidden attractor in system (1). Note that L. Chua himself, analyzing in the work

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(Chua & Lin, 1990) different cases of attractor existence in Chua's circuit, does not admit the existence of such hidden attractor.

2 ANALYTICAL-NUMERICAL FOR ATTRACTORS LOCALIZATION

Consider a system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \psi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n, \tag{3}$$

where $\mathbf{P}$ is a constant $n \times n$-matrix, $\psi(\mathbf{x})$ is a continuous vector-function, and $\psi(0) = 0$.

Define a matrix $\mathbf{K}$ in such a way that the matrix

$$\mathbf{P}_0 = \mathbf{P} + \mathbf{K} \tag{4}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$) and the rest of its eigenvalues have negative real parts. We assume that such $\mathbf{K}$ exists. Rewrite system (3) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \varphi(\mathbf{x}), \tag{5}$$

where $\varphi(\mathbf{x}) = \psi(\mathbf{x}) - \mathbf{K}\mathbf{x}$.

Introduce a finite sequence of functions $\varphi^0(\mathbf{x}), \varphi^1(\mathbf{x}), \ldots, \varphi^m(\mathbf{x})$ such that the graphs of neighboring functions $\varphi^j(\mathbf{x})$ and $\varphi^{j+1}(\mathbf{x})$ slightly differ from one another, the function $\varphi^0(\mathbf{x})$ is small, and $\varphi^m(\mathbf{x}) = \varphi(\mathbf{x})$. Using a smallness of function $\varphi^0(\mathbf{x})$, we can apply and mathematically strictly justify (Leonov, 2009; Leonov, 2009; Leonov, 2010; Leonov et. al., 2010a; Leonov et. al., 2010b) the method of harmonic linearization (describing function method) for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \varphi^0(\mathbf{x}), \tag{6}$$

and determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$. For the localization of attractor of original system (5), we shall follow numerically the transformation of this periodic solution (a starting *oscillating attractor* — an attractor, not including equilibria, denoted further by $\mathcal{A}_0$) with increasing j. Here two cases are possible: all the points of $\mathcal{A}_0$ are in an attraction domain of attractor $\mathcal{A}_1$, being an oscillating attractor of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \varphi^j(\mathbf{x}) \tag{7}$$

with $j = 1$, or in the change from system (6) to system (7) with $j = 1$ it is observed a loss of stability bifurcation and the vanishing of $\mathcal{A}_0$. In the first case the solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of system (7) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. If in the process of computation the solution $\mathbf{x}^1(t)$ has not fallen to an equilibrium and it is not increased indefinitely (here a sufficiently large computational interval $[0, T]$ should always be considered), then this solution reaches an attractor $\mathcal{A}_1$. Then it is possible to proceed to system (7) with $j = 2$ and to perform a similar procedure of computation of $\mathcal{A}_2$, by starting a trajectory of system (7) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ and computing the trajectory $\mathbf{x}^2(t)$.

Proceeding this procedure and sequentially increasing j and computing $\mathbf{x}^j(t)$ (being a trajectory of system (7) with initial data $\mathbf{x}^{j-1}(T)$) we either arrive at the computation of $\mathcal{A}_m$ (being an attractor of system (7) with $j = m$, i.e. original system (5)), either, at a certain step, observe a loss of stability bifurcation and the vanishing of attractor.

To determine the initial data $\mathbf{x}^0(0)$ of starting periodic solution, system (6) with nonlinearity $\varphi^0(\mathbf{x})$ can be transformed by linear nonsingular transformation $\mathbf{S}$ to the form

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + \varepsilon\varphi_1(x_1, x_2, \mathbf{x}_3), \\ \dot{x}_2 &= \omega_0 x_1 + \varepsilon\varphi_2(x_1, x_2, \mathbf{x}_3), \\ \dot{\mathbf{x}}_3 &= \mathbf{A_3}\mathbf{x}_3 + \varepsilon\varphi_3(x_1, x_2, \mathbf{x}_3) \end{aligned} \tag{8}$$

Here $\mathbf{A_3}$ is a constant $(n-2) \times (n-2)$ matrix, all eigenvalues of which have negative real parts, φ_3 is an $(n-2)$-dimensional vector-function, φ_1, φ_2 are certain scalar functions. Without loss of generality, it may be assumed that for the matrix $\mathbf{A_3}$ there exists positive number $\alpha > 0$ such that

$$\mathbf{x}_3^*(\mathbf{A_3} + \mathbf{A_3}^*)\mathbf{x}_3 \le -2\alpha|\mathbf{x}_3|^2, \quad \forall \mathbf{x}_3 \in \mathbb{R}^{n-2} \tag{9}$$

Introduce the following describing function

$$\Phi(a) = \int_0^{2\pi/\omega_0} \Big[\varphi_1\left((\cos\omega_0 t)a, (\sin\omega_0 t)a, 0\right)\cos\omega_0 t + \\ +\varphi_2\left((\cos\omega_0 t)a, (\sin\omega_0 t)a, 0\right)\sin\omega_0 t \Big] dt.$$

Theorem 1. *(Leonov et. al., 2010b) If it can be found a positive a_0 such that*

$$\Phi(a_0) = 0, \tag{10}$$

then there is a periodic solution in system (6) *with the initial data $\mathbf{x}^0(0) = \mathbf{S}(y_1(0), y_2(0), \mathbf{y}_3(0))^*$*

$$y_1(0) = a_0 + O(\varepsilon), y_2(0) = 0, \mathbf{y}_3(0) = \mathbf{O_{n-2}}(\varepsilon). \tag{11}$$

Here $\mathbf{O_{n-2}}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$.

3 LOCALIZATION OF HIDDEN ATTRACTOR IN CHUA'S SYSTEM

We now apply the above algorithm to analysis of Chua's system with scalar nonlinearity. For this purpose, rewrite Chua's system (1) in the form (3)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{12}$$

Here

$$\mathbf{P},\mathbf{q},\mathbf{r} = \begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$\psi(\sigma) = (m_0 - m_1)\mathrm{sat}(\sigma)$.

Introduce the coefficient k and small parameter ε, and represent system (12) as (6)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varepsilon\varphi(\mathbf{r}^*\mathbf{x}), \tag{13}$$

where

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^* = \begin{pmatrix} -\alpha(m_1+1+k) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix},$$

$\lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega_0,\ \lambda_3^{\mathbf{P}_0} = -d,$

$\varphi(\sigma) = \psi(\sigma) - k\sigma = (m_0 - m_1)\mathrm{sat}(\sigma) - k\sigma.$

In practice, to determine k and ω_0 it is used the transfer function $W(p)$ of system (3):

$$W_{\mathbf{P}}(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q},$$

where p is a complex variable. Then $\mathrm{Im}W(i\omega_0) = 0$ and k is computed then by formula $k = -(\mathrm{Re}W(i\omega_0))^{-1}$.

By nonsingular linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ system (13) can be reduced to the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} + \mathbf{b}\varepsilon\varphi(\mathbf{c}^*\mathbf{y}), \tag{14}$$

where

$$\mathbf{A},\mathbf{b},\mathbf{c} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}.$$

The transfer function $W_{\mathbf{A}}(p)$ of system (14) can be represented as

$$W_{\mathbf{A}}(p) = W_{\mathbf{P}}(p).$$

Further, using the equality of transfer functions of systems (13) and (14), we obtain

$$W_{\mathbf{A}}(p) = \mathbf{r}^*(\mathbf{P}_0 - p\mathbf{I})^{-1}\mathbf{q}.$$

This implies the following relations

$$\begin{aligned} k &= \frac{-\alpha(m_1 + m_1\gamma + \gamma) + \omega_0^2 - \gamma - \beta}{\alpha(1+\gamma)}, \\ d &= \frac{\alpha + \omega_0^2 - \beta + 1 + \gamma + \gamma^2}{1+\gamma}, \\ h &= \frac{\alpha(\gamma + \beta - (1+\gamma)d + d^2)}{\omega_0^2 + d^2}, \\ b_1 &= \frac{\alpha(\gamma + \beta - \omega_0^2 - (1+\gamma)d)}{\omega_0^2 + d^2}, \\ b_2 &= \frac{\alpha\big((1+\gamma-d)\omega_0^2 + (\gamma+\beta)d\big)}{\omega_0(\omega_0^2 + d^2)}. \end{aligned} \tag{15}$$

System (13) can be reduced to the form (14) by the nonsingular linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$. Having solved the following matrix equations

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}, \quad \mathbf{b} = \mathbf{S}^{-1}\mathbf{q}, \quad \mathbf{c}^* = \mathbf{r}^*\mathbf{S}, \tag{16}$$

one can obtain the transformation matrix

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

By (11), for small enough ε we determine initial data for the first step of multistage localization procedure

$$\mathbf{x}(0) = \mathbf{S}\mathbf{y}(0) = \mathbf{S}\begin{pmatrix} a_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 s_{11} \\ a_0 s_{21} \\ a_0 s_{31} \end{pmatrix}.$$

Returning to Chua's system denotations, for determining the initial data of starting solution of multistage procedure we have the following formulas

$$\begin{aligned} x(0) &= a_0,\ y(0) = a_0(m_1+1+k), \\ z(0) &= a_0\frac{\alpha(m_1+k) - \omega_0^2}{\alpha}. \end{aligned} \tag{17}$$

Consider system (13) with the parameters

$$\begin{aligned} \alpha = 8.4562,\ \beta = 12.0732,\ \gamma = 0.0052, \\ m_0 = -0.1768,\ m_1 = -1.1468. \end{aligned} \tag{18}$$

Note that for the considered values of parameters there are three equilibria in the system: a locally stable zero equilibrium and two saddle equilibria.

Now we apply the above procedure of hidden attractors localization to Chua's system (12) with parameters (18). For this purpose, compute a starting frequency and a coefficient of harmonic linearization. We have

$$\omega_0 = 2.0392, \quad k = 0.2098.$$

Then, compute solutions of system (13) with nonlinearity $\varepsilon\varphi(x) = \varepsilon(\psi(x) - kx)$, sequentially increasing ε from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$ with the step 0.1.

By (15) and (17) we obtain the initial data

$$x(0) = 9.4287, y(0) = 0.5945, z(0) = -13.4705$$

for the first step of multistage procedure for the construction of solutions. For the value of parameter $\varepsilon_1 = 0.1$, after transient process the computational procedure reaches the starting oscillation $\mathbf{x}^1(t)$. Further, by the sequential transformation $\mathbf{x}^j(t)$ with increasing the parameter ε_j, using the numerical procedure, for original Chua's system (12) the set $\mathcal{A}_{\text{hidden}}$ is computed. This set is shown in Fig. 3.

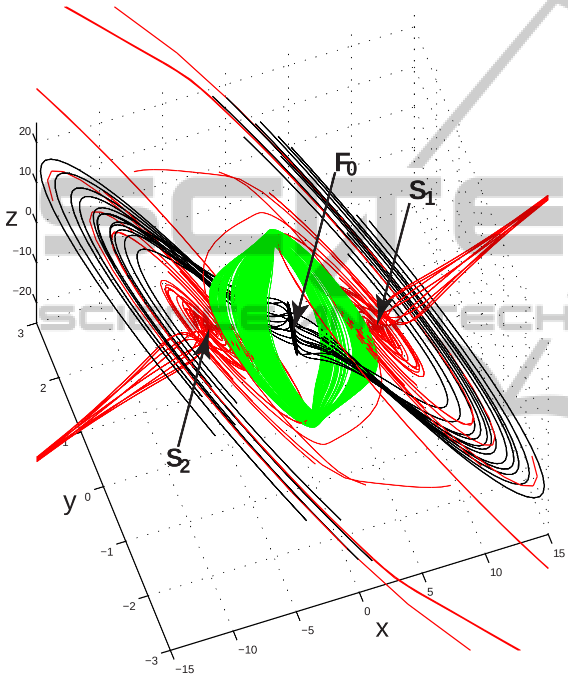

Figure 2: Equilibrium, stable manifolds of saddles, and localization of hidden attractor.

The considered system has three stationary points: the stable zero point F_0 and the symmetric saddles S_1 and S_2. To zero equilibrium F_0 correspond the eigenvalues $\lambda_1^{F_0} = -7.9591$ and $\lambda_{2,3}^{F_0} = -0.0038 \pm 3.2495i$ and to the saddles S_1 and S_2 correspond the eigenvalues $\lambda_1^{S_{1,2}} = 2.2189$ and $\lambda_{2,3}^{S_{1,2}} = -0.9915 \pm 2.4066i$. The behavior of trajectories of system in a neighborhood of equilibria is shown in Fig. 3.

We remark that here positive Lyapunov exponent (Leonov & Kuznetsov, 2007) corresponds to the computed trajectories.

By the above and with provision for the remark on the existence, in system, of locally stable zero equilibrium and two saddle equilibria, we arrive at the conclusion that in $\mathcal{A}_{\text{hidden}}$ a hidden strange attractor is computed.

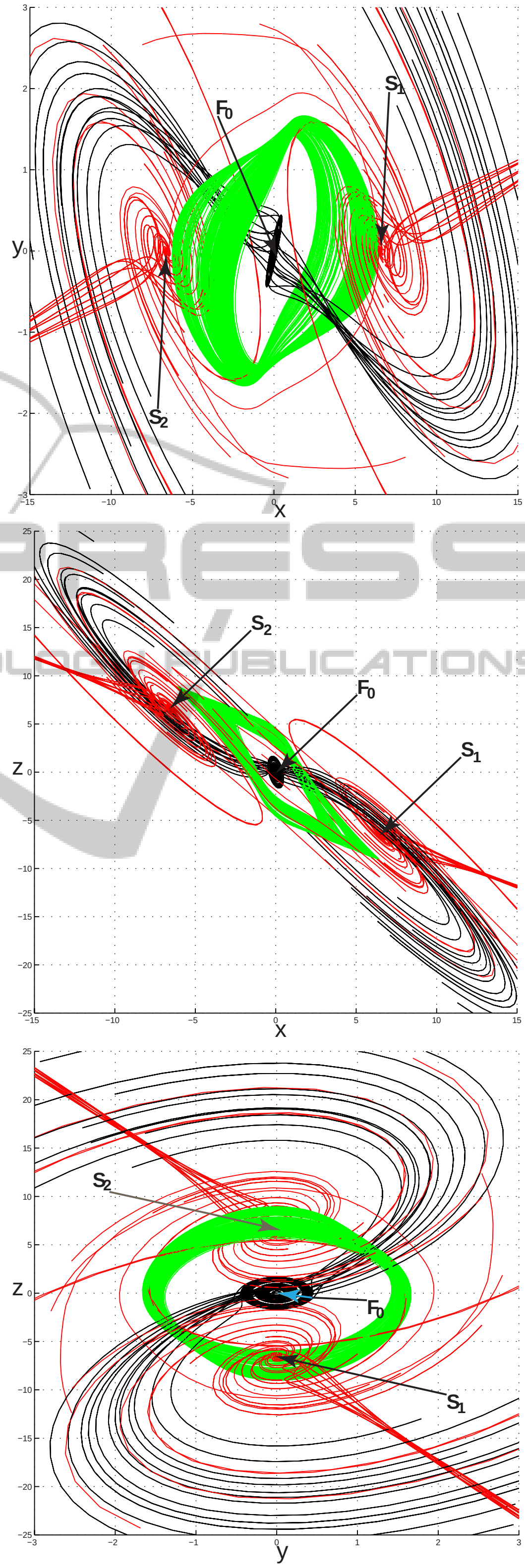

Figure 3: Hidden attractor projections on (x,y), (x,z), and (y,z).

4 CONCLUSIONS

In the present work the application of special analytical-numerical algorithm for hidden attractor localization is discussed. The existence of such hidden attractor in classical Chua's circuits is demonstrated.

It is also can be noted that to obtain existence of hidden attractor in Chua's circuit one can artificially stabilized (Suykens et al., 1997; Savaci & Gunel, 2006; Leonov et. al., 2010a) zero stationary point by inserting small stable zone around zero stationary point into nonlinearity (Chua diode characteristics).

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REFERENCES

Lorenz E. N. (1963). Deterministic nonperiodic flow. J. Atmos. Sci. V.20. pp. 130–141.

Rossler O. E. (1976). An Equation for Continuous Chaos. Physics Letters. V.57A. N5. pp. 397–398.

Chua L. O., Lin G. N. (1990). Canonical Realization of Chua's Circuit Family. IEEE Transactions on Circuits and Systems. V.37. N4. pp. 885–902.

Chen G. & Ueta T. (1999). Yet another chaotic attractor. Int. J. Bifurcation and Chaos. N9, pp. 1465–1466.

Leonov G. A., Kuznetsov N. V., Vagaytsev V. I. (2011). Localization of hidden Chua's attractors. Physics Letters A. 375 (35), pp. 2230-2233 (doi:10.1016/j.physleta.2011.04.037)

Leonov G. A. (2009). On the method of harmonic linearization. Automation and remote controle. 70(5), pp. 800–810.

Leonov G. A. (2009). On the Aizerman problem. Automation and remote control. 70(7), pp. 1120–1131.

Leonov G. A. (2010). Effective methods for periodic oscillations search in dynamical systems. App. math. & mech. 74(1), pp. 24–50.

Leonov G. A., Vagaitsev V. I., Kuznetsov N. V. (2010a). Algorithm for localizing Chua attractors based on the harmonic linearization method. Doklady Mathematics. 82(1), pp. 663–666.

Leonov G. A., Bragin V. O., Kuznetsov N. V. (2010b). Algorithm for Constructing Counterexamples to the Kalman Problem. Doklady Mathematics. 82(1), pp. 540–542.

Kuznetsov N. V., Leonov G. A. (2008). Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems. Journal of Vibroengineering. Vol. 10, Iss. 4, pp. 460–467.

Leonov G. A., Kuznetsov, N. V. (2010). Limit cycles of quadratic systems with a perturbed weak focus of order 3 and a saddle equilibrium at infinity Doklady Mathematics. 82(2), pp. 693–696.

Leonov G. A., Kuznetsov N. V., and Kudryashova E. V. (2011). A Direct Method for Calculating Lyapunov Quantities of Two-Dimensional Dynamical Systems. Proceedings of the Steklov Institute Of Mathematics. Vol. 272 (SUPPL. 1), pp. S119–S127.

Chua L. O. (1992). A Zoo of Strange Attractors from the Canonical Chua's Circuits. Proceedings of the IEEE 35th Midwest Symposium on Circuits and Systems (Cat. No.92CH3099-9). V.2. pp. 916–926.

Chua L. O. (1995). A Glimpse of Nonlinear Phenomena from Chua's Oscillator. Philosophical Transactions: Physical Sciences and Engineering. V.353. N1701. pp. 3–12.

Bilotta E., Pantano P. (2008). A gallery of Chua attractors. World scientific series on nonlinear science, Series A.

Shi Z., Hong S., and Chen K. (2008). Experimental study on tracking the state of analog Chua's circuit with particle filter for chaos synchronization. Physics Letters A, Vol. 372, Iss. 34, pp. 5575–5580.

Leonov G. A., Kuznetsov N. V. (2007). Time-Varying Linearization and the Perron effects. International Journal of Bifurcation and Chaos. Vol. 17, No. 4, pp. 1079–1107.

Savaci F. A., Gunel S. (2006). Harmonic Balance Analysis of the Generalized Chua's Circuit. International Journal of Bifurcation and Chaos. Vol. 16, No. 8, pp. 2325–2332.

Suykens J. A. K., Huang A., Chua L. O. (1997). A Family of n-scroll Attractors from a Generalized Chuas Circuit. AEU-International Journal of Electronics & Communications. Vol. 51, No. 3, pp. 131–138.