ANALYSIS AND COMPUTATION OF OPTIMAL BOUNDS OF BI-DIRECTIONAL FRAMES

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Abstract: Frames are mathematical tools which can represent redundancies in many application problems. In the studies of frames, the frame bounds and frame bound ratio are very important indices characterizing the robustness and numerical performance of frame systems. In this paper, the frame bounds of a class of frame, which can be modeled by the bi-directional impulse response of linear time systems, are analyzed and computed. By using the state space approach, the tightest lower and upper frame bounds can be directly and efficiently computed.

1 INTRODUCTION

In the study of vector spaces, frame is a more flexible tool compared with basis. The frame elements are linearly dependent, so to provide redundancies in frames, and all elements in the vector space can be written as a linear combination of the frame elements. Frames, as a mathematical theory were introduced by Duffin and Schaeffer (Duffin and Schaeffer, 1952) in 1950s. Since 1986, frames have played an important role in signal processing, see (Daubechies et al., 1986) written by Daubechies, Grossman, and Meyer. Some particular classes of frames have been extensively studied, for example, Gabor frames, which are also called Weyl-Heisenberg frames described in (Heil and Walnut, 1989) and (Casazza, 2001), and wavelet frames, which are introduced in (Daubechies et al., 1986), (Daubechies, 1990), and (Daubechies, 1992). Frames, or redundant representations, can also be found in pyramid coding (Burt and Adelson, 1983); source coding (Benedettom et al., 2006); denoising (Dragotti et al., 2003); robust transmission (Bernardini and Rinaldo, 2006); CDMA systems; multiantenna code design; segmentation; classification; restoration and enhancement; signal reconstruction; and so on.

The theory of frames is a powerful means for the analysis and design of the oversampled uniform filter banks (FBs). (Vetterli and Cvetkovic, 1996) and (Cvetkovic and Vetterli, 1998) studied properties of oversampled FBs. Necessary and sufficient conditions on a FB to implement a frame or a tight frame in $l^2(\mathbb{Z})$ were given in terms of the properties of the corresponding polyphase analysis matrix. The frame-theoretic analysis is based on the fact that the polyphase matrix provides matrix representation of a frame operator, i.e. $S(z) = E^*(z)E(z)$, which can be found in (Bolcskei et al., 1998) (Bolcskei and Hlawatsch, 2001) (Mertins, 2003) etc. Recently (Chai et al., 2007) presented a direct computational method for the frame-theory-based analysis and design of oversampled FBs, which employed the state space representation of the polyphase matrix $E(z)$.

In the studies of frames, the frame bounds and frame bound ratio are very important indices characterizing the robustness and numerical performance of frame systems. The quantification and computation of frame bounds have been actively investigated in past years. The classic approach to obtain frame bounds of multirate FBs is in the frequency domain, for example, (Bolcskei et al., 1998), (Bolcskei and Hlawatsch, 2001), (Mertins, 2003), (Stanhill and Yehoshua Zeevi, 1998). (Bayram and Selesnick, 2009) stated the frame bounds of iterated FBs making use the wavelet frame bounds computed in the frequency domain. The frequency approach to compute the frame bounds is an approximation method which samples the polyphase matrix of the frame operator over the frequency range $\omega \in [0, 2\pi)$ and then performs eigenanalysis on the sampled matrices. Such sampling approach can be

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very tedious when the frequency grid is dense and the polyphase matrix is nondiagonal and of infinite impulse response. Moreover, the error due to the frequency domain sampling of the polyphase matrix cannot be precisely quantified and predicted by the density of the frequency grid for generic oversampled FBs. The existing literature shows that wavelet frame bounds are computed in the frequency domain see for example (Daubechies, 1992) and (Christensen, 2003). The frequency approach to approximate wavelet frame bounds is even more complex and more tedious since it involves much denser frequency grids.

Frame bounds can also be obtained in the time domain via linear matrix inequality (LMI) technique, see (Chai et al., 2008), which is an application of the KYP lemma stated in (Rantzer, 1996). This method avoids the frequency-domain sampling and approximation, but is only applicable to causal filter bank (FB) frames in the forward direction.

In this paper, motivated by the limitation of existing techniques, we propose a direct state space approach to the analysis and computation of the frame bounds of a more general class of frames. This class of frames can be bi-directional with mixed causal-anticausal realizations and may not necessarily be in the form of multirate FBs. A state space approach is presented to the modeling of the bi-directional frames. The LMI solution, based on the state space model, can then provide accurate and efficient computation of the optimal frame bounds.

The rest of the paper is organized as follows. Section 2 presents notations followed by the fundamentals of frames. Different representations of mixed causal-anticausal LTI systems are introduced, including the state space representation and the transfer function representation in Section 2. Section 3 presents a direct state space approach to compute the frame bounds of a class of frame which can be modeled as mixed causal-anticausal linear systems. Examples are given in Section 4 to illustrate how to obtain the bounds of frames that are modeled as stable mixed causal-anticausal LTI systems. The paper is concluded by Section 5.

2 PRELIMINARIES

2.1 Notations

$\mathbb{R}(\mathbb{C})$ denotes the set of real (complex) numbers, $\mathbb{R}^{q\times p}(\mathbb{C}^{q\times p})$ denotes the set of real (complex) matrix with size $q \times p$. Let $\mathbb{Z}$ denote the set of integer numbers. $\mathbb{H}$ denotes the Hilbert space, which is a real or complex inner product space. Let $(.)^T$ denote the transpose of a matrix or vector, and $(.)^*$ denote the Hermitian (conjugate) transpose of a matrix or vector or function, which is also known as the adjoint of $(.)$, $(.)^{-1}$ denote the inverse of $(.)$, and $(.)^\dagger$ the left pseudo-inverse of $(.)$. $l^2(\mathbb{Z})$ is the space of square summable scalar sequences with an index set $\mathbb{Z}$ defined as

$$l^2(\mathbb{Z}) := \left\{ x_m \in \mathbb{C}, m \in \mathbb{Z} | \sum_{m\in\mathbb{Z}} |x_m|^2 \right\}.$$

The l^2 norm is a vector norm defined for a complex column vector $x = \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix}$ by

$$\|x\| = \sqrt{x^* x}.$$

The $l^2(\mathbb{Z})$ space is a Hilbert space with respect to the inner product

$$< x, y >= x^* y = y^* x.$$

2.2 Fundamental of Frames

A bi-directional frames is defined as:

Definition 1. *A sequence* $\{f_k \in \mathbb{R}^{\infty\times 1}\}_{k\in\mathbb{Z}}$ *of elements in* $\mathbb{H}$ *is a frame for* $\mathbb{H}$ *if there exist constants* $\alpha, \beta > 0$ *such that*

$$\alpha \|f\|^2 \le \sum_{k\in\mathbb{Z}} |< f, f_k >|^2 \le \beta \|f\|^2, \forall f \in \mathbb{H}.$$

The constants α *and* β *are called frame bounds.*

A vector space can be represented in terms of frames and elements in such vector space can be written as a linear combination of frame elements. In this paper, we consider a class of bi-directional infinite dimensional frames.

The optimal lower frame bound is the supremum over all possible lower frame bounds, and the optimal upper frame bound is the infimum over all possible upper frame bounds. Note that the optimal frame bounds are called the frame bounds in short in the rest of the paper. If the frame bounds satisfy $\alpha = \beta$, the frame is called a tight frame.

For a frame $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathbb{H}$, the pre-frame operator or the synthesis operator is given by

$$T : l^2(\mathbb{Z}) \to \mathbb{H}, T\{c_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} c_k f_k.$$

The adjoint of the pre-frame operator is given by

$$T^* : \mathbb{H} \to l^2(\mathbb{Z}), T^* f = \{< f, f_k >\}_{k\in\mathbb{Z}}.$$

By composing T with its adjoint T^*, we obtain the frame operator

$$S:\mathbb{H}\to\mathbb{H}, Sf = TT^*f = \sum_{k\in\mathbb{Z}} < f, f_k > f_k.$$

The frame operator S is bounded by the frame bounds α and β, invertible, self-adjoint, and positive.

If $\{f_k\}_{k\in\mathbb{Z}}$ is a frame for $\mathbb{H}$, the pre-frame operator (synthesis operator) can be shown as an infinite dimensional matrix

$$T = \begin{bmatrix} \cdots & f_{-1} & f_0 & f_1 & f_2 & \cdots & f_k & \cdots \end{bmatrix},$$

i.e. the infinite dimensional matrix has the vectors f_k as columns. The adjoint of the pre-frame operator can be shown as

$$T^* = \begin{bmatrix} \vdots \\ f_{-1}^T \\ f_0^T \\ f_1^T \\ f_2^T \\ \vdots \\ f_k^T \\ \vdots \end{bmatrix},$$

i.e. the infinite dimensional matrix T^* has the vectors f_k^T as rows.

The frame $\{g_k\}_{k\in\mathbb{Z}} = \{S^{-1}f_k\}_{k\in\mathbb{Z}}$ is called the canonical dual frame of $\{f_k\}_{k\in\mathbb{Z}}$, which satisfies

$$f = \sum_{k\in\mathbb{Z}} < f, g_k > f_k, \forall f \in \mathbb{H}.$$

If α, β are the optimal bounds for $\{f_k\}_{k\in\mathbb{Z}}$, then the bounds β^{-1}, α^{-1} are optimal for the canonical dual frame $\{S^{-1}f_k\}_{k\in\mathbb{Z}}$. Hence, the lower bound of a frame is equivalent to the inverse of the upper bound of the canonical dual frame.

2.3 Fundamental of LTI Systems

State space equations of causal and anticausal linear time-invariant (LTI) systems are given as

$$\begin{aligned} x_{m+1} &= Ax_m + Bu_m, \\ x'_{m-1} &= A'x'_m + B'u_m, \\ y_m &= Cx_m + C'x'_m + (D+D')u_m, \end{aligned} \quad (1)$$

where $x_m \in \mathbb{R}^{n_c}$ is the forward state variable, $x'_m \in \mathbb{R}^{n_{ac}}$ is the backward state variable, $u_m \in \mathbb{R}^p$ is the system input, $y_m \in \mathbb{R}^q$ is the system output. $A \in \mathbb{R}^{n_c\times n_c}, B \in \mathbb{R}^{n_c\times p}, C \in \mathbb{R}^{q\times n_c}, D \in \mathbb{R}^{q\times p}$, $A' \in \mathbb{R}^{n_{ac}\times n_{ac}}, B' \in \mathbb{R}^{n_{ac}\times p}, C' \in \mathbb{R}^{q\times n_{ac}}$ and $D' \in \mathbb{R}^{q\times p}$ are the state space matrices of the causal-anticausal system. We use

$$E_c(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c = D + C(zI - A)^{-1}B$$

to represent the causal system and

$$E_{ac}(z) = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]_{ac} = D' + C'(z^{-1}I - A')^{-1}B'$$

to represent the anticausal system. Hence the mixed causal-anticausal LTI system has transfer function matrix

$$\begin{aligned} E(z) = & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]_{ac} \\ = & D + D' + C(zI - A)^{-1}B \\ & + C'(z^{-1}I - A')^{-1}B'. \end{aligned}$$

For consistency of the symbols, we use E to denote the system operator for the causal-anticausal LTI system (1), which gives

$$y = Eu.$$

In the above equation,

$$u = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}, \quad y = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix}, \quad (2)$$

and

$$E = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & D+D' & C'B' & C'A'B' & \cdots \\ \cdots & CB & D+D' & C'B' & \cdots \\ \cdots & CAB & CB & D+D' & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3)$$

The entries of the system operator E (i^{th} row and j^{th} column, $i, j \in \mathbb{Z}$) can be shown as

$$E_{ij} = \begin{cases} CA^{i-j-1}B, & i > j, \\ D + D', & i = j, \\ C'(A')^{j-i-1}B', & i < j. \end{cases}$$

An LTI system is causal if the system operator E is left lower-triangular and anticausal if E is right upper-triangular.

The mixed causal-anticausal LTI system (1) is stable if and only if

$$|\rho(A)| < 1 \text{ and } |\rho(A')| < 1,$$

where $\rho(.)$ indicates the largest eigenvalue of (.), which means that $E(z)$ has no poles on the unit circle.

The operator norm (induced l^2 norm) of stable mixed causal-anticausal systems is defined by:

$$\|E\| = sup_{u\in l^2(\mathbb{Z}), u\neq 0} \frac{\|Eu\|}{\|u\|}.$$

It is equivalent to the square of the upper frame bound if E implements a frame.

The following lemma from (Shu and Chen, 1996) is very useful to obtain the state space description of the cascaded causal-anticausal discrete time LTI systems. It is essential in the state space analysis of LTI systems.

Lemma 1. *Assume the compatibility of the operators. We have the state space matrices of the cascaded causal-anticausal discrete time systems as*

$$\begin{aligned}
&\left[\begin{array}{c|c} A_2' & B_2' \\ \hline C_2' & D_2' \end{array}\right]_{ac} \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{c} \\
&= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline D_2'C_1 + C_2'XA_1 & D_2'D_1 + C_2'XB_1 \end{array}\right]_{c} \\
&+ \left[\begin{array}{c|c} A_2' & B_2'D_1 + A_2'XB_1 \\ \hline C_2' & 0 \end{array}\right]_{ac}, \\
&\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{c} \left[\begin{array}{c|c} A_1' & B_1' \\ \hline C_1' & D_1' \end{array}\right]_{ac} \\
&= \left[\begin{array}{c|c} A_2 & B_2D_1' + A_2YB_1' \\ \hline C_2 & D_2D_1' + C_2YB_1' \end{array}\right]_{c} \\
&+ \left[\begin{array}{c|c} A_1' & B_1' \\ \hline D_2C_1' + C_2YA_1' & 0 \end{array}\right]_{ac},
\end{aligned}$$

where X, Y are given by the Sylvester equations:

$$\begin{aligned} A_2'XA_1 - X + B_2'C_1 = 0, \\ A_2YA_1' - Y + B_2C_1' = 0. \end{aligned} \tag{4}$$

Let E_1 be a causal-anticausal LTI system such that $E_1(z) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{c} + \left[\begin{array}{c|c} A_1' & B_1' \\ \hline C_1' & D_1' \end{array}\right]_{ac}$, and E_2 be another causal-anticausal LTI system such that $E_2(z) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{c} + \left[\begin{array}{c|c} A_2' & B_2' \\ \hline C_2' & D_2' \end{array}\right]_{ac}$.

Lemma 2 below presents the necessary and sufficient condition for an LTI system to be a frame. Its proof can be found in (Cvetkovic and Vetterli, 1998).

Lemma 2. *Let* $E(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{c} + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]_{ac} \in \mathbb{C}^{q\times p}$ *be the transfer function of a stable causal-anticausal LTI system. Then for all $u \in l^2$, there exist constants α and β such that*

$$\alpha\|u\|^2 \le \|Eu\|^2 \le \beta\|u\|^2$$

if and only if $E(e^{j\omega})$ is full column rank on $[0, 2\pi)$.

The frame bounds of frames that modeled as causal LTI systems are computed in (Chai et al., 2008), and this result is stated in lemma 3.

Lemma 3. *Given a causal LTI system E with state space representation* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{c}$, *the problem of finding the minimum β and maximum α over $\omega \in [0, 2\pi)$ is equivalent to*

$$\min_P \beta$$

subject to:

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \beta I \end{bmatrix} \le 0,$$

$$P = P^T, \beta > 0.$$

and

$$\max_Q \alpha$$

subject to

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD + \alpha I \end{bmatrix} \le 0,$$

$$Q = Q^T, \alpha > 0.$$

The result makes use of the KYP lemma stated in (Rantzer, 1996).

3 FRAMES IN LTI STATE SPACE REALIZATIONS

Let $\{h(m) \in \mathbb{R}^{q\times p}\}$ be the impulse response of a stable causal-anticausal LTI system E. Then the output signal of E is given by

$$y = Eu.$$

The bi-directional infinite sequence $h_k = \{h_k(m)\}$ can be represented by the state space matrices as

$$h_k(m) = \begin{cases} CA^{k-m-1}B, & k > m, \\ D + D', & k = m, \\ C'(A')^{m-k-1}B', & k < m, \end{cases} \tag{5}$$

resulting in

$$y_k = \sum_{m\in\mathbb{Z}} h_k(m)u_m.$$

For a frame whose elements $\{f_k(m)\}$ can be written as the bi-directional infinite impulse responses of the linear system E that maps the input u to the output y, i.e. $\{f_k(m) = h_k^T(m)\}$, we say that the frame can be modeled as a mixed causal-anticausal LTI system with the state space realization. For such frames, its canonical dual frame is actually the pseudo-inverse system. Hence the lower frame bound can be found as the inverse of the upper frame bound of the canonical dual frame, which is constructed by the impulse response of the pseudo-inverse system.

Theorem 1. *Let $\{f_k\}_{k\in\mathbb{Z}}(\{h_k^T\}_{k\in\mathbb{Z}})$ be a frame implemented by the impulse response $\{h_k^T\}_{k\in\mathbb{Z}}$ of a stable mixed causal-anticausal LTI system E. Then its canonical dual frame is given by the impulse response of $E^\dagger$, the pseudo inverse system of E. The square of the operator norm of $E^\dagger$ equals the inverse of the lower frame bound of $\{f_k\}_{k\in\mathbb{Z}}$.*

Proof: There exists a dual frame $\{g_k\}_{k\in\mathbb{Z}}$ for given frame $\{f_k\}_{k\in\mathbb{Z}}$ for which

$$u = \sum_{k=-\infty}^{\infty} < u, g_k > f_k = \sum_{k=-\infty}^{\infty} < u, f_k > g_k. \qquad (6)$$

The frame element represents the bi-directional infinite impulse responses of a linear system, hence $f_k = h_k^T$ results in $E = T^*$, where T^* is the adjoint of the synthesis operator of the frame $\{f_k\}_{k\in\mathbb{Z}}$. The linear system operator E can be factorized into:

$$E = Q^* U R_{lr} V$$

where Q is inner, $U^*U = I$ and $VV^* = I$ and R_{lr} is the left and right outer. More details of the inner factor and the outer factor can be found in (Dewilde and Van Der Veen, 1998). E has a Moore-Penrose (pseudo-) inverse

$$E^\dagger = V^* R_{lr}^{-1} U^* Q.$$

The pre-frame operator for $\{g_k\}_{k=-\infty}^{\infty}$ is denoted by $\tilde{T}$. The canonical dual frame is obtained by

$$\{g_k\}_{k=-\infty}^{\infty} = \{S^{-1} f_k\}_{k=-\infty}^{\infty},$$

which implies

$$\tilde{T} = S^{-1} T = (TT^*)^{-1} T,$$

hence $\tilde{T}T^* = I$. The canonical dual frame has the minimum l^2 norm among the dual frames.

Since $E = T^*$, the pre-frame operator of the canonical dual frame can be obtained:

$$\begin{aligned} \tilde{T} &= (TT^*)^{-1} T \\ &= (V^* R_{lr}^* U^* Q Q^* U R_{lr} V)^{-1} V^* R_{lr}^* U^* Q \\ &= V^* R_{lr}^{-1} U^* Q Q^* U (R_{lr}^*)^{-1} V V^* R_{lr}^{-1} U^* Q \\ &= V^* R_{lr}^{-1} U^* Q. \end{aligned} \qquad (7)$$

We can see that

$$\tilde{T} = E^\dagger.$$

Thus $\tilde{T}T^* = I$ and $E^\dagger E = I$.

Since $E^\dagger$ equals the pre-frame operator $\tilde{T}$ of the dual frame, the square of the operator norm of $E^\dagger$ is equivalent to the upper frame bound of the dual frame, following the definition of the operator norm of a linear system given in Section 2.3. The inverse of the lower bound of the frame is equivalent to the upper bound of the canonical dual frame, which is equivalent to the square of the operator norm of the left pseudo-inverse system $E^\dagger$. □

Lemma 3 presents the LMI approach to obtain the frame bounds of the frames modeled as causal LTI systems. In this paper, we propose a direct method to obtain the frame bounds of frames modeled as mixed causal-anticausal LTI systems. This method avoids a large amount of computations to convert the stable mixed causal-anticausal realization into unstable causal realization.

Theorem 2. *Given a frame that can be modeled as a stable mixed causal-anticausal LTI system E with $E(e^{j\omega})$ being full column rank on $\omega \in [0, 2\pi)$, the optimal upper frame bound is the infimum (minimum) of all possible β satisfying*

$$\|Eu\|^2 \le \beta \|u\|^2.$$

The problem of finding the minimum β is equivalent to the following optimization problem:

$$\min_{P,Q,X} \beta$$

subject to:

$$\begin{bmatrix} A^T PA - P + C^T C & A^T X - XA' + C^T C' & A^T PB + C^T D - XB' \\ X^T A - A'^T X^T + C'^T C & A'^T QA' - Q + C'^T C' & A'^T QB' + C'^T D + X^T B \\ B^T PA + D^T C - B'^T X^T & B'^T QA' + D^T C' + B^T X & B^T PB + B'^T QB' + D^T D - \beta I \end{bmatrix} \le 0,$$

where $P = P^T > 0$, $Q = Q^T > 0$ and X is a general matrix.

Proof: The class of frames modeled by the mixed causal-anticausal LTI system has the rational transfer function as

$$\begin{aligned} &E(z) \\ &= D + C(zI - A)^{-1} B + C'(z^{-1} I - A')^{-1} B' \\ &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array}\right]_{ac}, \end{aligned}$$

and $\|Eu\|^2 \le \beta \|u\|^2$ implies $E^*(z)E(z) \le \beta I$.

$$\begin{aligned} &E^*(z)E(z) \\ &= \left(\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]_{ac} + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array}\right]_c \right) \\ &\times \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array}\right]_{ac} \right) \\ &= \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c \\ &+ \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array}\right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array}\right]_{ac} \\ &+ \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array}\right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_c \\ &+ \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array}\right]_{ac} \end{aligned}$$

There are four terms in the above expression, each term is a cascaded system. By using lemma 1, we can represent the first term as:

$$
\begin{aligned}
&\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{c} \\
&= \left[\begin{array}{c|c} A & B \\ \hline D^T C + B^T PA & D^T D + B^T PB \end{array}\right]_{c} \\
&\quad + \left[\begin{array}{c|c} A^T & C^T D + A^T PB \\ \hline B^T & 0 \end{array}\right]_{ac}
\end{aligned}
$$

where $A^T PA - P + C^T C = 0$, yielding $B^T (z^{-1}I - A^T)^{-1}(A^T PA - P + C^T C)(zI - A)^{-1}B = 0$. The second term can be rewritten as:

$$
\begin{aligned}
&\left[\begin{array}{c|c} A^{'T} & C^{'T} \\ \hline B^{'T} & 0 \end{array}\right]_{c} \left[\begin{array}{c|c} A^{'} & B^{'} \\ \hline C^{'} & 0 \end{array}\right]_{ac} \\
&= \left[\begin{array}{c|c} A^{'T} & A^{'T} QB^{'} \\ \hline B^{'T} & B^{'T} QB^{'} \end{array}\right]_{c} + \left[\begin{array}{c|c} A^{'} & B^{'} \\ \hline B^{'T} QA^{'} & 0 \end{array}\right]_{ac}
\end{aligned}
$$

where $A^{'T} QA^{'} - Q + C^{'T} C^{'} = 0$, yielding $B^{'T}(zI - A^{'T})^{-1}(A^{'T} QA^{'} - Q + C^{'T} C^{'})(z^{-1}I - A^{'})^{-1}B^{'} = 0$. The third term can be rewritten as:

$$
\begin{aligned}
&\left[\begin{array}{c|c} A^{'T} & C^{'T} \\ \hline B^{'T} & 0 \end{array}\right]_{c} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{c} \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ C^{'T} C & A^{'T} & C^{'T} D \\ \hline 0 & B^{'T} & 0 \end{array}\right]_{c} \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A^{'T} & C^{'T} D + UB \\ \hline -B^{'T} U & B^{'T} & 0 \end{array}\right]_{c}
\end{aligned}
$$

where $UA - A^{'T}U + C^{'T}C = 0$, yielding $B^{'T}(zI - A^{'T})^{-1}(UA - A^{'T}U + C^{'T}C)(zI - A)^{-1}B = 0$. The fourth term can be rewritten as:

$$
\begin{aligned}
&\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]_{ac} \left[\begin{array}{c|c} A^{'} & B^{'} \\ \hline C^{'} & 0 \end{array}\right]_{ac} \\
&= \left[\begin{array}{cc|c} A^T & C^T C^{'} & 0 \\ 0 & A^{'} & B^{'} \\ \hline B^T & D^T C^{'} & 0 \end{array}\right]_{ac} \\
&= \left[\begin{array}{cc|c} A^T & 0 & -VB^{'} \\ 0 & A^{'} & B^{'} \\ \hline B^T & D^T C^{'} + B^T V & 0 \end{array}\right]_{ac}
\end{aligned}
$$

where $A^T V - VA^{'} + C^T C^{'} = 0$, yielding $B^T(z^{-1}I - A^T)^{-1}(A^T V - VA^{'} + C^T C^{'})(z^{-1}I - A^{'})^{-1}B^{'} = 0$. The third term condition $UA - A^{'T}U + C^{'T}C = 0$ and fourth term condition $A^T V - VA^{'} + C^T C^{'} = 0$ result in $V = U^T, U = V^T$.

Hence we can represent $E^*(z)E(z) - \beta I$ as

$$
\begin{aligned}
&E^*(z)E(z) - \beta I \\
&= (D^T D + B^T PB + B^{'T} QB^{'} - \beta I) + (D^T C \\
&+ B^T PA)(zI - A)^{-1}B + B^T(z^{-1}I - A^T)^{-1}(C^T D \\
&+ A^T PB) + B^T(z^{-1}I - A^T)^{-1}(A^T PA - P + C^T C) \\
&\times (zI - A)^{-1}B + B^{'T}(zI - A^{'T})^{-1}(A^{'T}QB^{'}) \\
&+ (B^{'T}QA^{'})(z^{-1}I - A^{'})^{-1}B^{'} + B^{'T}(zI - A^{'T})^{-1} \\
&\times (A^{'T}QA^{'} - Q + C^{'T}C^{'})(z^{-1}I - A^{'})^{-1}B^{'} - B^{'T}U \\
&\times (zI - A)^{-1}B + B^{'T}(zI - A^{'T})^{-1}(C^{'T}D + UB) \\
&+ B^{'T}(zI - A^{'T})^{-1}(UA - A^{'T}U + C^{'T}C)(zI - A)^{-1} \\
&\times B - B^T(z^{-1}I - A^T)^{-1}VB^{'} + (D^T C^{'} + B^T V) \\
&\times (z^{-1}I - A^{'})^{-1}B^{'} + B^T(z^{-1}I - A^T)^{-1} \\
&\times (A^T V - VA^{'} + C^T C^{'})(z^{-1}I - A^{'})^{-1}B^{'},
\end{aligned}
$$

We can further rewrite $E^*(z)E(z) - \beta I \leq 0$ as:

$$
\begin{aligned}
&E^*(z)E(z) - \beta I \\
&= \begin{bmatrix} B^T(z^{-1}I - A^T)^{-1} & B^{'T}(zI - A^{'T})^{-1} & I \end{bmatrix} \\
&\times \begin{bmatrix} A^T PA - P + C^T C & A^T V - VA^{'} + C^T C^{'} & C^T D + A^T PB - VB^{'} \\ UA - A^{'T}U + C^{'T}C & A^{'T}QA^{'} - Q + C^{'T}C^{'} & A^{'T}QB^{'} + C^{'T}D + UB \\ D^T C + B^T PA - B^{'T}U & B^{'T}QA^{'} + D^T C^{'} + B^T V & D^T D + B^T PB + B^{'T}QB^{'} - \beta I \end{bmatrix} \\
&\times \begin{bmatrix} (zI - A)^{-1}B \\ (z^{-1}I - A^{'})^{-1}B^{'} \\ I \end{bmatrix} \leq 0
\end{aligned}
$$

Thus the matrix

$$
\begin{bmatrix} A^T PA - P + C^T C & A^T V - VA^{'} + C^T C^{'} & C^T D + A^P B - VB^{'} \\ UA - A^{'T}U + C^{'T}C & A^{'T}QA^{'} - Q + C^{'T}C^{'} & A^{'T}QB^{'} + C^{'T}D + UB \\ D^T C + B^T PA - B^{'T}U & B^{'T}QA^{'} + D^T C^{'} + B^T V & D^T D + B^T PB + B^{'T}QB^{'} - \beta I \end{bmatrix} \leq 0
$$

is a negative definite matrix. Letting $X = V$, which means that $X^T = U$, we prove the theorem. □

Theorem 2 can be used to obtain the upper bound of the causal LTI system by setting $A^{'} = 0, B^{'} = 0, C^{'} = 0, D^{'} = 0$, resulting $X = 0$. In this case, theorem 2 can be found in lemma 3. The theorem can also be used to obtain the upper bound of the anticausal LTI system by setting $A = 0, B = 0, C = 0, D = 0$, hence $X = 0$.

Similarly, we have the lower bound computed by the following theorem.

Theorem 3. *For a frame whose elements are the bi-infinite dimensional impulse responses of a stable mixed causal-anticausal LTI systems E with $E(e^{j\omega})$ being full column rank in $\omega \in [0, 2\pi)$, the optimal lower frame bound is the supermum (maximum) of all possible α satisfying*

$$
\alpha \|u\|^2 \leq \|Eu\|^2.
$$

The problem of finding the maximum α is equivalent to the following optimization problem

$$\min_{P,Q,X} -\alpha$$

subject to:

$$\begin{bmatrix} A^TPA-P+C^TC & A^TX-XA'+C^TC' & A^TPB+C^TD-XB' \\ X^TA-A'^TX^T+C'^TC & A'^TQA'-Q+C'^TC' & A'^TQB'+C'^TD+X^TB \\ B^TPA+D^TC-B'^TX^T & B'^TQA'+D^TC'+B^TX & B^TPB+B'^TQB'+D^TD-\alpha I \end{bmatrix} \geq 0,$$

where $P = P^T > 0$, $Q = Q^T > 0$ *and* X *is a general matrix.*

The proof is analogue to the proof of theorem 2.

4 EXAMPLES

Example 1. We make use of the IIR Butterworth filters shown in (Herley and Vetterli, 1993). The low pass filter $H(z)$ is given as

$$H(z) = \frac{\sum_{i=0}^{N} \begin{pmatrix} N \\ i \end{pmatrix} z^{-i}}{\sqrt{2}\sum_{l=0}^{(N-1/2)} \begin{pmatrix} N \\ 2l \end{pmatrix} z^{-2l}},$$

where N is the order of the filter and the high pass filter $G(z)$ is given as:

$$G(z) = z^{2n-1}H(-z^{-1}).$$

We give an example with order $N = 7$ and $n = 0$:

$$H(z) = \frac{1+7z^{-1}+21z^{-2}+35z^{-3}+35z^{-4}+21z^{-5}+7z^{-6}+z^{-7}}{\sqrt{2}(1+21z^{-2}+35z^{-4}+7z^{-6})},$$

$$G(z) = z^{-1}\frac{1-7z^{1}+21z^{2}-35z^{3}+35z^{4}-21z^{5}+7z^{6}-z^{7}}{\sqrt{2}(1+21z^{2}+35z^{4}+7z^{6})},$$

The causal-anticausal state space representations the low pass filter $H(z)$ and high pass filter $G(z)$ are shown as:

$$H(z) = \left[\begin{array}{ccc|c} 0 & -0.2319 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.3499 & 0 & 0.0234 & 0.7071 \end{array}\right]_c + \left[\begin{array}{cccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 \\ 1 & 0 & 0 & 0 & -6.6787 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 16.3918 \\ \hline 0 & 0 & 0 & 0.0331 & 0 \end{array}\right]_{ac},$$

and

$$G(z) = \left[\begin{array}{ccccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.7071 & -0.5431 & 0.4865 & -0.1524 & 0.0234 & -0.3499 \end{array}\right]_c + \left[\begin{array}{cc|c} 0 & -0.2319 & -0.0404 \\ 1.001 & 0 & 0 \\ \hline -0.001 & -1.4273 & 0 \end{array}\right]_{ac}.$$

The frame bounds of the Butterworth FB frame are shown in table 1. The FB constructs a approximate tight frame. The dual frame is realized by:

$$\tilde{H}(z) = H(z^{-1}), \tilde{G}(z) = G(z^{-1}).$$

Table 1: Frame bounds of Butterworth FB frame.

Decimation Factor	1	2
β	2	1.0008
α	1.9985	0.9984

Example 2. The scaling filter $H(z)$ and wavelet filter $G(z)$ are given as

$$H(z) = \frac{1}{\sqrt{2}}\left(\frac{1}{8}z^2 + \frac{1}{2}z + \frac{3}{4} + \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2}\right),$$

$$G(z) = 6\sqrt{\frac{70}{1313}}\left(-\frac{1}{2}z + 1 - \frac{1}{2}z^{-1}\right).$$

The state space representations of the low pass and high pass filters are given as

$$H(z) = \left[\begin{array}{cc|c} 0 & 0 & 0.5 \\ 1 & 0 & 0 \\ \hline 0.7071 & 0.1768 & 0.5303 \end{array}\right]_c + \left[\begin{array}{cc|c} 0 & 0 & 0.5 \\ 1 & 0 & 0 \\ \hline 0.7071 & 0.1768 & 0 \end{array}\right]_{ac}$$

and

$$G(z) = \left[\begin{array}{c|c} 0 & 1 \\ \hline -0.6927 & 1.3854 \end{array}\right]_c + \left[\begin{array}{c|c} 0 & 1 \\ \hline -0.6927 & 0 \end{array}\right]_{ac}$$

The frame bounds of the FB frame are given in table 2.

Table 2: Frame bounds of FIR FB frame.

Decimation Factor	1	2
β	7.6773	3.8385
α	1.1090	0

5 CONCLUSIONS AND FUTURE WORK

A class of frames, with elements in the form of bi-directional infinite impulse responses of an LTI system, can be equivalently modeled as mixed causal-anticausal LTI systems. This paper presents a direct state space approach to the analysis and computation of optimal frame bounds of this class of frames. It is shown that the lower frame bound is also equal to the inverse of the square of the operator norm of the left pseudoinverse system which achieves perfect reconstruction. Accurate and efficient computation of the frame bounds has been achieved using the LMI technique and the obtained results are demonstrated by examples.

The results obtained in this paper are applicable to a class of frames which are governed by exponential type perfoamance hahavior and can be modeled by LTI system responses in the time and frequency domains. Currently, the authors are extending the LTI state space approach presented in this paper to linear time varying (LTV) state space modeling, analysis and computation of frames. This study will enable deeper understanding and more efficient evaluation of a more general class of frames which may not be properly analyzed and evaluated in the conventinoal frequency domain.

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