CONSENSUS PROBLEM OF MULTI-AGENT SYSTEMS WITH MARKOVIAN COMMUNICATION FAILURE

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Abstract: This paper studies the consensus problem of multi-agent systems with Markovian communication failure which may be caused by limited communication capacity. The occurrence of the failures is modeled by a discrete-time Markov chain. A consensus sufficiency condition is established in terms of linear matrix inequalities (LMIs). Based on this condition, a new controller design method is provided. A numerical example is utilized to illustrate the effectiveness of the proposed approach.

1 INTRODUCTION

In the past few years, multi-agent system (MAS) has sparked the interest of researchers. Up to now, results of networked MASs have been broadly applied to biology, physics and engineering, such as the study of swarming behavior (Liu and Passino, 2004), automated highway systems (AHSs) (Bender, 1991) and congestion control in communication (Paganini et al., 2005). In the cooperative behaviors, consensus, which means making a group of agents to reach an agreement on certain quantity of interest that depends on the states of all agents, is a fundamental topic in MASs fields and has been studied recently (Olfati-Saber et al., 2007).

Due to the special property of MASs, the interconnected communication network among agents plays an important role in the consensus reaching problem and is usually described by Laplacian graph. Depending on applications, the network topologies of multiple agents are either fixed or switched, while the latter is more practical due to the limited or imperfect communication channel, noises or some special objectives. Results on switching topology have been provided in recent articles such as (Olfati-Saber and Murray, 2004).The physical systems are usually of big complexity, thus some dynamic processes are described by time-varying linear model. Particularly, for systems subject to randomly changing parameters, Markov jump linear system (MJLS), which is a hybrid system composed of a finite number of subsystem modes, is an appropriate class of models and has been extensively studied (Xiong and Lam, 2007). However, to the best of our knowledge, although MASs are usually treated as networked systems, the issue of Markovian topology switching processes has not been fully investigated and fruitful results of MJLS were not applied to MASs until now.

In this paper, we investigate the consensus control problem of MASs with communication failure. By modeling the communication process in a Markovian process, a new sufficiency condition of the consensus problem is established in terms of linear matrix inequalities (LMIs) which can be easily solved. Based on this condition, a state-feedback controller is designed such that the consensus of the closed-loop system is mean square stable (MSS) with known communication failure processes.

Notation. Throughout this paper, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{S}^{n\times n}$ represent the n-dimensional Euclidean space, the set of all $n\times m$ real matrices and the $n\times n$ real symmetric positive definite matrices, respectively; $\mathbb{Z}_+$ is the set of non-negative integers; $(\Omega, \mathcal{F}, P)$ denotes a complete probability space; the superscript "T" represents the transpose; for Hermitian matrices $X = X^T \in \mathbb{R}^{n\times n}$ and $Y = Y^T \in \mathbb{R}^{n\times n}$, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-

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definite (respectively, positive definite); I_n is the $n\times n$ identity matrix; $\mathbf{E}(\cdot)$ denotes the expectation operator with respect to some probability measure; $(M)_{ij}$ refers to the ith row, jth column element of matrix M; $\|\cdot\|$ represents the Euclidean norm for a vector and the spectral norm for a matrix; the symbol $\otimes$ denotes the Kronecker product; trace$(\cdot)$ is the trace of matrix; diag$(M_1,M_2,\ldots,M_N)$ is a block-diagonal matrix with diagonal blocks $M_1,M_2,\ldots,M_N$; $\mathbf{1}_N$ is defined as $\mathbf{1}_N=(1,1,\ldots,1)^T\in\mathbb{R}^N$ and $\mathbf{0}_N$ is defined similarly, that is, $\mathbf{0}_N=[0,0,\ldots,0]^T\in\mathbb{R}^N$.

2 PRELIMINARIES

A *directed graph* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ consists of a finite *vertex set* $\mathcal{V}$ and an *edge set* $\mathcal{E}\subset\mathcal{V}^2$. Suppose there are n vertices in $\mathcal{V}$, then the graph has an *order* n and each vertex can be uniquely labeled by an integer i belonging to a finite index set $\mathcal{I}=\{1,2,\ldots,n\}$. Each edge can be denoted by an ordered pair of distinct vertices (v_i,v_j) where v_j is the head and v_i is the tail, that is, the edge points from v_i to v_j with no self-loop. Each edge $(v_i,v_j)\in\mathcal{E}$ corresponds to the information transmission from agent j to agent i. The graph with the property that for any $(v_i,v_j)\in\mathcal{E}\Leftrightarrow(v_j,v_i)\in\mathcal{E}$ is said to be *symmetric* or *undirected*. The *in (out)-degree* of v_i, denoted by $d_i(v_i)$ $(d_o(v_i))$, is the number of edges with v_i as its tail(head). If $(v_i,v_j)\in\mathcal{E}$, then v_j is one of the *neighbors* of v_i. The set of neighbors of v_i is denoted by $\mathcal{N}_i=\{v_j\in\mathcal{V}:(v_i,v_j)\in\mathcal{E}\}$. An *adjacency matrix* of graph $\mathcal{G}$ with order n is an $n\times n$ matrix $\mathcal{A}=\{a_{ij}\}$ defined as

$$a_{ij}=\begin{cases}1 & \text{if } (v_i,v_j)\in\mathcal{E},\\ 0 & \text{otherwise.}\end{cases}$$

An *in-degree matrix* of graph $\mathcal{G}$ with order n is an $n\times n$ matrix $\mathcal{D}=\text{diag}\{d_{11},d_{22},\ldots,d_{nn}\}$ where $d_{ii}=\sum_{v_j\in\mathcal{N}_i}a_{ij}$. A *Laplacian matrix* $\mathcal{L}$ of graph $\mathcal{G}$ with order n is an $n\times n$ matrix defined as follows:

$$\mathcal{L}=\mathcal{D}-\mathcal{A}.$$

Let us consider a MAS with n agents. The discrete-time linear dynamics of agent i can be described by the following equation:

$$x_i(k+1)=Ax_i(k)+Bu_i(k),\quad i\in\mathcal{I} \tag{1}$$

where $x_i(k)\in\mathbb{R}^m$ is the system state, $u_i(k)\in\mathbb{R}^l$ is the control input, $A\in\mathbb{R}^{m\times m}, B\in\mathbb{R}^{m\times l}$, $k\in\mathbb{Z}_+$ is the time step, $x_i(0)\triangleq x_{i0}$ is the initial state.

Suppose the communication failure between agents i and j of the MAS (1) behaves in an independent way, that is, agent i can receive data from agent j does not necessarily mean agent j can receive data from agent i. The control input of the ith agent with communication failure is

$$u_i(k)=K\sum_{v_j\in\mathcal{N}_i}\gamma_{ij}(k)(x_i(k)-x_j(k)) \tag{2}$$

where $K\in\mathbb{R}^{l\times m}$ is the controller gain to be designed, $\gamma_{ij}(k)$ denotes the communication status from agent j to i at time k (1 for successful communication, 0 for unsuccessful communication). The communication status process is assumed to be a discrete-time homogeneous Markov chain taking values in a finite set $\mathcal{W}=\{0,1\}$ with transition probability matrix

$$\Pi_{ij}=\begin{bmatrix}1-\beta_{ij} & \beta_{ij}\\ \alpha_{ij} & 1-\alpha_{ij}\end{bmatrix}, \tag{3}$$

where $0\le\Pr(\gamma_{ij}(k+1)=0\,|\,\gamma_{ij}(k)=1)=\alpha_{ij}\le1$ and $0\le\Pr(\gamma_{ij}(k+1)=1\,|\,\gamma_{ij}(k)=0)=\beta_{ij}\le1$ are called the failure probability and the recovery probability, respectively. To simplify the expression, $\mathscr{A}\triangleq(\alpha_{ij})$, $\mathscr{B}\triangleq(\beta_{ij})$ are used to denote the failure probability matrix and recovery probability matrix, respectively. Notice that the communication failure model in (2) indicates that the error signal $x_i(k)-x_j(k)$ will not be employed by the controller u_i at time k when the j communication channel fails.

Under the above formulation, if there is no communication channel from agent j to i, that is, no edge (v_i,v_j) in the graph and $a_{ij}=0$ in the Laplacian matrix, then the absence channel is treated as an 'ineffective' channel with $\gamma_{ij}(k)=0$ for all k, with the failure and recovery probabilities assigned to be $\alpha_{ij}=1$ and $\beta_{ij}=0$, respectively. By treating the absent communication channels this way, the original problem is equivalent to considering the communication failure problem of an MAS with a complete graph governed by known communication failure probability $\gamma_{ij}(k)$ in the communication channel from j to i at time k. Consequently, the Laplacian matrix at time k can be rewritten as

$$L(k)=\begin{bmatrix}\sum_{j\neq1}\gamma_{1j}(k) & -\gamma_{12}(k) & \ldots & -\gamma_{1n}(k)\\ \ldots & \ldots & \ldots & \ldots\\ -\gamma_{n1}(k) & -\gamma_{n2}(k) & \ldots & \sum_{j\neq n}\gamma_{nj}(k)\end{bmatrix} \tag{4}$$

where $L(k)\in\mathcal{L}^0\triangleq\{\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_d\}$ such that $\mathcal{L}^0$ contains all possible Laplacian matrices of the MAS. Here, $\max d=2^{\tilde{d}}$ where $\tilde{d}\triangleq\sum_{i=1}^n d_{ii}$ is the total number of effective communication channels in the graph (that is, the number of edges of the complete graph subtracting those 'ineffective' edges).

Definition 1. *A communication failure process is said to be Markovian if it is a discrete-time homogenous Markov chain defined in a complete probability space*

$(\Omega, \mathcal{F}, P)$, and takes value in $\mathcal{W}$ with known transition probability matrix $\Pi_1 \triangleq (\lambda_{ij}) \in \mathbb{R}^{s_0 \times s_0}$.

Denote by X the concatenation of vectors $x_1, x_2, \ldots, x_n$, that is, $X = (x_1^T, x_2^T, \ldots, x_n^T)^T$, by (1) and (2), the closed-loop dynamics of MASs (1) can be rewritten in a matrix form

$$X(k+1) = (\bar{A} + \bar{B}(k))X(k), \tag{5}$$

where $\bar{A} = I_n \otimes A$, $\bar{B}(k) = L(k) \otimes (BK)$, $L(k)$ takes values from $\mathcal{L}^0$. In this paper, we always assume that every agent could receive information of some other agents with a non-zero probability during the whole control process.

The mode transition probability $\pi_{\mathcal{L}_a\mathcal{L}_b}$ from $\mathcal{L}_a$ to $\mathcal{L}_b$ is given by

$$\pi_{\mathcal{L}_a\mathcal{L}_b} = \prod_{i,j=1, i\neq j}^{n} (\Gamma_{\mathcal{L}_a\mathcal{L}_b})_{ij} \tag{6}$$

for all $a, b = 1, \ldots, d$, which satisfies $\sum_{b=1}^{d} \pi_{\mathcal{L}_a\mathcal{L}_b} = 1$, where

$$(\Gamma_{\mathcal{L}_a\mathcal{L}_b})_{ij} = \begin{cases} 1-\alpha_{ij} & \text{if } (\mathcal{L}_b)_{ij} = (\mathcal{L}_a)_{ij} = -1, \\ 1-\beta_{ij} & \text{if } (\mathcal{L}_b)_{ij} = (\mathcal{L}_a)_{ij} = 0, \\ \alpha_{ij} & \text{if } (\mathcal{L}_b)_{ij} = 0, (\mathcal{L}_a)_{ij} = -1, \\ \beta_{ij} & \text{if } (\mathcal{L}_b)_{ij} = -1, (\mathcal{L}_a)_{ij} = 0. \end{cases}$$

3 STABILITY ANALYSIS AND CONTROLLER SYNTHESIS

In this section, we first give definitions on stability and consensus. Then we consider the consensus reaching and controller design problems for MASs with communication failure characteristics described in Section 2.

Definition 2. *MAS (5) with Markovian communication failure process (2) is said to be mean square stable (MSS) if*

$$\lim_{k\to\infty} \mathbf{E}(\|X(k)\|^2 \mid X(0)) = 0 \tag{7}$$

for any initial state $X(0) \in \mathbb{R}^{mn}$.

Definition 3. *Agents of MAS (5) with Markovian communication failure process (2) are said to reach consensus if*

$$\lim_{k\to\infty} \mathbf{E}(x_i(k) - x_j(k)) = 0 \tag{8}$$

for all $i, j \in \mathcal{I}$.

Now we are in the position to present the main contribution of this paper.

Theorem 1. *Consider MAS (5) with Markovian communication failure (2), given the controller gain matrix K, consensus is reached if there exist real matrices $P_{\mathcal{U}} \in \mathbb{S}^{(n-1)m\times(n-1)m}$, $\mathcal{U} \in \mathcal{L}^0$, such that*

$$\sum_{\mathcal{V}\in\mathcal{L}^0} \pi_{\mathcal{U}\mathcal{V}} \Phi_{\mathcal{V}}^T P_{\mathcal{V}} \Phi_{\mathcal{V}} - P_{\mathcal{U}} < 0, \quad \forall \mathcal{U} \in \mathcal{L}^0, \tag{9}$$

where $\Phi_{\mathcal{V}} = I_{n-1} \otimes A + \Lambda_{\mathcal{V}} \otimes BK$, $\Lambda_{\mathcal{V}} = T_o^T \mathcal{V} T_o$ and T_o is the orthogonal basis for the null space of $\mathbf{1}_n$.

Proof. Construct orthogonal matrix $T = \begin{bmatrix} \frac{1}{\sqrt{n}}\mathbf{1}_n & T_o \end{bmatrix} \in \mathbb{R}^{n\times n}$, where T_o is the orthogonal complement of $\mathbf{1}_n$ satisfying $T_o^T T_o = \mathbf{I}_{n-1}$. Since T_o is the orthogonal complement of $\mathbf{1}_n$, then

$$\mathbf{1}_n^T T_o = \mathbf{0}_{n-1}^T, T^T T = \begin{bmatrix} \frac{1}{\sqrt{n}}\mathbf{1}_n^T \\ T_o^T \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}}\mathbf{1}_n & T_o \end{bmatrix} = \mathbf{I}_n;$$

which means T is an orthogonal matrix and $T^T \mathcal{L} T$ is a similarity transformation. Partition the Laplacian matrix $\mathcal{L}$ and matrix T^T conformably:

$$T^T = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}}\mathbf{1}_{n-1}^T \\ T_{o1}^T & T_{o2}^T \end{bmatrix},$$

then

$$T^T \mathcal{L} T = \begin{bmatrix} 0 & A_{\mathcal{L}} \\ \mathbf{0}_{n-1} & \Lambda_{\mathcal{L}} \end{bmatrix}, \tag{10}$$

where $A_{\mathcal{L}} = \frac{1}{\sqrt{n}}\mathbf{1}_n^T \mathcal{L} T_o$ and $\Lambda_{\mathcal{L}} = T_o^T \mathcal{L} T_o$.

If the graph is undirected, that is, $\mathcal{L} = \mathcal{L}^T$. Thus

$$T^T \mathcal{L} T = \begin{bmatrix} 0 & (\frac{1}{\sqrt{n}} T_o^T \mathcal{L} \mathbf{1}_n)^T \\ \mathbf{0}_{n-1} & T_o^T \mathcal{L} T_o \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0}_{n-1}^T \\ \mathbf{0}_{n-1} & \Lambda_{\mathcal{L}} \end{bmatrix},$$

Denote $\tilde{X}(k) = (T \otimes I_m)^T X(k)$, the dynamics of the closed-loop system can be described by

$$\begin{aligned} &\tilde{X}(k+1) \\ &= (I_n \otimes A)\tilde{X}(k) + \begin{bmatrix} 0 \otimes I_m & A_{L(k)} \otimes (BK) \\ \mathbf{0}_{n-1} \otimes I_m & \Lambda_{L(k)} \otimes (BK) \end{bmatrix} \tilde{X}(k), \end{aligned}$$

where $A_{L(k)} = \frac{1}{\sqrt{n}}\mathbf{1}_n^T L(k) T_o$, $\Lambda_{L(k)} = T_o^T L(k) T_o$.

Define

$$\tilde{X}(k+1) = \begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix},$$

where

$$X_1(k+1) = AX_1(k) + (A_{L(k)} \otimes (BK))X_2(k), \tag{11}$$
$$X_2(k+1) = \Phi_{L(k)} X_2(k). \tag{12}$$

where $\Phi_{L(k)} = I_{n-1} \otimes A + \Lambda_{L(k)} \otimes BK$. Let

$$z_i(k) = \sum_{v_j\in\mathcal{N}_i} \gamma_{ij}(k)(x_i(k) - x_j(k)),$$

$$Z(k) = (z_1^T(k), z_2^T(k), \ldots, z_n^T(k))^T,$$

then

$$Z(k) = (L(k) \otimes I_m)X(k)$$

where $L(k)$ is given by (4), and

$$(T^T \otimes I_m)Z(k) = \begin{bmatrix} A_{L(k)} \otimes I_m \\ \Lambda_{L(k)} \otimes I_m \end{bmatrix} X_2(k),$$

where $T^T L(k) T = \begin{bmatrix} 0 & A_{L(k)} \\ \mathbf{0}_{n-1} & \Lambda_{L(k)} \end{bmatrix}$. Thus, $Z(k)$ is MSS if $X_2(k)$ is MSS, that is, the consensus of MASs (5) is reached if (12) is MSS.

Define a Lyapunov function as follows:

$$V(k, L(k)) = X_2^T(k) P_{L(k)} X_2(k)$$

where $P_{L(k)} > 0$ are matrices to be determined.

Let $\mathcal{U} = L(k)$, $\mathcal{V} = L(k+1)$, then

$$\begin{aligned} & \mathbf{E}(V(k+1, L(k+1)) | L(k) = \mathcal{U}) - V(k, \mathcal{U}) \\ = \; & \mathbf{E}(X_2^T(k+1) P_{L(k+1)} X_2(k+1) | L(k) = \mathcal{U}) \\ & - X_2^T(k) P_{\mathcal{U}} X_2(k) \\ = \; & X_2^T(k) \left[\sum_{\substack{\mathcal{V} \in \mathcal{L}^0 \\ j=1,\ldots,s_0}} \pi_{\mathcal{U}\mathcal{V}} \Phi_{\mathcal{V}}^T P_{\mathcal{V}} \Phi_{\mathcal{V}} - P_{\mathcal{U}} \right] X_2(k) \\ < \; & 0 \end{aligned}$$

for any $X_2(k) \neq 0$ if inequality (9) holds. Hence $\lim_{k\to\infty} \mathbf{E}(V(k, L(k))) = 0$, which ensures

$$\lim_{k\to\infty} \mathbf{E}(\|X_2(k; x_0)\|^2) = 0,$$

that is, system (12) is MSS. This completes the proof. □

Now, based on Theorem 1 and Schur complement Lemma, a controller design method is readily derived in the following theorem.

Theorem 2. *Consider MAS (5) of n agents with Markovian communication failure (2), consensus of the closed-loop system is reached if there exist $\mathcal{X}_i \in \mathbb{S}^{(n-1)m\times(n-1)m}$, $i = 1, 2, \ldots, d$, $\mathcal{Y} \in \mathbb{R}^{l\times m}$ and $G \in \mathbb{R}^{m\times m}$ such that $\forall i = 1, 2, \ldots, d$,*

$$\begin{bmatrix} -I_{n-1} \otimes (G + G^T) + \mathcal{X}_i & \Theta_i^T \\ \Theta_i & -\Lambda \end{bmatrix} < 0 \qquad (13)$$

where

$$\begin{aligned} \Lambda &= diag(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_d), \\ \Theta_i &= \begin{bmatrix} \sqrt{\pi_{i\mathcal{L}_1}} \Xi_{\mathcal{L}_1}^T & \sqrt{\pi_{i\mathcal{L}_2}} \Xi_{\mathcal{L}_2}^T & \cdots & \sqrt{\pi_{i\mathcal{L}_d}} \Xi_{\mathcal{L}_d}^T \end{bmatrix}^T, \\ \Xi_{\mathcal{V}} &= I_{n-1} \otimes AG + \Lambda_{\mathcal{V}} \otimes B\mathcal{Y}, \\ \Lambda_{\mathcal{V}} &= T_o^T \mathcal{V} T_o, \quad \mathcal{V} \in \mathcal{L}^0 \end{aligned}$$

and T_o is the orthogonal basis for the null space of $\mathbf{1}_n$. Moreover, a consensus controller gain matrix in (2) is given by $K = \mathcal{Y} G^{-1}$.

Proof. Define
$\Psi_i \triangleq \begin{bmatrix} \sqrt{\pi_{i\mathcal{L}_1}} \Phi_{\mathcal{L}_1}^T & \sqrt{\pi_{i\mathcal{L}_2}} \Phi_{\mathcal{L}_2}^T & \cdots & \sqrt{\pi_{i\mathcal{L}_d}} \Phi_{\mathcal{L}_d}^T \end{bmatrix}^T$,
where $\Phi_{\mathcal{V}} = I_{n-1} \otimes A + \Lambda_{\mathcal{V}} \otimes BK$. Pre- and post-multiplying inequality (13) by $\begin{bmatrix} \Psi_i & I_{(n-1)md} \end{bmatrix}$ and its transpose, respectively, then

$$\Psi_i \mathcal{X}_i \Psi_i^T - \Lambda < 0,$$

that is,

$$\begin{bmatrix} -\mathcal{X}_i^{-1} & \Psi_i^T \\ \Psi_i & -\Lambda \end{bmatrix} < 0. \qquad (14)$$

Thus, inequality (14) is equivalent to inequality (9) by replacing P_i by $\mathcal{X}_i^{-1}$. This completes the proof. □

4 CONCLUSIONS

In this paper, a consensus control problem of MASs with communication failure between agents has been studied. A sufficient condition on consensus reaching problem is established in terms of the feasibility of some LMIs. In addition, a state-feedback consensus controller is designed to make the closed-loop system reach consensus. A numerical example has been given to demonstrate the effectiveness of the proposed results.

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