

# HOMOTHETIC APPROXIMATIONS FOR STOCHASTIC PN

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Keywords: Stochastic Petri Nets, Continuous Petri Nets, Fluidification, Steady State, Reliability Analysis.

Abstract: Reliability analysis is often based on stochastic discrete event models like stochastic Petri nets. For complex dynamical systems with numerous components, analytical expressions of the steady state are tedious to work out because of the combinatory explosion with discrete models. For this reason, fluidification is an interesting alternative to estimate the asymptotic behaviour of stochastic processes with continuous Petri nets. Unfortunately, the asymptotic mean marking of stochastic and continuous Petri nets are mainly often different. This paper proposes a geometric approach that leads to a homothetic approximation of the stochastic steady state in specific regions of the marking space.

## 1 INTRODUCTION

Reliability analysis is a major challenge to improve the safety of industrial processes. For complex dynamical systems with numerous interdependent components, such studies are mainly based on stochastic discrete event models like Markov models (Rausand et al., 2004) or stochastic Petri nets (SPNs) (Molloy, 1982). Such models are mathematically well founded and lead either to analytical results or numerical simulations. But in case of large systems, the combinatory explosion limits their use. In this context, fluidification can be discussed as a relaxation method.

This paper is about the approximation of the SPNs asymptotic mean markings and average throughputs by mean of continuous Petri nets (CPNs) under infinite server semantic (Vazquez et al., 2008; Lefebvre et al., 2009). The limits of the fluidification of SPNs are discussed according to the partition in regions of the reachability state space. A characterization of the regions is proposed that leads to a homothetic approximation of the stochastic steady state. The proposed results are not constructive but concern the existence of solutions. They may be helpful to investigate the properties of a considered SPN and they may lead for example to the design of observers or controllers for stochastic processes.

## 2 FLUIDIFICATION OF SPN

### 2.1 Stochastic Petri Nets

A Petri net (PN) is defined as  $\langle P, T, W_{PR}, W_{PO} \rangle$  where  $P = \{P_i\}$  is a set of  $n$  places and  $T = \{T_j\}$  is a set of  $q$  transitions,  $W = W_{PO} - W_{PR} \in (\mathbf{Z})^{n \times q}$  is the incidence matrix,  $M(t)$  is the PN marking vector and  $M_I$  the PN initial marking (David et al., 1992). Depending on the incidence matrix, PNs may have P-semiflows. A P-semiflow  $y \in (\mathbf{Z}^+)^n$  is a non-zero solution of equation  $y^T \cdot W = 0$ . Let define  $Y = \{y_1, \dots, y_h\}$  as a basis of  $W^T$  kernel, composed of  $h$  minimal P-semiflows. For simplicity, the basis  $Y$  will be represented as a matrix  $Y \in (\mathbf{Z}^+)^{n \times h}$  that satisfies (1):

$$Y^T \cdot M(t) = Y^T \cdot M_I = C, \quad t \geq 0 \quad (1)$$

Let define, for each minimal P-semiflow  $y_i \in Y$ , its support as the subset  $P(y_i) \subset P$  of places that belong to the corresponding marking invariant. For each  $P(y_i)$  it is possible to select a single place and to recover the marking of this place from the marking of the other places in  $P(y_i)$ . Let define as a consequence the subset  $P_2 \subset P$  of  $h$  places whose markings may be recovered from  $Y$ , and the subset  $P_1 \subset P$  of  $n - h$  other places. The permutation matrix  $D$  defined according to  $P_1$  and  $P_2$  leads to the marking  $M' = ((M')_1^T (M')_2^T)^T = D \cdot M$ .

A stochastic Petri net (SPN) is a timed PN whose transitions firing periods are characterized a firing

rate vector  $\mu = (\mu_j) \in (\mathbf{R}^+)^q$  (Molloy, 1982). The marking and mean marking vectors of a SPN at time  $t$  will be referred as  $M_s(t)$  and  $MM_s(t)$ . The SPNs considered in this paper are bounded, reinitialisable, with infinite server semantic, race policy and resampling memory. As a consequence, the considered SPNs have a reachability graph with a finite number  $N$  of states and their marking process is mapped into a Markov model with state space isomorphic to the reachability graph (Bobbio *et al.*, 1998). The Markov model has an asymptotic state propability vector  $\Pi_{ss} = (\pi_{ss\ k}) \in [0, 1]^{1 \times N}$  and the asymptotic mean marking  $M_{mms}$  of SPNs depends from  $\Pi_{ss}$ :

$$m_{mms\ i} = \sum_{k=1, \dots, N} m_{k\ i} \cdot \pi_{ss\ k}, i = 1, \dots, n \quad (2)$$

## 2.2 Continuous Petri Nets and Regions

CPNs have been developed in order to provide continuous approximations of the discrete behaviours of PNs (David *et al.*, 1992; Silva and Recalde, 2004). A CPN is defined as  $\langle PN, X_{max} \rangle$  where  $PN$  is a Petri nets and  $X_{max} = \text{diag}(x_{max\ j}) \in (\mathbf{R}^+)^{q \times q}$  is the diagonal matrix of maximal firing speeds  $x_{max\ j}$ ,  $j = 1, \dots, q$ .  $M_c(t)$  is the marking vector and  $X_c(t) = (x_{c\ j}(t)) \in (\mathbf{R}^+)^q$  is the firing speeds vector that satisfy  $dM_c(t) / dt = W \cdot X_c(t)$ . For CPNs with infinite server semantic,  $X_c(t)$  depends continuously on the marking of the places according to  $x_{c\ j}(t) = x_{max\ j} \cdot \min(m_k(t) / w^{PR}_{kj})$ , for all  $P_k \in {}^\circ T_j$ , where  ${}^\circ T_j$  stands for the set of  $T_j$  upstream places. A marked CPN has a steady state if the marking vector  $M_c(t)$  tends to a finite limit  $M_{mms}$  in long run.

According to the function “min(.)”, the marking space of CPNs is divided into  $K$  regions  $\mathbf{A}_k$  (eventually empty) with  $K = |{}^\circ T_1| \times \dots \times |{}^\circ T_q|$ . Each region  $\mathbf{A}_k$  is defined by its PT-set (Julvez *et al.* 2005) defined according to (3):

$$\text{PT-set}(\mathbf{A}_k) = \{(P_i, T_j) \text{ s.t. } \forall M_c(t) \in \mathbf{A}_k, x_{c\ j}(t) = x_{max\ j}(t) \cdot m_{c\ i}(t) / w^{PR}_{ij}\} \quad (3)$$

The place  $P_i$  such that  $i = \text{argmin}(m_k(t) / w^{PR}_{kj})$  for all  $P_k \in {}^\circ T_j$  is the critical place for transition  $T_j$  at time  $t$ . A constraint matrix  $A_k = (a^k_{ij}) \in (\mathbf{R}^+)^{q \times n}$ ,  $k = 1, \dots, K$ ,  $i = 1, \dots, q$  and  $j = 1, \dots, n$  is defined for each region  $\mathbf{A}_k$  according to the corresponding PT-set:  $a^k_{ij} = 1/w^{PR}_{ji}$  if  $(P_i, T_j) \in \text{PT-set}(\mathbf{A}_k)$  and  $a^k_{ij} = 0$  elsewhere. For each region  $\mathbf{A}_k$ , equation (4) holds:

$$\forall M_c(t) \in \mathbf{A}_k, dM_c(t) / dt = W \cdot X_{max} \cdot A_k \cdot M_c(t) \quad (4)$$

**Definition:** A region  $\mathbf{A}_k$  is critical if there exists two transitions  $T_j$  and  $T_k$  that have the same critical place  $P_i$  in region  $\mathbf{A}_k$ .

**Proposition 1:** Marking  $M_c \in \mathbf{A}_k$  iff  $M_c$  satisfies (5):

$$\begin{pmatrix} -I_n \\ A(k) \\ Y^T \\ -Y^T \end{pmatrix} \cdot M_c \leq \begin{pmatrix} 0 \\ 0 \\ C \\ -C \end{pmatrix} \quad (5)$$

with  $I_n$  and the identity matrix of size  $n$  and:

$$A(k) = \begin{bmatrix} \dots \end{bmatrix} \in (\mathbf{Z}^+)^{q \cdot (K-1) \times n} \quad (6)$$

**Proof:** Equation (5) results from the definition of PT-sets and P-semiflows of the PN. The equation  $-I_n \cdot M_c \leq 0$  stands for the positivity of the marking. The equation  $A(k) \cdot M_c = 0$  defines the region borders according to the “min” functions. Finally, the equation  $Y^T \cdot M_c \leq C$  and  $-Y^T \cdot M_c \leq -C$  result from the P-semiflows.  $\square$

## 2.3 Continuous Approximation of SPNs

Numerous structural and behavioural properties are not preserved with fluidification (Silva and Recalde 2004). The average throughput and mean marking of a CPN are mainly not identical to the ones of a discrete PN (Julvez *et al.*, 2005, Lefebvre *et al.*, 2009). Concerning SPNs, the steady state is mainly often different from the one of a CPN with same parameters ( $x_{max\ j} = \mu_j$ ,  $j = 1, \dots, q$ ). The asymptotic mean markings of SPNs can be approximated with the steady state of CPNs if all transitions remain enabled with degree at least 1 in long run and the marking vector does not leave the region of initial marking in long run (Vasquez *et al.*, 2008). These conditions limit strongly the interest of fluidification. In our preceding works, we have investigated the limit of fluidification for the approximation of SPNs. CPNs with a modified set of maximal firing speed can be used to approximate the mean marking in non critical regions (Lefebvre and Leclercq 2010). In the next section, we continue this investigation for critical regions.

## 3 HOMOTHETIC ESTIMATION

Let consider the problem to reach  $M_{mms}$  when  $M_{mms} \in \mathbf{A}_i$  (eventually critical) and  $M_i \in \mathbf{A}_k$  (non critical) with  $\mathbf{A}_i \neq \mathbf{A}_k$ . The proposition 2 provides conditions to work out admissible but partial

homothetic transformations of ratio  $\alpha$  such that  $(\alpha \cdot (M'_{mms1})^T (M''_{mms2})^T)^T \in \mathbf{A}_k$ . Then a CPN with modified constant maximal firing speeds ( $x_{maxj} \neq \mu_j, j = 1, \dots, q$ ) is worked out with proposition 3. This CPN approximates  $\alpha \cdot M'_{mms1}$ .

**Proposition 2:** Let define  $M'_{mms} = D \cdot M_{mms}$  and  $M''_{mms2}$  such that the  $Y^T \cdot (\alpha \cdot (M'_{mms1})^T (M''_{mms2})^T)^T = C$ . The condition  $(\alpha \cdot (M'_{mms1})^T (M''_{mms2})^T)^T \in \mathbf{A}_k$  holds if  $\alpha$  satisfies (7):

$$\begin{pmatrix} -I_{n-h} \\ A(k) \\ Y^T \\ -Y^T \end{pmatrix} \cdot D^{-1} \cdot \begin{pmatrix} \alpha \cdot M'_{mms1} \\ M''_{mms2} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ C \\ -C \end{pmatrix} \quad (7)$$

**Proof:** Proposition 2 results from proposition 1 by replacing  $M_c$  by  $D^{-1} \cdot D \cdot M_c$  and by considering the partial homothetic transformations of ratio  $\alpha$ . Proposition 2 characterises the intersection of the region  $\mathbf{A}_k$  and the direction  $M_{mms}$ .

**Proposition 3:** Consider a SPN with  $M_I \in \mathbf{A}_k$  (non critical) and  $M_{mms} \in \mathbf{A}_i$  (eventually critical) with  $\mathbf{A}_i \neq \mathbf{A}_k$ . Let define the CPN with same structure and initial marking.  $M_c(t)$  tends asymptotically to  $M_{mmc}$  such that  $M'_{mmc1} = \alpha \cdot M'_{mms1}$  if there exist  $X_{max}$  such that  $M_c(t)$  satisfies the proposition 1 for all  $t \geq 0$  and equation (8) holds:

$$W \cdot X_{max} \cdot A_k \cdot D^{-1} \cdot \begin{pmatrix} \alpha \cdot M'_{mms1} \\ M''_{mms2} \end{pmatrix} = 0 \quad (8)$$

**Proof:** Proposition 3 results from the steady state solution of equation (4) and from the partial homothetic transformations of ratio  $\alpha$ .

The propositions 2 and 3 lead to a 4-stages algorithm for estimating  $M_{mms}$  in critical regions.

Work out the transformation matrix  $D$ .

List the conditions to be satisfied by  $\alpha$ , so that the partial homothetic transformation of  $M_{mms}$  and  $M_I$  are in the same non critical region.

Work out the modified constant firing speeds that drive  $M_c(t)$  to  $M_{mmc}$  st  $M'_{mmc1} = \alpha \cdot M'_{mms1}$ .

Recover the asymptotic stochastic mean marking  $M_{mms}$  with (1).

## 4 EXAMPLE

Consider for example the marked SPN described in fig.1 (Julvez *et al.* 2005). This PN has 2 P-semiflows:  $Y = ((0 \ 0 \ 0 \ 1 \ 1)^T (1 \ 1 \ 2 \ 1 \ 0)^T)^T$  and  $C = (4 \ 5)^T$ . The subsets of places  $\mathbf{P}_1 = \{P_1, P_2, P_3\}$  and  $\mathbf{P}_2 = \{P_4, P_5\}$  are defined according to  $Y$  and lead to the trivial transformation matrix  $D = I_5$  (i.e.  $M' = M$ ).

The fig. 2 illustrates stochastic mean markings that are reached from  $M_I = (5 \ 0 \ 0 \ 0 \ 4)^T$  according to various transitions firing rate vectors  $\mu \in [0 : 10]^4$ .

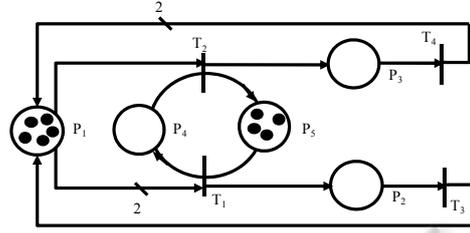


Figure 1: An example of SPN with  $M_I = (5 \ 0 \ 0 \ 0 \ 4)^T$ .

If the PN of fig. 1 is considered as a CPN, 4 regions  $\mathbf{A}_1$  to  $\mathbf{A}_4$  exist. The regions are defined by the constraint matrices  $A_1$  to  $A_4$ .

$$A_1 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The regions are also depicted in figures 2 to 4 according to the full lines (reachable area limits) and dotted lines (regions intersections).

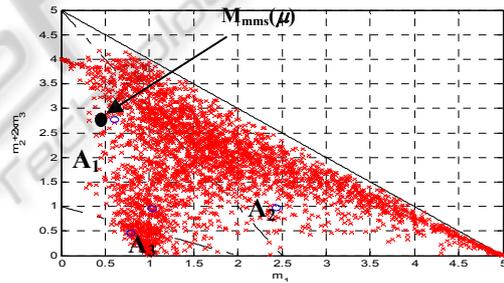


Figure 2: Projection in plan  $(m_1, m_2 + 2 \cdot m_3)$  of  $M_{mms}$  for the SPN of fig. 1 with various vectors  $\mu \in [0 : 10]^4$ .

The CPN as a single critical region  $\mathbf{A}_1$  and  $M_I \in \mathbf{A}_2$ . The fig. 3 illustrates the asymptotic continuous mean markings that are reached from  $M_I$  and according to various maximal firing speeds  $x_{maxj} \in [0.1 : 10]$ ,  $j = 1, \dots, 4$ . Some areas in the critical region  $\mathbf{A}_1$  are reachable with SPNs and not with CPNs. For examples, the asymptotic mean markings  $M_{mms}(\mu) = (0.6 \ 0.2 \ 1.3 \ 1.6 \ 2.4)^T$  obtained with  $\mu = (6 \ 2 \ 3 \ 0.5)^T$  is in critical region  $\mathbf{A}_1$  and is not reachable with CPN (figs. 2 and 3):  $X_{max} = \text{diag}(6 \ 2 \ 3 \ 0.5)^T$  leads to  $M_{mmc}(X_{max}) = (0.1 \ 0.1 \ 0.4 \ 3.9 \ 0.1)^T$  and any other maximal firing speeds also fail.

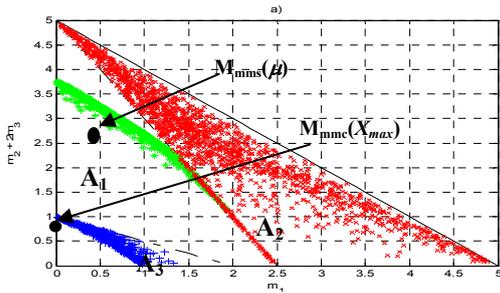


Figure 3: Projection in plan  $(m_1, m_2+2.m_3)$  of  $M_{mmc}$  for the CPN of fig. 1 with various  $x_{maxj} \in [0.1 : 10], j=1, \dots, 4$ .

The propositions 2 and 3 are used to work out the admissible ratio  $\alpha$  and the maximal firing speeds that lead to homothetic approximations of  $M_{mms1}(SPN)$ . In the region  $A_2$ , (7) leads to (10):

$$\begin{pmatrix} -I_5 \\ -1 & 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & -1 \\ Y^T \\ -Y^T \end{pmatrix} \begin{pmatrix} \alpha m'_{mms1} \\ \alpha m'_{mms2} \\ \alpha m'_{mms3} \\ m''_{mms4} \\ m''_{mms5} \end{pmatrix} \leq \begin{pmatrix} 0_{5 \times 1} \\ 0 \\ 0 \\ 4 \\ 5 \\ -4 \\ -5 \end{pmatrix} \quad (9)$$

and then to the admissible interval  $\alpha \in [5/(2.m_{mms1} + m_{mms2} + 2.m_{mms3}) : 5/(m_{mms1} + m_{mms2} + 2.m_{mms3})]$ . For the considered example  $\alpha \in [1.26 : 1.48]$ . The figure 4 illustrates various homothetic marking trajectories for SPN obtained for some values of parameter  $\alpha$  in admissible interval in order to reach  $\alpha.M'_{mms1}(\mu)$ .

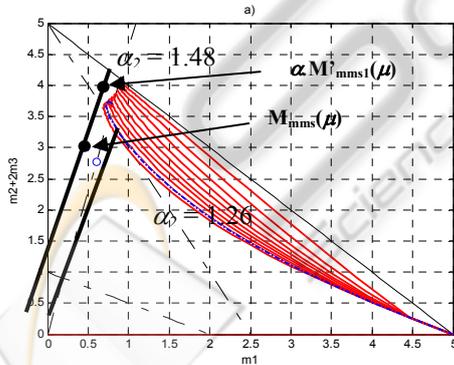


Figure 4: Projection in plan  $(m_1, m_2+2.m_3)$  of the homothetic convergence to  $\alpha.M'_{mms1}(\mu)$ .

The proposition 3 is used to work out the set of the admissible maximal firing speeds that depend on the parameter  $\alpha$  such that the CPN with same structure and initial marking tends to  $M_{mms}$ :

$$\begin{aligned} x_{max1} &= 2.x_{max4}.m_{mms3} / m_{mms1} \\ x_{max2} &= \alpha.x_{max4}.m_{mms3} / (5-\alpha.(m_{mms1}+m_{mms2}+2.m_{mms3})) \\ x_{max3} &= x_{max4}.m_{mms3} / m_{mms2} \end{aligned} \quad (10)$$

where  $x_{max4}$  is a dof. For example, consider the particular homothetic ratio  $\alpha = 4/3$ . The trajectory (dotted line in figure 4), obtained for  $X_{max} = (4.25, 3.41, 6, 1)$  results in asymptotic marking  $m'_{mmc1} = 0.80$ ,  $m'_{mmc2} = 0.28$ ,  $m'_{mmc3} = 1.71$ . From this approximation, it is easy to recover the asymptotic stochastic mean marking  $M_{mms}(\mu)$ .

## 5 CONCLUSIONS

This paper has proposed partial homothetic transformations of the SPN mean marking to approximate. The proposed results concern the existence of solutions but are not constructive in the sense that the asymptotic stochastic mean markings to estimate by CPNs must a priori be known. The selection of the best projectors and ratios will be investigated in our further works. Our future work is also to investigate continuous approximations directly derived from the SPNs transition firing rates.

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