

# AN INVESTIGATION INTO THE DISTRIBUTION OF MEMBERSHIP GRADES FOR NON-STATIONARY FUZZY SETS

Pragnesh A. Gajjar

*The University of Nottingham, School of Mathematical Sciences, Nottingham NG7 2RD, U.K.*

Jonathan M. Garibaldi

*The University of Nottingham, School of Computer Science, Nottingham NG8 1BB, U.K.*

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**Abstract:** In this paper we study some properties related to the distribution of membership grades for non-stationary fuzzy sets. We obtain the formulation for the distribution, where the non-stationary fuzzy sets are obtained by generating instantiations about the center values. The two cases considered are for Triangular and Gaussian underlying membership functions. The analytical results obtained are then compared with computer generated results, for completeness.

## 1 INTRODUCTION

Fuzzy sets were introduced and studied by Zadeh (Zadeh, 1965) to model uncertainty inherent in assigning membership of elements to real world sets, such as the set of *old* people or the set of *tall* people. However, as pointed out by Klir and Folger (Klir and Folger, 1988), in reality, these type-1 fuzzy sets have some limitations - that they are certain and actually do not have any fuzziness. In fact, Zadeh (Zadeh, 1975) addressed this problem and proposed ‘fuzzy sets with fuzzy membership functions’, and defined fuzzy sets of type- $n$ ,  $n = 2, 3, \dots$ , for which membership functions range over fuzzy sets of type  $(n - 1)$ .

Dubois and Prade (Dubois and Prade, 1980), Yager (Yager, 1980), Mizumoto and Tanaka (Mizumoto and Tanaka, 1976) subsequently advocated their use. But their use in practice was limited due to significant computational requirements associated with their implementation. Recently, due to efforts of Mendel (Mendel and John, 2002) and also due to increase in computational power, type-2 fuzzy sets have received renewed interest. Mendel (Mendel, 2001) restricted the class of type-2 fuzzy sets and called it Interval Type-2 fuzzy sets which are characterized by having secondary membership functions which only take the values 0 or 1. Garibaldi *et al* (Garibaldi and Ozen, 2007) pointed out that although type-2 fuzzy sets capture the concept of uncertainty in membership

functions, they do not capture the notion of variability which is very natural in human decision making. To incorporate this variability into decision making in the context of fuzzy expert system, they proposed the notion of ‘non-deterministic fuzzy reasoning’ in which variability is introduced into the membership function of a fuzzy system through random alterations to the parameters of the functions. Garibaldi *et al* (Garibaldi *et al.*, 2008) formalized this notion and called it a non-stationary fuzzy sets which is a set (collection) of type-1 fuzzy sets, obtained by instantiations of the underlying fuzzy membership function. The variations in the non-stationary fuzzy sets were generated as a result of perturbations in the underlying membership function. Various approaches to perturb the underlying membership function viz. variation in location, variation in width, noise variation, were discussed in (Garibaldi *et al.*, 2008). They discussed two case studies which revealed that the distribution of membership grades was following some pattern which was far from straight-forward and dependent on choice of underlying membership function and also the point of interest. This motivated us to investigate the distribution of membership grades for non-stationary fuzzy sets.

Although the present paper does not consider all the options for generating instantiations such as discussed in Garibaldi *et al* (Garibaldi *et al.*, 2008), the main focus is on instantiations generated by variation

in location. The analysis was carried out to obtain formulations for the distribution of membership grades resulting from Normalized perturbations in the center values of the underlying membership function of a non-stationary fuzzy set. The two cases studied are when the underlying fuzzy membership function is (i) Triangular (TMF), and (ii) Gaussian (GMF).

The paper is divided into five sections. Basic definitions and results are given in the next Section. In Section 3, we obtain analytic expressions for the frequency distribution of membership grades for non-stationary fuzzy sets with TMF and GMF as underlying membership functions. Case studies for each of the two types are discussed, followed by results and discussions. Finally the conclusions are drawn.

## 2 PRELIMINARIES

*Definition:* Non-stationary Fuzzy Sets (Garibaldi et al., 2008)

Let  $X$  be a universe of discourse and  $A$  denote a fuzzy set characterized by a membership function  $\mu_A$ . Let  $\mathbb{T} = \{t_i; \forall i\}$  be set of time points and  $f : \mathbb{T} \rightarrow \mathbb{R}$  be the perturbation function.

A non-stationary fuzzy set  $\hat{A}$  of the universe of discourse  $X$  is characterized by a non-stationary membership function  $\mu_{\hat{A}} : \mathbb{T} \times X \rightarrow [0, 1]$  that associates with each element  $(t, x) \in \mathbb{T} \times X$ .

In simple terms, for a given (standard) fuzzy set  $A$  and a set of time points  $\mathbb{T}$ , a non-stationary fuzzy set  $\hat{A}$  is a set of duplicates of  $A$  varied over time.

The time duplication of  $A$  is termed as an instantiation and is denoted by  $\hat{A}_t$ . Thus, at any given moment of time  $t \in \mathbb{T}$ , the non-stationary fuzzy set  $\hat{A}$  instantiates the (standard) fuzzy set  $\hat{A}_t$ . The standard fuzzy set  $A$  is then termed as the underlying fuzzy set, and its associated membership function  $\mu_A(x)$  the underlying membership function.

The following three alternative approaches for the generation of instantiations were suggested in (Garibaldi et al., 2008).

- 1) variation in location (center/mean)

$$\mu_{\hat{A}}(t, x) = \mu_A(x + a(t)) \quad \forall t \in \mathbb{T}. \quad (1)$$

where  $a(t)$  is constant for any given  $t$ .

In this case, the membership function is shifted, as a whole, on right or left, depending on whether  $a(t) > 0$  or  $a(t) < 0$ , relative to the underlying membership function.

- 2) variation in width (spread)

$$|\hat{A}_{t, \alpha+}| = |A_{\alpha+}| + a_{\alpha}(t) \quad \forall t \in \mathbb{T}, \alpha \in [0, 1]. \quad (2)$$

In this case, the cardinalities of all strong  $\alpha$ -cut sets relative to the underlying membership function are increased or decreased, depending on whether  $a(t) > 0$  or  $a(t) < 0$ , relative to the underlying membership function.

- 3) noise variation

$$\mu_{\hat{A}}(t, x) = \mu_A(x) + a(t) \quad \forall t \in \mathbb{T}. \quad (3)$$

where  $a(t)$  is constant for any given  $t$ .

In this case, the membership function is shifted upward or downward, depending on whether  $a(t) > 0$  or  $a(t) < 0$ , relative to the underlying membership function.

For the transformation of random variables, we shall use the following result from the probability theory (VijayaKumar et al., 2005).

*Result:* If a random variable  $X$  is transformed to a new variable  $Y$  by the mapping  $T$ , that is,  $Y = T(X)$ , then the probability density function (p.d.f.) of  $Y$  depends on the p.d.f. of  $X$  as well as the mapping  $T$ , and can be obtained by first finding the connection between their cumulative distribution functions (c.d.f.s.) and then taking the derivatives to determine relation between the two p.d.fs.

In particular, if  $T$  is one-to-one, then the probability that the random variable  $X$  takes on a value in an elemental interval  $dx$  centered at  $x$  is the same as the probability that the random variable  $Y$  takes on a value in an elemental interval  $dy$  centered at  $y$ .

that is,

$$f_X(x) |dx| = f_Y(y) |dy| \quad (4)$$

given that the sizes  $dx$  and  $dy$  are given by  $T$ , that is,

$$dy = \left| \frac{dT}{dx} \right| dx \quad (5)$$

$\therefore$  from (4),

$$f_Y(y) = \left. \frac{f_X(x)}{\frac{dT(x)}{dx}} \right|_{x=T^{-1}(y)} \quad (6)$$

It is a straightforward exercise to show that if  $X$  follows a Normal distribution with mean  $\omega$  and variance  $\sigma^2$  and  $Y = aX + b$  is an affine transformation, where  $a$  and  $b$  are constants, then  $Y$  also follows a Normal distribution, but with mean  $a\omega + b$  and variance  $a^2\sigma^2$ .

Symbolically,

$$X \sim N(\omega, \sigma^2) \Rightarrow Y = aX + b \sim N(a\omega + b, a^2\sigma^2). \quad (7)$$

This result can be easily verified by fitting a Normal curve on a histogram drawn with  $Y$  as data values.

### 3 MAIN RESULTS

#### 3.1 Instantiations with Underlying Triangular Membership Function

The triangular membership function is defined by

$$(x_L, x_C, x_R)_T = \begin{cases} \frac{x_L - x}{x_L - x_C} & ; x_L \leq x \leq x_C \\ \frac{x_R - x}{x_R - x_C} & ; x_C \leq x \leq x_R \\ 0 & ; \text{otherwise} \end{cases}$$

Because of the symmetry of the underlying TMF (assuming equal spreads), and for simplicity, we consider the left tail of a TMF where the membership function  $\phi$  is expressed as an affine transformation of  $x$ , say  $y = \phi(x) = mx + c$ , where

$$m = \text{slope} = \frac{1}{x_C - x_L} \quad \text{and} \quad c = -\frac{x_L}{x_C - x_L} = -mx_L.$$

What is interesting and important to investigate is that for a fixed  $x^* \in X$ ; what can be said about the distribution of  $y$  values at  $x^*$ ; given that center  $x_C$  follows Normal distribution?

Let  $\varepsilon_t = \varepsilon(t)$  be variations in the center  $x_C$  at moment  $t$ , when the instantiations are produced, and suppose  $\varepsilon_t \sim N(\omega, \sigma^2)$ .

Then,

$$x_C \leftrightarrow x_C - \varepsilon_t, \quad x_L \leftrightarrow x_L - \varepsilon_t$$

whereas the slope  $= m = \frac{1}{x_C - x_L}$  remains unchanged (see Figure 1).

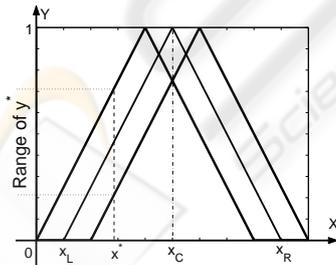


Figure 1: Instantiations with TMF.

Thus, for a fixed  $x = x^*$ , we obtain

$$(y_t(x^*))_L = \phi(x^* - \varepsilon_t) = m x^* - (x_L + \varepsilon_t) m \quad (8)$$

Now, as  $\varepsilon_t \sim N(\omega, \sigma^2)$ , by (7), for the left tail,

$$(y_t(x^*))_L \sim N(m(x^* - x_L - \omega), m^2 \sigma^2) \quad (9)$$

Thus, the frequency function of  $(y_t)_L$  is

$$\Psi[(y_t(x^*))_L] = \frac{1}{m\sigma\sqrt{2\pi}} e^{-\frac{[y_t - m(x^* - x_L - \omega)]^2}{2m^2\sigma^2}} \quad (10)$$

Similarly, for the right tail,

$$(y_t)_R = \phi^*(x - \varepsilon_t) = n x - (x_R + \varepsilon_t) n \quad (11)$$

where,  $n = \frac{1}{x_C - x_R}$  = slope of the right tail.

Hence, for fixed  $x = x^*$ , if  $\varepsilon_t \sim N(\omega, \sigma^2)$ , then

$$(y_t(x^*))_R \sim N(n(x^* - x_R - \omega), n^2 \sigma^2) \quad (12)$$

and so,

$$\Psi^*[(y_t(x^*))_R] = \frac{1}{n\sigma\sqrt{2\pi}} e^{-\frac{[y_t - n(x^* - x_R - \omega)]^2}{2n^2\sigma^2}} \quad (13)$$

#### 3.2 Instantiations with Underlying Gaussian Membership Function

In this case, we consider  $\mu(x; c, \sigma)$ , the underlying Gaussian membership function to be

$$\mu(x; c, \sigma) = e^{-\frac{(x-c)^2}{2\sigma^2}} ; x \in \mathbb{X} \quad (14)$$

where  $c$  is mean,  $\sigma^2$  is variance, and  $\mathbb{X}$  is the universe of discourse.

As in the case of triangular membership function, we first consider the left tail of the Gaussian curve. To generate instantiations, we assume that  $\sigma$  remains constant throughout the process (as we are only considering variation in location parameter), and let the instantiations be generated due to Normalized variation in location parameter ( $M$ , say), that is,  $M \sim N(0, v^2)$  (Figure 2), so that the p.d.f. of  $M$  is given by,

$$f(M; 0, v) = \frac{1}{v\sqrt{2\pi}} e^{-\frac{M^2}{2v^2}} ; M \in \mathbb{M} \quad (15)$$

where  $\mathbb{M} \subset \mathbb{X}$  is the set of center values obtained by perturbing  $M$ .

For such variations in center value  $M$ , we wish to study the distribution of membership grades at some fixed  $x^* \in \mathbb{X}$ ,  $x^* < M$  (for left tail of Gaussian). A close observation reveals that this is same as obtaining the distribution of  $y^* = \mu(M; c, \sigma)$  for the

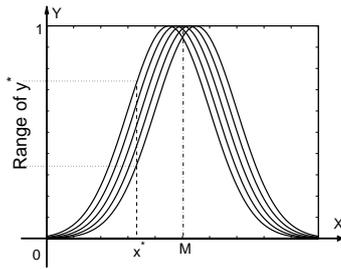


Figure 2: Instantiations with GMF.

underlying membership function, as  $M$  varies.

We define,

$$y^* = \mu(M; c, \sigma) = e^{-\frac{(M-c)^2}{2\sigma^2}} \quad (16)$$

Let  $T: \mathbb{M} \rightarrow \mathbb{Y}$  be a transformation defined by

$$T(\xi) = \mu(\xi; c, \sigma) = e^{-\frac{(\xi-c)^2}{2\sigma^2}} \quad (17)$$

so that

$$T(M) = y^* \quad (18)$$

Clearly,  $T$  is one-to-one as it maps each  $m_o \in \mathbb{M}$  uniquely to a  $y_o \in \mathbb{Y}$  (as we have considered only the left tail of Gaussian).

Further, suppose that  $G$  is c.d.f. of  $y^*$ . Then, as  $T^{-1}$  is decreasing,

$$\begin{aligned} G(y^*) &= P(y \leq y^*) \\ &= \begin{cases} 1 - P(T^{-1}(y) \leq M) & ; y^* \in (0, \tau) \\ P(T^{-1}(y) \leq M) & ; y^* \in (\tau, 1) \end{cases} \end{aligned} \quad (19)$$

where  $\tau = \mu(0; c, \sigma)$ .

Using (6), for  $0 < y^* < 1$ ,

$$\begin{aligned} g(y^*) &= \text{p.d.f. of } y^* \\ &= \left| \frac{d}{dy^*} G(y^*) \right| \\ &= \left| \mu(M; c, \sigma) \frac{dM}{dy^*} \right|_{M=T^{-1}(y^*)} \end{aligned} \quad (20)$$

where,  $\mu(M; c, \sigma)$  is defined in (16).

Also, from (16),  $y^* = \mu(M; c, \sigma)$ ,

$$\begin{aligned} \therefore (M-c)^2 &= -2\sigma^2 \ln(y^*) \\ \Rightarrow M = T^{-1}(y^*) &= c \pm \sigma \sqrt{-2 \ln(y^*)} \\ &= c \pm \sigma \theta \end{aligned} \quad (21)$$

where

$$\theta = \sqrt{-2 \ln(y^*)} \quad (22)$$

is well defined for  $y^* \in (0, 1)$

For  $M$  given by (21),

$$\frac{dM}{dy^*} = \mp \frac{\sigma}{y^* \theta} \quad (23)$$

Now,

$$M \geq \mu \Rightarrow y^* \in (0, \mu(0; c, \sigma))$$

$$\text{and } M < \mu \Rightarrow y^* \in (\mu(0; c, \sigma), 1). \quad (24)$$

Thus,  $g(y^*)$

$$\begin{aligned} &= \begin{cases} \mu(M; c, \sigma) \frac{\sigma}{y^* \theta} & ; y^* \in (0, 1) \\ 0 & ; \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\sigma}{y^* \theta} e^{-\frac{(\mu + \sigma \theta - c)^2}{2\sigma^2}} & ; y^* \in (0, \mu(0; c, \sigma)) \\ \frac{\sigma}{y^* \theta} e^{-\frac{(\mu - \sigma \theta - c)^2}{2\sigma^2}} & ; y^* \in (\mu(0; c, \sigma), 1) \\ 0 & ; \text{otherwise} \end{cases} \end{aligned} \quad (25)$$

The same formulation is obtained by considering  $x^* \in \mathbb{X}$ ,  $x^* > M$  (for right tail of Gaussian).

## 4 CASE STUDIES

In this Section, two case studies are described which were carried out to validate the formulations for the distribution of membership grades, obtained in Section 3, when the underlying membership function is (i) Triangular and (ii) Gaussian. In both the cases considered, the non-stationary fuzzy sets were constructed by generating instantiations about the center values.

### 4.1 Case Study-I

In the first case study, instantiations were generated with underlying TMF. For simplicity in writing codes, the triangular fuzzy number  $(x_L, x_C, x_R)_T = (-10, 0, 10)_T$  was considered as the underlying membership function, and 5000 instantiations were generated. The center  $x_C$  of the underlying TMF was perturbed using Normally distributed random numbers with mean 0 and standard deviation 0.5.

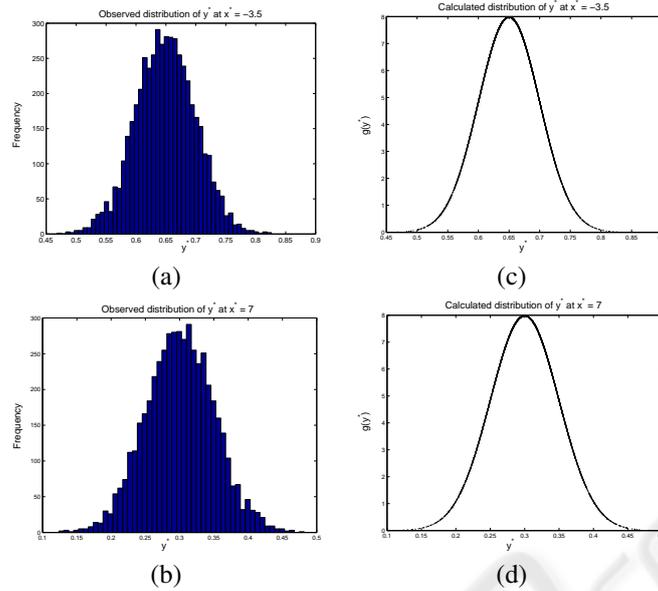


Figure 3: (a),(b) Observed distribution of  $y^*$  at  $x^* = -3.5$  and  $7.0$  respectively (c), (d) Calculated distribution of  $y^*$  at  $x^* = -3.5$  and  $7.0$  respectively, using formulations given by (10) and (13).

#### 4.1.1 Results

The results were obtained for variation in location of the underlying TMF by considering 5000 instantiations around the center. The distribution of membership grades of the inputs over time, for the values of  $x^* = -3.5$  and  $7.0$  (selected values are shown for illustrative purposes) were obtained as shown in Figure 3. The membership grades were found to be Normally distributed as established analytically in Section 3.1. The range of  $y^*$  values were  $(0.46837, 0.82610)$  and  $(0.12390, 0.48163)$ . Not surprisingly, the length of the intervals in each case remained fixed as  $0.35773$ .

#### 4.2 Case Study-II

In the second case study, instantiations were generated with underlying GMF defined by (14). The center of the underlying GMF was perturbed using Normally distributed random numbers as per (15). These numbers depend on the choice of the standard deviation  $\sigma$  of the underlying GMF and were chosen such that the fixed  $x^*$  always remain less (greater) than  $M$  for left (right) tail, when the instantiations are generated. Further, we choose  $v \ll \sigma$ , say,  $v \approx \frac{\sigma}{10}$ . The dependence of  $v$  on  $\sigma$  is out of scope of this paper and is presently being investigated.

#### 4.2.1 Results

The Normalized random numbers generated as per above were in the range  $(-0.45, 0.45)$ . This means that  $x^*$  should be chosen such that  $x^* < -0.45$  or  $x^* > 0.45$ , but in both cases should be in the interval  $(-3, 3) \subset \mathbb{X}$ . For values outside this interval, we have to generate other set of instantiations with different center value  $c$ . With center at  $c = 0$ , 5000 instantiations were generated and membership grades  $y^*$  evaluated at  $x^* = -1.50$  and  $2.25$  (selected values are shown for illustrative purposes). The corresponding ranges for  $y^*$  values were respectively,  $(0.15837, 0.53630)$  and  $(0.031174, 0.18735)$ . The histograms for the observed values of  $y^*$  were plotted as shown in Figures 4 (a),(b), whereas the histograms for the distribution of  $y^*$  values given by the analytical formula (25) were plotted as shown in Figures 4 (c),(d). The figures showed good similarity between the observed and calculated distributions.

### 5 DISCUSSION

The goal of this paper is to study the relationship between (horizontal) perturbations of non-stationary fuzzy sets and the (vertical) distribution of membership grades. This is part of ongoing efforts/research into the relationships between non-stationary fuzzy sets and type-2 fuzzy sets. The distribution of membership grades of non-stationary fuzzy sets cor-

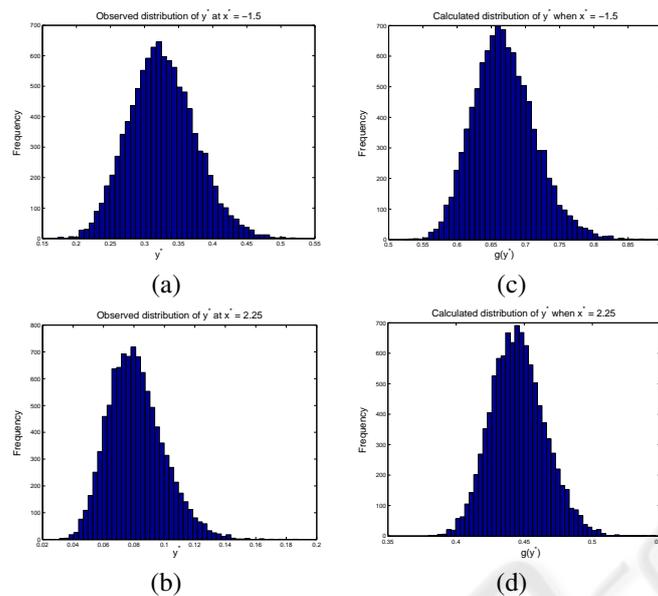


Figure 4: (a), (b) Observed distribution of  $y^*$  at  $x^* = -1.50$  and  $2.25$  respectively (c), (d) Calculated distribution of  $y^*$  at  $x^* = -1.50$  and  $2.25$  respectively, given by (25).

respond (informally) to the secondary membership function of general type-2 fuzzy sets. The two cases investigated were for variations in center values. The case of non-symmetrical TMF (with varied spread) can be dealt with by an approach similar to the one described in Section 3.1; provided the slopes of left (right) tails of non-stationary fuzzy sets remains constant. More work needs to be done to understand the behaviour of membership grades when the center is perturbed using other forms of distributions, such as, uniform, and also by generating instantiations by varying parameters other than the location (mean), for instance, by varying the slope in TMF or variance in GMF. Further research can also be carried out for other types of underlying membership functions. These areas will be further explored in our future research, in addition to seeing the application areas for non-stationary fuzzy sets.

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