SIMPLE GENETIC ALGORITHM WITH GENERALISED $\alpha^\star$-SELECTION

Dynamical System Model, Fixed Points, and Schemata

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Abstract: The dynamical system model proposed by VOSE provides a theory of genetic algorithms as specific random heuristic search (RHS) algorithms by describing the stochastic trajectory of a population with the help of a deterministic heuristic function and its fixed points. In order to simplify the mathematical analysis and to enable the explicit calculation of the fixed points the simple genetic algorithm (SGA) with α-selection has been introduced where the best or α-individual is mated with individuals randomly chosen from the population with uniform probability. This selection scheme also allows to derive a simple coarse-grained system model based on the equivalence relation imposed by schemata.

In this paper, the α-selection scheme is generalised to $\alpha^\star$-selection by allowing the β best individuals of the current population instead of the single best α-individual to mate with other individuals randomly chosen from the population. It is shown that most of the results obtained for α-selection can be transferred to the SGA with generalised $\alpha^\star$-selection, e.g. the explicit calculation of the fixed points of the heuristic function or the derivation of a coarse-grained system model based on schemata.

1 INTRODUCTION

As specific instances of random heuristic search (RHS), genetic algorithms mimic biological evolution and molecular genetics in simplified form. Genetic algorithms process populations of individuals which evolve according to selection and genetic operators like crossover and mutation. The algorithm's stochastic dynamics can be described with the help of a dynamical system model introduced by VOSE et al. (Reeves and Rowe, 2003; Vose, 1999a; Vose, 1999b). The population trajectory is attracted by the fixed points of an underlying deterministic heuristic function which also yields the expected next population. However, even for moderate problem sizes the calculation of the fixed points is difficult.

The simple genetic algorithm (SGA) with α-selection allows to explicitly derive the fixed points of the heuristic function as well as to formulate a simple coarse-grained system model based on the equivalence relation imposed by schemata (Neubauer, 2008a; Neubauer, 2008b). In this selection scheme, the best or α-individual is mated with individuals randomly chosen from the current population with uniform probability. This paper extends the α-selection scheme to generalised $\alpha^\star$-selection by allowing the β best individuals of the current population instead of the single best α-individual to mate with other individuals randomly chosen from the current population. It is shown that most results obtained for the SGA with α-selection can be transferred to the SGA with generalised $\alpha^\star$-selection by redefining the system matrix of the dynamical system model, e.g. the explicit calculation of the fixed points of the respective heuristic function or the derivation of a simple coarse-grained system model based on schemata.

The paper is organised as follows. In section 2, the SGA with α-selection is defined, the dynamical system model, the corresponding heuristic function and its fixed points are formulated, and a simple coarse-grained system model based on the equivalence relation imposed by schemata is described. In section 3, these results are extended to the SGA with generalised $\alpha^\star$-selection. A brief conclusion is given in section 4.

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2 SGA WITH α-SELECTION

The SGA with α-selection, crossover and mutation defined by masks is described in this section following (Neubauer, 2008a; Neubauer, 2008b) and the notation and definition of the SGA in (Vose, 1999a). In the present context, the genetic algorithm is used for the maximisation of a fitness function $f : \Omega \to \mathbb{R}$ which is defined over the search space $\Omega = \mathbb{Z}_2^\ell = \{0,1\}^\ell$. Each binary ℓ-tuple $(a_0, a_1, \ldots, a_{\ell-1})$ will be identified with the integer $a = a_0 \cdot 2^{\ell-1} + a_1 \cdot 2^{\ell-2} + \ldots + a_{\ell-1} \cdot 2^0$ leading to the search space $\Omega = \{0, 1, \ldots, n-1\}$ with cardinality $|\Omega| = n = 2^\ell$. With this binary number representation, the bitwise modulo-2 addition $a \oplus b$, modulo-2 multiplication $a \otimes b$ and binary complement $\overline{a}$ are defined. The integer a is also viewed as a column vector $(a_0, a_1, \ldots, a_{\ell-1})^{\mathrm{T}}$; the integer $n - 1 = 2^\ell - 1$ corresponds to the all-one ℓ-tuple $\mathbf{1}$. Finally, the indicator function $[i = j]$ is defined by $[i = j] = 1$ if $i = j$ and 0 if $i \neq j$.

The SGA with α-selection formulated in (Neubauer, 2008a; Neubauer, 2008b) works over populations of r individual binary ℓ-tuples $a \in \Omega$. In each generation, offspring individuals are generated by genetic operators like crossover χ_Ω and mutation μ_Ω which are applied to selected parental individuals. In the α-*selection* scheme, the best individual or α-*individual* b in the current population is mated with individuals randomly chosen from the current population with uniform probability r^{-1} (see Fig. 1).

```
initialise population;
while end of iteration ≠ true do
    select α-individual b as first parent;
    for the creation of r offspring do
        select second parent c randomly;
        create offspring a = μΩ (χΩ (b,c));
    end
end
```

Figure 1: SGA with α-selection.

The *crossover* operator $\chi_\Omega : \Omega \times \Omega \to \Omega$ randomly generates an offspring ℓ-tuple $a = (a_0, a_1, \ldots, a_{\ell-1})$ according to $a = \chi_\Omega(b, c)$ with crossover probability χ from two ℓ-tuples $b = (b_0, b_1, \ldots, b_{\ell-1})$ and $c = (c_0, c_1, \ldots, c_{\ell-1})$. With the crossover mask $m \in \Omega$ the ℓ-tuples $a = b \otimes m \oplus \overline{m} \otimes c$ or $a = b \otimes \overline{m} \oplus m \otimes c$ are generated one of which is chosen as offspring a with equal probability 2^{-1}. The crossover mask m is randomly chosen from Ω according to the probability distribution vector $\chi = (\chi_0, \chi_1, \ldots, \chi_{n-1})^{\mathrm{T}}$.

The *mutation* operator $\mu_\Omega : \Omega \to \Omega$ randomly flips each bit of the ℓ-tuple $a = (a_0, a_1, \ldots, a_{\ell-1})$ with mutation probability μ. It can be equivalently formulated with the help of the mutation mask $m \in \Omega$ according to $\mu_\Omega(a) = a \oplus m$. The mutation mask m is randomly chosen from Ω according to the probability distribution vector $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})^{\mathrm{T}}$.

2.1 Dynamical System Model

The dynamical system model of the SGA with α-selection can be compactly formulated with the population vector $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})^{\mathrm{T}}$. Each component p_i gives the proportion of element $i \in \Omega$ in the current population. The population vector $\mathbf{p}$ is an element of the simplex $\Lambda = \{\mathbf{p} \in \mathbb{R}^n : p_i \geq 0 \wedge \sum_{i \in \Omega} p_i = 1\}$.

The SGA with α-selection is an instance of RHS $\tau : \Lambda \to \Lambda$. The RHS τ is equivalently represented by a heuristic function $\mathcal{G} : \Lambda \to \Lambda$ according to $\mathbf{q} = \tau(\mathbf{p})$ with the expected next generation population vector $\mathbf{q}$ (see Fig. 2). For a given population vector $\mathbf{p}$ the heuristic function $\mathcal{G}$ yields the probability distribution $\mathcal{G}(\mathbf{p})_i = \Pr\{\text{individual } i \text{ is sampled from } \Omega\}$ which underlies the generation of the next population. The stochastic trajectory $\mathbf{p}$, $\tau(\mathbf{p})$, $\tau^2(\mathbf{p})$, ... approximately follows the trajectory $\mathbf{p}$, $\mathcal{G}(\mathbf{p})$, $\mathcal{G}^2(\mathbf{p})$, ... of the deterministic dynamical system defined by $\mathcal{G}$. The RHS τ behaves like the dynamical system model in the limit of infinite populations (Vose, 1999a).

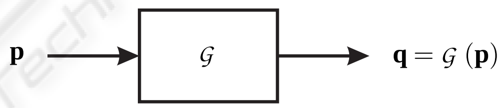

Figure 2: Dynamical system model of the SGA.

2.1.1 Heuristic

In the α-selection scheme, the α-individual b is selected as the first parent whereas the second parent is chosen uniformly at random from the current population. The heuristic function $\mathcal{G}(\mathbf{p})$ is then given by (Neubauer, 2008a; Neubauer, 2008b)

$$\mathbf{q} = \mathcal{G}(\mathbf{p}) = A \cdot \mathbf{p} \tag{1}$$

with the *system matrix*

$$\boxed{A = \sigma_b \cdot M^* \cdot \sigma_b} \ . \tag{2}$$

Here, $(\sigma_b)_{i,j} = [i \oplus j = b]$ denotes the permutation matrix. The $n \times n$ *mixing matrix* is defined by (Vose, 1999a)

$$M_{i,j} = \sum_{u,v \in \Omega} \mu_v \cdot \frac{\chi_u + \chi_{\overline{u}}}{2} \cdot [i \otimes u \oplus \overline{u} \otimes j = v] \ . \tag{3}$$

The twist M^* of the symmetric mixing matrix $M = M^{\mathrm{T}}$ is given by $(M^*)_{i,j} = M_{i \oplus j, i}$. The components of the $n \times n$ system matrix are given by

$$\boxed{A_{i,j} = M_{i \oplus b, i \oplus j}} \quad . \tag{4}$$

Compared to the SGA in (Vose, 1999a), the α-selection scheme yields a simpler heuristic function $\mathcal{G}$ which is completely described by the α-individual b and the mixing matrix M. This dynamical system model is illustrated in Fig. 3.

$$\mathbf{p} \longrightarrow \boxed{\sigma_b} \longrightarrow \boxed{M^*} \longrightarrow \boxed{\sigma_b} \longrightarrow \mathbf{q}$$

Figure 3: Dynamical system model of the SGA with α-selection.

2.1.2 Fixed Points

For a given α-individual b the heuristic function $\mathcal{G}(\mathbf{p}) = A \cdot \mathbf{p}$ of the SGA with α-selection is linear. The fixed points $\omega = \mathcal{G}(\omega) = A \cdot \omega$ are obtained from the eigenvectors of the system matrix A to eigenvalue 1. There exists a single eigenvalue $\lambda_0 = 1$ with corresponding eigenvector ω whereas the remaining $n-1$ eigenvalues fulfill $0 \leq \lambda_i \leq 1 - 2\mu$. The eigenvector ω yields the unique fixed point of the heuristic function $\mathcal{G}$ for a given α-individual b.

The fixed point ω can be determined explicitly with the help of the WALSH transform. For the matrix A the WALSH transform is $\widehat{A} = W \cdot A \cdot W$ with the symmetric and orthogonal $n \times n$ WALSH matrix $W_{i,j} = n^{-1/2} \cdot (-1)^{i^{\mathrm{T}} j}$ (Vose and Wright, 1998). The WALSH transform of the vector ω is $\widehat{\omega} = W \cdot \omega$. A and its WALSH transform $\widehat{A}$ have the same eigenvalues with eigenvectors which are also related by the WALSH transform, especially yielding $\omega = A \cdot \omega \Leftrightarrow \widehat{\omega} = \widehat{A} \cdot \widehat{\omega}$. The WALSH transform of the system matrix is given by

$$\widehat{A}_{i,j} = \widehat{M}_{i \oplus j, j} \cdot (-1)^{b^{\mathrm{T}}(i \oplus j)} \quad . \tag{5}$$

For 1-point crossover χ_Ω and mutation μ_Ω the WALSH transform of the mixing matrix M is formulated in (Vose, 1999a). Because the WALSH transform of the mixing matrix fulfills $\widehat{M}_{i,j} \propto [i \otimes j = 0]$ the WALSH transform $\widehat{A}$ is a lower triangular matrix (Neubauer, 2008a; Neubauer, 2008b). Due to the relation $\widehat{\omega} = \widehat{A} \cdot \widehat{\omega}$ the WALSH transform of the fixed point can be iteratively determined from

$$\boxed{\widehat{\omega}_i = \frac{1}{1 - \widehat{A}_{i,i}} \cdot \sum_{j=0}^{i-1} \widehat{A}_{i,j} \cdot \widehat{\omega}_j} \tag{6}$$

for $1 \leq i \leq n-1$ starting with $\widehat{\omega}_0 = n^{-1/2}$ which ensures $\sum_{i \in \Omega} \omega_i = 1$. The fixed point $\omega = W \cdot \widehat{\omega}$ is finally obtained via the inverse WALSH transform.

2.2 Schemata

Following (Vose, 1999a) *schemata* can be considered as specific equivalence relations in which two equivalent individuals $i \equiv j$ in the search space Ω belong to the same equivalence class $[i] = \{j \in \Omega : j \equiv i\}$. With the help of the *quotient map* $\Xi_{[i],j} = [i \equiv j]$ this can be expressed as $i \equiv j$ if and only if $\Xi_{[i],j} = 1$. Two populations are equivalent if the proportions of individuals in each equivalence class $[i] \in \Omega/\!\equiv$ with $i \in \Omega$ are the same in both populations. With population vectors $\mathbf{p}$ and $\mathbf{p}'$ this corresponds to the condition $\Xi\mathbf{p} = \Xi\mathbf{p}'$.

A *schemata family* is defined by the ℓ-tuple $\xi \in \Omega$ via the quotient map $\Xi_{[i],j} = [j \otimes \xi = i]$ with $i \in \Omega_\xi = \{i \in \Omega : i \otimes \overline{\xi} = 0\}$ and $j \in \Omega$ leading to the $2^{\mathbf{1}^{\mathrm{T}}\xi} \times 2^\ell$ matrix Ξ. Two individuals $i, j \in \Omega$ are equivalent if they agree on the defining positions of the schemata family according to $i \equiv j \Leftrightarrow i \otimes \xi = j \otimes \xi$. The cardinality of Ω_ξ is $|\Omega_\xi| = 2^{\mathbf{1}^{\mathrm{T}}\xi}$ with the number of defining positions $\mathbf{1}^{\mathrm{T}}\xi$.

2.2.1 Schema Heuristic

A dynamical system $\mathcal{G}$ is consistently modeled by the simplified coarse-grained system $\widetilde{\mathcal{G}}$ implied by the equivalence relation $\equiv$ if the diagram in Fig. 4 commutes, i.e. for two equivalent population vectors $\mathbf{p}$ and $\mathbf{p}'$ the population vectors in the next generation $\mathcal{G}(\mathbf{p})$ and $\mathcal{G}(\mathbf{p}')$ must also be equivalent.

$$\begin{tikzcd}
\mathbf{p} \arrow[r, "\mathcal{G}"] \arrow[d, "\Xi"'] & \mathcal{G}(\mathbf{p}) \arrow[d, "\Xi"] \\
\widetilde{\mathbf{p}} \arrow[r, dashed, "\widetilde{\mathcal{G}}"'] & \Xi\mathcal{G}(\mathbf{p})
\end{tikzcd}$$

Figure 4: Commutativity diagram with quotient map Ξ.

For the SGA with α-selection, crossover and mutation the proportion of the expected next population representing schema $[i] = i \oplus \Omega_{\overline{\xi}}$ with $i \in \Omega_\xi$ is given by (Neubauer, 2008a; Neubauer, 2008b)

$$\Xi G(\mathbf{p}) = A_\xi \cdot \Xi\mathbf{p} \quad . \tag{7}$$

The $2^{\mathbf{1}^{\mathrm{T}}\xi} \times 2^{\mathbf{1}^{\mathrm{T}}\xi}$ *schema system matrix*

$$\boxed{(A_\xi)_{[i],[j]} = (M_\xi)_{[i \oplus b],[i \oplus j]}} \tag{8}$$

with $i, j \in \Omega_\xi$ is defined with the help of the $2^{\mathbf{1}^{\mathrm{T}}\xi} \times$

$2^{\mathbf{1}^{\mathrm{T}}\xi}$ *schema mixing matrix*

$$(M_\xi)_{[i],[j]} = \sum_{u,v\in\Omega_\xi} (\Xi\mu)_{[v]} \cdot \frac{(\Xi\chi)_{[u]} + (\Xi\chi)_{[\overline{u}]}}{2} \cdot [i\otimes u \oplus \overline{u}\otimes j = v] \quad . \tag{9}$$

The schema system matrix A_ξ can be obtained from system matrix A and quotient map Ξ according to

$$\boxed{A_\xi = \frac{2^{\mathbf{1}^{\mathrm{T}}\xi}}{n} \cdot \Xi \cdot A \cdot \Xi^{\mathrm{T}}} \quad . \tag{10}$$

The schema heuristic function $\widetilde{\mathcal{G}}$ is defined according to $\widetilde{\mathcal{G}}(\Xi\mathbf{p}) = A_\xi \cdot \Xi\mathbf{p}$. Since the schema system matrix A_ξ depends on the α-individual b the heuristic function $\mathcal{G}$ is not compatible with the equivalence relation imposed by schemata in the strict sense. If the α-individual b is lost or a better individual is sampled from the search space Ω in the next generation the schema system matrix A_ξ and the schema heuristic function $\widetilde{\mathcal{G}}$ change. The α-individual b can be considered as an exogenous parameter to the coarse-grained system model (see Fig. 5).

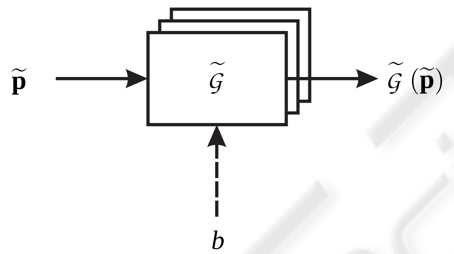

Figure 5: Coarse-grained system model of the SGA with α-selection depending on the α-individual b.

2.2.2 Schema Fixed Points

As for the dynamical system model and the corresponding heuristic function $\mathcal{G}$, there exists a unique fixed point of the schema heuristic function $\widetilde{\mathcal{G}}$ which can be calculated from the WALSH transform $\widehat{A}_\xi = W_\xi \cdot A_\xi \cdot W_\xi$ of the schema system matrix A_ξ. Here, the $2^{\mathbf{1}^{\mathrm{T}}\xi} \times 2^{\mathbf{1}^{\mathrm{T}}\xi}$ WALSH matrix W_ξ is defined over Ω_ξ. The WALSH transform $\widehat{A}_\xi$ of the schema system matrix A_ξ is given by

$$(\widehat{A}_\xi)_{[i],[j]} = (\widehat{M}_\xi)_{[i\oplus j],[j]} \cdot (-1)^{b^{\mathrm{T}}(i\oplus j)} \tag{11}$$

with $i, j \in \Omega_\xi$. $\widehat{A}_\xi$ is obtained from $\widehat{A}$ by choosing rows and columns with indices in Ω_ξ, i.e.

$$\boxed{(\widehat{A}_\xi)_{[i],[j]} = \widehat{A}_{i,j}} \quad . \tag{12}$$

Similar to the system matrix A it can be shown that the WALSH transform $\widehat{A}_\xi$ of the schema system matrix A_ξ is a lower triangular matrix with an eigenvalue $\lambda_{[0]} = (\widehat{A}_\xi)_{[0],[0]} = 1$ leading to the unique schema fixed point $\widetilde{\omega} = A_\xi \cdot \widetilde{\omega}$ which is related to the fixed point ω according to $\widetilde{\omega} = \Xi\omega$. Taking into account $\widehat{\widetilde{\omega}} = \widehat{A}_\xi \cdot \widehat{\widetilde{\omega}}$ the WALSH transform of the schema fixed point can be iteratively determined from

$$\boxed{\widehat{\widetilde{\omega}}_{[i]} = \frac{1}{1-\widehat{A}_{i,i}} \cdot \sum_{j\in\Omega_\xi\cap\{0,1,\ldots,i-1\}} \widehat{A}_{i,j} \cdot \widehat{\widetilde{\omega}}_{[j]}} \tag{13}$$

for $i \in \Omega_\xi$ starting with $\widehat{\widetilde{\omega}}_{[0]} = 2^{-\mathbf{1}^{\mathrm{T}}\xi/2}$. The schema fixed point $\widetilde{\omega} = W_\xi \cdot \widehat{\widetilde{\omega}}$ is finally obtained via the inverse WALSH transform over Ω_ξ.

3 SGA WITH GENERALISED $\alpha^\star$-SELECTION

The α-selection scheme can be generalised by allowing the β best individuals of the current population instead of the single best α-individual to mate with other individuals randomly chosen from the current population. Most of the theoretical results obtained for α-selection with a single α-individual are transferrable to the SGA with generalised $\alpha^\star$-selection.

The SGA with generalised $\alpha^\star$-selection is illustrated in Fig. 6. For the *generalised* $\alpha^\star$*-selection* scheme the β best individuals b_0, b_1, …, $b_{\beta-1}$ in the current population are mated with individuals randomly chosen from the current population. For the creation of each offspring individual one of the β best individuals b_0, b_1, …, $b_{\beta-1}$ is chosen with uniform probability β^{-1} as the first parent b whereas the second parent c is chosen uniformly at random from the current population with probability r^{-1}.

initialise population;
while end of iteration $\neq$ true **do**
  select β best individuals $b_0, b_1, \ldots, b_{\beta-1}$;
  for the creation of r offspring **do**
    select first parent b from β
      best individuals randomly;
    select second parent c from
      population randomly;
    create offspring $a = \mu_\Omega(\chi_\Omega(b,c))$;
  end
end

Figure 6: SGA with generalised $\alpha^\star$-selection.

3.1 Dynamical System Model

In this section, the dynamical system model, the corresponding heuristic function and its fixed points are derived for the SGA with generalised $\alpha^\star$-selection.

3.1.1 Heuristic

In the generalised $\alpha^\star$-selection scheme, one of the β best individuals is selected from the set $\{b_k\}_{0\leq k\leq \beta-1}$ as the first parent with uniform probability β^{-1} whereas the second parent is chosen uniformly at random from the current population according to the probability distribution $\Pr\{\text{individual } j \text{ is selected}\} = p_j$ with $j \in \Omega$. The heuristic function $\mathcal{G}$ is given by

$$\mathcal{G}(\mathbf{p})_i = \sum_{k=0}^{\beta-1} \frac{1}{\beta} \sum_{j=0}^{n-1} p_j \cdot \Pr\{\mu_\Omega(\chi_\Omega(b_k, j)) = i\} \quad .$$

The mixing operation comprises crossover χ_Ω and mutation μ_Ω. With the help of the probability distributions for crossover and mutation this leads to

$$\begin{aligned}
&\Pr\{\mu_\Omega(\chi_\Omega(b_k, j)) = i\} \\
&\quad = \sum_{u\in\Omega} \mu_u \cdot \Pr\{\chi_\Omega(b_k, j) = i \oplus u\} \\
&\quad = \sum_{u\in\Omega} \mu_u \sum_{v\in\Omega} \frac{\chi_v + \chi_{\overline{v}}}{2} \cdot [b_k \otimes v \oplus \overline{v} \otimes j = i \oplus u] \\
&\quad = M_{i\oplus b_k, i\oplus j}
\end{aligned}$$

with $n \times n$ mixing matrix M according to (3). The heuristic function is

$$\mathcal{G}(\mathbf{p})_i = \sum_{j=0}^{n-1} p_j \cdot \frac{1}{\beta} \sum_{k=0}^{\beta-1} M_{i\oplus b_k, i\oplus j} \quad .$$

With the $n \times n$ *system matrix*

$$\boxed{A_{i,j} = \frac{1}{\beta} \sum_{k=0}^{\beta-1} M_{i\oplus b_k, i\oplus j}} \tag{14}$$

this leads to the linear system of equations for the expected next population vector

$$q_i = \mathcal{G}(\mathbf{p})_i = \sum_{j=0}^{n-1} A_{i,j} \cdot p_j \tag{15}$$

or equivalently

$$\mathbf{q} = \mathcal{G}(\mathbf{p}) = A \cdot \mathbf{p} \tag{16}$$

which corresponds to the heuristic function in (1) for the SGA with α-selection. By making use of the permutation matrix σ_b and the twist M^* of the mixing matrix the system matrix A can be expressed as

$$\boxed{A = \frac{1}{\beta} \sum_{k=0}^{\beta-1} \sigma_{b_k} \cdot M^* \cdot \sigma_{b_k}} \quad . \tag{17}$$

The corresponding dynamical system model is illustrated in Fig. 7.

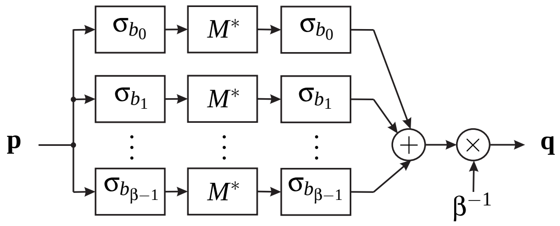

Figure 7: Dynamical system model of the SGA with generalised $\alpha^\star$-selection.

3.1.2 Fixed Points

Similar to the SGA with α-selection the fixed points ω of the heuristic function $\mathcal{G}$ are obtained from the eigenvectors of the system matrix A to eigenvalue 1 due to the linear relation $\mathcal{G}(\mathbf{p}) = A \cdot \mathbf{p}$ for a given set $\{b_k\}_{0\leq k\leq\beta-1}$ of β best individuals. Since the system matrix A and its WALSH transform $\widehat{A}$ have the same eigenvalues with eigenvectors, which are also related by the WALSH transform, the WALSH transform of the system matrix

$$\widehat{A}_{i,j} = \widehat{M}_{i\oplus j,j} \cdot \frac{1}{\beta} \sum_{k=0}^{\beta-1} (-1)^{b_k^{\mathrm{T}}(i\oplus j)} \tag{18}$$

is derived. The system matrix A as well as its WALSH transform $\widehat{A}$ depend on the β best individuals.

The WALSH transform $\widehat{A}$ is a lower triangular matrix with eigenvalues λ_i given by the diagonal elements $\lambda_i = \widehat{A}_{i,i} = \widehat{M}_{0,i}$ leading to

$$\lambda_i = \frac{(1-2\mu)^{\mathbf{1}^{\mathrm{T}} i}}{2} \cdot \sum_{k\in\Omega_{\overline{i}}} (\chi_k + \chi_{k\oplus i}) \quad . \tag{19}$$

Because of $\lambda_0 = 1$ and $0 \leq \lambda_i \leq 1 - 2\mu$ for $1 \leq i \leq n-1$ there exists a single eigenvector ω to eigenvalue 1 which is a fixed point of the heuristic function $\omega = \mathcal{G}(\omega) = A \cdot \omega$. Taking into account $\widehat{\omega} = \widehat{A} \cdot \widehat{\omega}$ with lower triangular matrix $\widehat{A}$ the WALSH transform $\widehat{\omega}$ of the fixed point can be recursively calculated according to (6). The fixed point is then obtained via the inverse WALSH transform $\omega = W \cdot \widehat{\omega}$.

3.2 Schemata

In correspondence to the SGA with α-selection, the schema heuristic function will be formulated for the SGA with generalised $\alpha^\star$-selection in this section.

3.2.1 Schema Heuristic

The proportion of the expected next population representing schema $[i] = i \oplus \Omega_{\overline{\xi}}$ with $i \in \Omega_\xi$ is given by

$$\Xi G(\mathbf{p}) = A_\xi \cdot \Xi \mathbf{p} \quad . \tag{20}$$

The $2^{\mathbf{1}^{\mathrm{T}}\xi} \times 2^{\mathbf{1}^{\mathrm{T}}\xi}$ *schema system matrix* is defined by

$$\boxed{(A_\xi)_{[i],[j]} = \frac{1}{\beta} \sum_{k=0}^{\beta-1} (M_\xi)_{[i \oplus b_k],[i \oplus j]}} \qquad (21)$$

with $i, j \in \Omega_\xi$ and the $2^{\mathbf{1}^{\mathrm{T}}\xi} \times 2^{\mathbf{1}^{\mathrm{T}}\xi}$ schema mixing matrix M_ξ defined in (9). As in (10), the schema system matrix A_ξ can be obtained from system matrix A and quotient map Ξ.

The schema system matrix A_ξ and the schema heuristic function $\widetilde{\mathcal{G}}$ defined by $\widetilde{\mathcal{G}}(\Xi \mathbf{p}) = A_\xi \cdot \Xi \mathbf{p}$ depend on the set $\{b_k\}_{0 \leq k \leq \beta-1}$ of β best individuals. This set (or the set $\{[b_k]\}_{0 \leq k \leq \beta-1}$ of their corresponding equivalence classes) thus acts like an exogenous parameter set to the coarse-grained system model, as illustrated in Fig. 8.

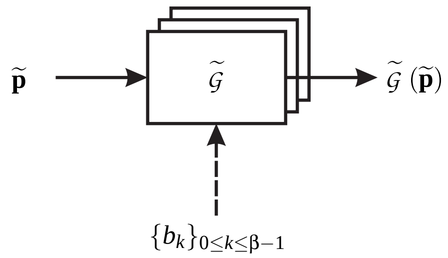

Figure 8: Coarse-grained system model of the SGA with generalised $\alpha^\star$-selection.

3.2.2 Schema Fixed Points

For a given set $\{b_k\}_{0 \leq k \leq \beta-1}$ of β best individuals there exists a unique fixed point of the schema heuristic function $\widetilde{\mathcal{G}}$ which again can be calculated from the WALSH transform $\widehat{A}_\xi$ of the schema system matrix A_ξ which is given by

$$(\widehat{A}_\xi)_{[i],[j]} = (\widehat{M}_\xi)_{[i \oplus j],[j]} \cdot \frac{1}{\beta} \sum_{k=0}^{\beta-1} (-1)^{b_k^{\mathrm{T}}(i \oplus j)} \qquad (22)$$

with $i, j \in \Omega_\xi$. As for the SGA with α-selection, $\widehat{A}_\xi$ is obtained from $\widehat{A}$ by choosing rows and columns with indices in Ω_ξ according to (12). By exploiting the lower triangularity of the WALSH transform $\widehat{A}_\xi$ of the schema system matrix A_ξ and the existence of an eigenvalue $\lambda_{[0]} = (\widehat{A}_\xi)_{[0],[0]} = 1$ the schema fixed point $\widetilde{\omega} = A_\xi \cdot \widetilde{\omega}$ can be determined as in (13). $\widetilde{\omega}$ can also be obtained from the fixed point ω according to $\widetilde{\omega} = \Xi \omega$.

4 CONCLUSIONS

The dynamical system model describes the stochastic dynamics of genetic algorithms with the help of the deterministic heuristic function $\mathcal{G}$ and its fixed points. Since for practical problem sizes the calculation of the fixed points is difficult the SGA with α-selection has been introduced in (Neubauer, 2008a; Neubauer, 2008b). For a given α-individual b the heuristic function $\mathcal{G}$ of the SGA with α-selection is defined by a linear system of equations with system matrix A. The unique fixed point ω can be calculated analytically from the WALSH transformed system matrix $\widehat{A}$.

As is shown in this paper, the theoretical results obtained for the SGA with α-selection can be transferred to the SGA with generalised $\alpha^\star$-selection. In this selection scheme, the β best individuals of the current population instead of the single best α-individual mate with other individuals randomly chosen from the current population. Generalised $\alpha^\star$-selection with $\beta > 1$ represents a weaker selection scheme than α-selection in the sense that not only the best individual b is used as the first parent but the β best individuals $b_0, b_1, \ldots, b_{\beta-1}$ are allowed to reproduce as the first parent. For a given set $\{b_k\}_{0 \leq k \leq \beta-1}$ of β best individuals the heuristic function $\mathcal{G}$ of the SGA with generalised $\alpha^\star$-selection is also formulated by a linear system of equations with a suitably redefined system matrix A. As for the SGA with α-selection, the SGA with generalised $\alpha^\star$-selection allows to explicitly determine a simple coarse-grained system model for a schemata family defined by the ℓ-tuple ξ. The corresponding RHS is defined by the schema system matrix A_ξ with similar properties as the system matrix A.

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