INFORMATION-THEORETIC VIEW OF CONTROL

Prateep Roy, Arben Çela
Université Paris-Est, ESIEE, Département Systèmes Embarqués, Paris, France

Yskandar Hamam
F'SATIE-TUT, Pretoria, South Africa

Keywords: Information Theory, Shannon Entropy, Mutual-Information, Control, Bode Sensitivity Integral.

Abstract: In this paper we are presenting the information-theoretic explanation of Bodé Sensitivity Integral, a fundamental limitation of control theory, controllability grammian and the issues of control under communication constraints. As resource-economic use of information is of prime concern in communication-constrained control problems, we need to emphasize more on informational aspect which has got direct relation with uncertainties in terms of Shannon Entropy and Mutual Information. Bode Integral which is directly related to the disturbances can be correlated with the difference of entropies between the disturbance-input and measurable output of the system. These disturbances due to communication channel-induced noises and limited bandwidth are causing the information packet-loss and delays resulting in degradation of control performances.

1 INTRODUCTION

In recent years, there has been an increased interest for the fundamental limitations in feedback control. Bode's sensitivity integral (Bode Integral, in short) is a well-known formula that quantifies some of the limitations in feedback control for linear time-invariant systems. In (Sandberg and Bernhardsson, 2005), it is shown that there is a similar formula for linear time-periodic systems.

In this paper, we focus on Bode integral of control theory and Shannon Entropy of information theory because the latter is a stronger metric for uncertainty which hinders control of a system.

It has been known that control theory and information theory share a common background as both theories study signals and dynamical systems in general. One way to describe their difference is that the focal point of information theory is the signals involved in systems while control theory focuses more on systems which represent the relation between the input and output signals. Thus, in a certain sense, we may expect that they have a complementary relation. For this reason, many researchers have consecrated studies on the interactions of the two theories : Control Theory and Information Theory.

In networked control systems, there are issues related to both control and communication since communication channels with data losses, time delays, and quantization errors are employed between the plants and controllers (Antsaklis and Baillieul, 2007). To guarantee the overall control performance in such systems, it is important to evaluate the quantity of information that the channels can transfer. Thus, for the analysis of networked control systems, information theoretic approaches are especially useful, and notions and results from this theory can be applied. The results in (Nair and Evans, 2004) and (Tatikonda and Mitter, 2004) show the limitation in the communication rate for the existence of controllers, encoders, and decoders to stabilize discrete-time linear feedback systems.

The focus of information theory is more on the signals and not on their input-output relation. Thus, based on information theoretic approaches, we may expect to extend prior results in control theory. One such result can be found in (Martins et al., 2007), where a sensitivity property is analyzed and Bode's integral formula (Bode H., 1945) is extended to a more general class of systems. A fundamental limitation of sensitivity functions is presented in relation to the unstable poles of the plants.

Roy P., ÃĞela A. and Hamam Y.
INFORMATION-THEORETIC VIEW OF CONTROL.
DOI: 10.5220/0002166600050012
In *Proceedings of the 6th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2009)*, page
ISBN: 978-989-674-001-6

2 PROBLEM FORMULATION

Networked control systems suffer from the drawbacks of packet losses, delays and quantization in particular. These cause degradation of control performances and under some conditions instability. Uncertainties due to packet losses, delays, quantization, communication channel induced noises etc. have a great influence on the system systems performance. If we consider only the uncertainties induced by channel noise and quantization we may write:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t); \qquad (1)$$
$$y(t) = Cx(t) + Du(t) + v(t);$$

where $A \in R^{n\times n}$ is the system or plant matrix and $B \in R^{n\times q}$ is the control or input matrix. Also, $x(t)$ is the state, $u(t)$ is the control input, $y(t)$ is the output, C is the output or measurement matrix, D is the Direct Feed matrix, $w(t)$ and $v(t)$ are the external disturbances and noises of Gaussian nature respectively. Our aim is to achieve better control performance of system by tackling these uncertainties using Shannon's Mutual-Information, Information-Theoretic Entropy and Bode Sensitivity. We present the information-theoretic model of such uncertainties and their possible reduction using information measures.

3 PRELIMINARIES

By means of the connection between Bode integral and the entropy cost function, paper (Iglesias, 2001) provided a time-domain characterization of Bode integral. The traditional frequency domain interpretation is that, if the sensitivity of a closed-loop system is decreased over a particular frequency range typically the low frequencies the designer "pays" for this in increased sensitivity outside this frequency range. This interpretation is also valid for the time-domain characterization presented in (Iglesias, 2001) provided one deals with time horizons rather than frequency ranges. Time-domain characterization of Bode's integral shows how the frequency domain trade-offs translate into the time-domain. Under the usual connection between the time and frequency domains: low (high) frequency signals are associated with long (short) time horizons. In Bode's result, it is important to differentiate between the stable poles, which do not contribute to the Bode sensitivity integral and the unstable poles, which do. Time-varying systems which can be decomposed into stable and unstable components are said to possess an exponential dichotomy. What the exponential dichotomy says is that the norm of the projection onto the stable subspace of any orbit in the system decays exponentially as $t \to \infty$ and the norm of the projection onto the unstable subspace of any orbit decays exponentially as $t \to -\infty$, and furthermore that the stable and unstable subspaces are conjugate. The existence of an exponential dichotomy allows us to define a stability preserving state space transformation (a Lyapunov transformation) that separates the stable and unstable parts of the system.

3.1 Mutual Information

Shannon's Mutual information is just the information carried by one random variable about the other. It is a quantity in the time domain. Mutual Information $I(X;Y)$, between X as the input variable and Y as the output variable, has the lower and upper bounds given by the following:

$$R(D) = RateDistortion = MinI(X;Y) \qquad (2)$$

$$C = CommunicationChannelCapacity = MaxI(X;Y) \qquad (3)$$

where D is the distortion which happens when information is compressed (i.e. fewer bits are used to represent or code more frequent or redundant informations) and entropy is the limit to this compression i.e. if one compresses the information beyond the entropy limit there is a high probability that the information will be distorted or erroneous. This is as per Shannon's Source Coding Theorem. We code more frequently used symbols with fewer number of bits and vice-versa.

Mutual information is also the difference of entropies, where entropy is nothing but the measure of uncertainty. Just as entropy (Middleton, 1960) in physical systems tends to increase in the course of time, the reverse is true for information about an information source : as information about the source is processed, it tends to decrease with time, becoming more corrupt or noisy until it is evidently destroyed unless additional information is made available. Here, information refers to the case of desired messages.

3.2 Shannon Entropy

Shannon proposed a measure of uncertainty in a discrete distribution based on the Boltzmann entropy of classical statistical mechanics. He called it the entropy and defined as follows.

We have to take into account the statistics of the alternatives by replacing our original measure of the number of alternatives by the more general expression defining the entropy as follows:

$$H = -\sum_i p_i \log_2 (p_i) \qquad (4)$$

where p_i is the probability of the alternative i. The above quantity is known as the binary entropy in *bits* as we use logarithmic base of 2 (with logarithmic base e the entropy is in *nats*), and was shown by Shannon to correspond to the minimum average number of bits needed to encode a probabilistic source of N states distributed with probability p_i. Intuitively, H can also be considered as a measure of uncertainty : it is minimum, and is equal to zero, when one of the alternatives appears with probability one, whereas it is maximum and equals to $\log_2 N$ when all the alternatives are equiprobable so that $p_i = \frac{1}{N}$ for all i.

The term entropy is associated with the uncertainty or randomness whereas information is used to reduce this uncertainty. Uncertainty is the main hindrance to control and if we can reduce the uncertainty by getting the relevant information and utilizing the information properly so as to achieve the desired control performance of the system. Many researchers have posed the same question: *How much information is required for controlling the system based on observed informations in the case where these informations are passed through communication channels in a networked based system?*

Mutual Information $I(X;Y)$ and Entropies $H(X)$, $H(Y)$ and joint entropy $H(X,Y)$ are related as :

$I(X;Y) = H(X) + H(Y) - H(X,Y)$

where $H(X)$ is the uncertainty that X has about Y, $H(Y)$ is the uncertainty that Y has about X , and $H(X,Y)$ is the uncertainty that X and Y hold in common. Information value degrades over time and entropy value increases over time in general. The conditional version of the chain rule (Cover and Thomas, 2006) :

$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$; valid for any random variables X and Y.

Mutual information $I(X;Y)$ is the amount of uncertainty in X, minus the amount of uncertainty in X which remains after Y is known", which is equivalent to "the amount of uncertainty in X which is removed by knowing Y". This corroborates the intuitive meaning of mutual information as the amount of information (that is, reduction in uncertainty) that each variable is having about the other.

The conditional entropy $H(X|Y)$ or read as conditional entropy of X knowing Y or conditioned on Y, is often interpreted in communication theory as representing an information-loss (the so-called equivocation of Shannon (Shannon, 1948)), which results from subtracting the maximum noiseless capacity $I(X;X) = H(X)$ of a communication channel with input X and output Y from the actual capacity of that channel as measured by $I(X;Y)$.

3.3 Bode Integral

Physically an intrinsically stable system needs no information on its internal state or the environment to assure its stability. So, if we consider a well designed stable feedback control system with disturbances or/and noises as inputs and performance signals as outputs then it not needed to have extra feedback loop to assure its stability. We may say the same thing for systems which are intrinsically open-loop stable. For example, a pendulum with non-zero friction coefficient subject to a perturbation will return back to the equilibrium position after a transient period without any need of extra information. For unstable systems the mutual information between the initial state and the output of the system is related to its unstable poles.

The simplest (and perhaps the best known) result is that, for an open loop stable plant, the integral of the logarithm of the closed loop sensitivity is zero; i.e.

$$\int_0^\infty \ln |S_0(j\omega)| \, d\omega = 0$$

Where, S_0 and ω being the sensitivity function and frequency respectively.

Now, we know that the logarithm function has the property that it is negative if $|S_0| < 1$ and it is positive if $|S_0| > 1$. The above result implies that set of frequencies over which sensitivity reduction occurs (i.e. where $|S_0| < 1$) must be matched by a set of frequencies over which sensitivity magnification occurs (i.e. where $|S_0| > 1$). For a stable rational transfer function $L(j\omega)$, sensitivity is defined as $S(j\omega) = \frac{1}{1+L(j\omega)}$. This has been given a nice interpretation as thinking of sensitivity as a pile of dirt. If we remove dirt from one set of frequencies, then it piles up at other frequencies. Hence, if one designs a controller to have low sensitivity in a particular frequency range, then the sensitivity will necessarily increase at other frequencies - a consequence of the weighted integral always being a constant; this phenomenon has also been called the Water-Bed Effect (pushing down on the water bed in one area, raises it somewhere else).

For linear systems Bode Integral is the difference in the entropy rates between the input and output of the systems which is an information-theoretic interpretation. For nonlinear system (if the open loop system is globally exponentially stable and has fading memory) this difference is zero. Fading Memory Requirement is used to limit the contributions of the past values of the input on the output. Entropy of the signals in the feedback loop help provide another interpretation of the Bode integral formula (Zang and Iglesias, 2003)(Mehta et al., 2006) as follows. Shannon En-

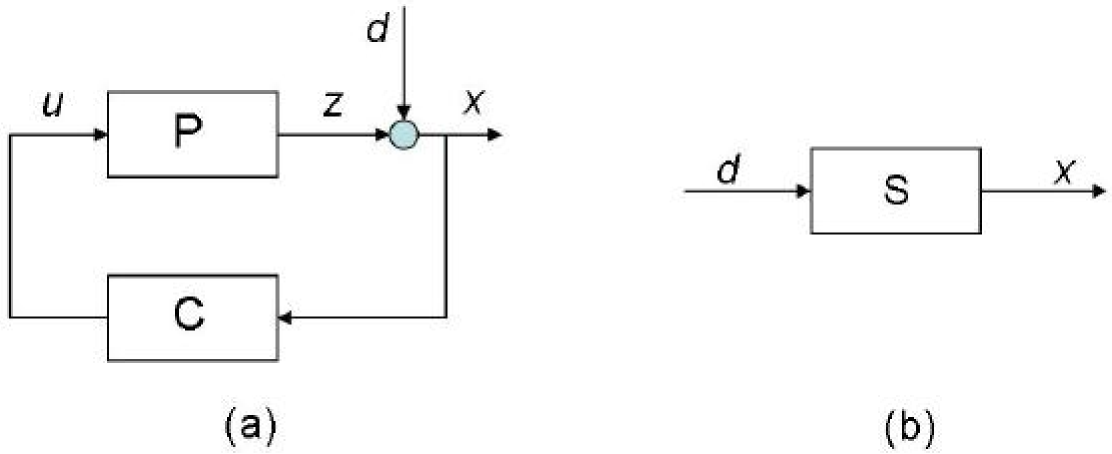

Figure 1: (a) Feedback loop and (b) Sensitivity function.

tropy - Bode Integral Relation can be rewritten as :

$$H_c(x) - H_c(d) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|S\left(e^{j\omega}\right)\right| d\omega = \sum_k \log\left(p_k\right) \tag{5}$$

Where $S\left(e^{j\omega}\right)$ is the transfer function of the feedback loop from the disturbance d to output x and p_k 's are unstable poles ($|p_k| > 1$) of the open-loop plant; S is referred to as the sensitivity function for an open-loop plant gain P and a stabilizing feedback controller gain C, S is given by $S = \frac{1}{1+PC}$. Sensitivity shows how much sensitive is the observable output state to input disturbance. Here, $H_c(x)$ and $H_c(d)$ denote the conditional entropy of the random processes associated with the output x and disturbance d respectively as per Figure1(Mehta et al., 2006).

Consider a random variable $x \in \Re^m$ of continuous type with entropy associated with this is given by

$H(x) := -\int_{\Re^m} p(x)\ln p(x)dx$;

where $p(x)$ being the probability density function of x and the conditional entropy of order n is defined as

$H(x_k|x_{k-1},\ldots,x_{k-n}) := -\int_{\Re^m} p(.)\ln p(.)dx$

where $p(.) = (x_k|x_{k-1},\ldots,x_{k-n})$.

This conditional entropy is a measure of the uncertainty about the value of x at time k under the assumption that its n most recent values have been observed. By letting n going to infinity, the conditional entropy of x_k is defined as

$H_c(x_k) := \lim_{n\to\infty} H(x_k|x_{k-1},\ldots,x_{k-n})$ assuming the limit exists. Thus the conditional entropy is a measure of the uncertainty about the value of x at time k under the assumption that its entire past is observed. Difference of conditional entropies between the output and input is nothing but the Bode sensitivity integral which equals the summation of logarithms of unstable poles.

For a stationary Markov process, conditional entropy (Cover and Thomas, 2006) is given by

$H(x_k|x_{k-1},\ldots,x_{k-n}) = H(x_k|x_{k-1})$.

4 RELATED WORK

It is well known that the sensitivity and complementary sensitivity functions represent basic properties of feedback systems such as disturbance attenuation, sensor-noise reduction, and robustness against uncertainties in the plant model. Researchers have worked earlier on the issues of relating the entropy and the Bode Integral and complementary sensitivity. Refer to work in (Sandberg and Bernhardsson, 2005), (Martins et al., 2007), (Bode H., 1945), (Freudenberg and Looze, 1988), (Zang, 2004), (Iglesias, 2002), (Iglesias, 2001), (Zang and Iglesias, 2003), (Mehta et al., 2006), (Sung and Hara, 1989), (Sung and Hara, 1988), (Okano et al., 2008), (Jialing Liu, 2006). In (Iglesias, 2002) the sensitivity integral is interpreted as an entropy integral in the time domain, i.e., no frequency-domain representation is used.

One has to gather relevant information, transmit the information to the relevant agent, process the information, if needed, and then use the information to control the system. The fundamental limitation in information transmission is the achievable information rate (i.e. a fundamental parameter of Information Theory), the fundamental limitation in information processing is the Cramer-Rao Bound (*CRB*) which deals with Fisher Information Matrix (*FIM*) in Estimation Theory, and the fundamental limitation in information utilization is the *Bode Integral* (i.e. a fundamental parameter of Control Theory), seemingly different and usually separately treated, are in fact three sides of the same entity as per the paper (Liu and Elia, 2006). Even Kalman et al. in their paper (Kalman et al., 1963) have stated that Controllability Grammian Matrix **W**-matrix is analogous to *FIM* and the determinant $\det \mathbf{W}$ is analogous to Shannon Information. These research work motivated us to investigate some important correlations amongst mutual information, entropy and design control parameters of practical importance rather than just concentrating on stability issues.

5 INFORMATION INDUCED BY CONTROLLABILITY GRAMMIAN

In general, from the viewpoint of the open-loop system, when the system is unstable, the system amplifies the initial state at a level depending on the size of the unstable poles (Okano et al., 2008). Hence, we can say that in systems having more unstable dynamics, the signals contain more information about the initial state. Using this extra information (in terms of mutual information between the control input and the initial state) we can reduce the entropy (uncertainty) and thus rendering more easy the observation of initial state.

Suppose that we have a feedback control system in which control signal is sent through a network with limited bandwidth. We will consider the case where the state of the system is measurable and the controller can send the state of the system without error. Under these conditions we may write:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu^*(t); \qquad (6) \\ u^*(t) &= -K_c x(t) + u^e(t); \end{aligned}$$

where K_c, $u^*(t)$ and $u^e(t)$, represent, respectively, the feedback controller gain, the applied control input and control error due to quantization noise of limited bandwidth network. In the sequel we are supposing that the control signal errors are caused by Gaussian White Noise which may be given by $u_i^e(t) = \sqrt{D_i}\delta(t)$. So we may write :

$$\begin{aligned} \dot{x}(t) &= (A - BK_c)x(t) + Bu^e(t); \qquad (7) \\ u^*(t) &= -K_c x(t) + u^e(t); \end{aligned}$$

or more compactly:

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + Bu^e(t); \qquad (8) \\ where, A - BK_c &= A_c. \\ u^*(t) &= -K_c x(t) + u^e(t); \end{aligned}$$

The feedback system (8) is a stable one which is perturbed by quantization errors or noises due to the bandwidth limitation.

Lemma : The controllability grammian matrix **W** of system (8) is related with the information-theoretic entropy H as follows (Mitra, 1969):

$$H(x,t) = \frac{1}{2}\ln\{\det \mathbf{W}(\mathbf{D},t)\} + \frac{n}{2}(1 + \ln 2\pi) \qquad (9)$$

(Where **D** being the Diagonal Matrix (positive definite symmetric matrix) with D_i being the ith diagonal element. Here unit impulse inputs are considered.)

= Average *apriori* uncertainty of the state x at time t for an order n of the system.

where

$$\mathbf{W}(\mathbf{D},\tau) = \int_0^{\tau} e^{A_c t} B \mathbf{D} B^T e^{A_c^T t} dt$$

for a system modeled as (8).

Proof of Eqn.(9): Referring to (Cover and Thomas, 2006) we are providing the proof. The input of (8) being Gaussian White Noise, the state of the system is with probability density having mean-value $\bar{x}(t) = e^{A_c t}x(0)$ and Covariance Matrix Σ at time t is given by

$$\Sigma = E\left\{(x - \bar{x})(x - \bar{x})^T\right\} = \mathbf{W}(\mathbf{D},t).$$

In a more detailed form :

$$x(t) = e^{A_c t}x(0) + \int_0^t e^{A_c t} Bu(t)dt$$

$$E\{x(t)\} = E\left\{e^{A_c t}x(0) + \int_0^t e^{A_c t} Bu(t)dt\right\}$$

$$Therefore, \bar{x}(t) = e^{A_c t}x(0)$$

where $\bar{x}(t)$ denotes the mean value of $x(t)$ and Covariance Matrix

$$\Sigma = E\left\{(x - \bar{x})(x - \bar{x})^T\right\}$$

$$\Rightarrow \Sigma = E[\left\{e^{A_c t}x(0) + \int_0^{\tau} e^{A_c t} Bu(t)dt - e^{A_c t}x(0)\right\} \left\{e^{A_c t}x(0) + \int_0^{\tau} e^{A_c t} Bu(t)dt - e^{A_c t}x(0)\right\}^T]$$

Therefore, $\Sigma = \int_0^\tau e^{A_c t} B\mathbf{D}B^T e^{A_c^T t} dt = \mathbf{W}(\mathbf{D},\tau)$.

where, $u_i(t) = \sqrt{D_i}\delta(t)$ i.e. weighted impulses and $\mathbf{D}$ being the Diagonal Matrix (positive definite symmetric matrix) with D_i being the ith diagonal element. Here unit impulse inputs are considered.

$$p(x,t) = \frac{1}{(2\pi)^{n/2}\{\det \mathbf{W}(\mathbf{D},t)\}^{1/2}} e^{[-1/2\left\{(x-\bar{x})^T \mathbf{W}^{-1}(\mathbf{D},t)(x-\bar{x})\right\}]} \tag{10}$$

Now, for multidimensional continuous case, entropy (precisely *differential entropy*) of a continuous random variable X with probability density function $f(x)$ (if $\int_{-\infty}^{\infty} f(x)dx = 1$) is defined (Cover and Thomas, 2006) as
Differential Entropy $h(X) = -\int_S f(x)\ln f(x)dx$;
where the set S for which $f(x) > 0$ is called the *support set* of X.
As in discrete case, the differential entropy depends only on the probability density of the random variable and therefore the differential entropy is sometimes written as $h(f)$ rather than $h(X)$. Here, we call differential entropy as $H(x,t)$ and $f(x)$ as $p(x,t)$ which are correlated as

$$H(x,t) = -\int p(x,t)\ln p(x,t)dx \tag{11}$$

Using equation (10) in equation (11) we get

$$H(x,t) = -\int p(x,t)[-\tfrac{1}{2}(x-\bar{x})^T \mathbf{W}^{-1}(\mathbf{D},t)(x-\bar{x}) - \ln\left\{(2\pi)^{n/2}\{\det \mathbf{W}(\mathbf{D},t)\}^{1/2}\right\}]dx$$

$$H(x,t) = \tfrac{1}{2}E\left[\sum_{i,j}\{(X_i-\bar{X}_i)(\mathbf{W}^{-1}(\mathbf{D},t))_{ij}(X_j-\bar{X}_j)\}\right] + \tfrac{1}{2}\ln[\{(2\pi)^n\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{1}{2}E\left[\sum_{i,j}\{(X_i-\bar{X}_i)(X_j-\bar{X}_j)(\mathbf{W}^{-1}(\mathbf{D},t))_{ij}\}\right] + \tfrac{1}{2}\ln[\{(2\pi)^n\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{1}{2}\sum_{i,j}[E\{(X_j-\bar{X}_j)(X_i-\bar{X}_i)\}(\mathbf{W}^{-1}(\mathbf{D},t))_{ij}] + \tfrac{1}{2}\ln[\{(2\pi)^n\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{1}{2}\sum_j\sum_i(\mathbf{W}(\mathbf{D},t))_{ji}(\mathbf{W}^{-1}(\mathbf{D},t))_{ij} + \tfrac{1}{2}\ln[\{(2\pi)^n\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{1}{2}\sum_j\{(\mathbf{W}(\mathbf{D},t))(\mathbf{W}^{-1}(\mathbf{D},t))\}_{jj} + \tfrac{1}{2}\ln[\{(2\pi)^n\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{1}{2}\sum_j \mathbf{I}_{jj} + \tfrac{1}{2}\ln[\{(2\pi)^n\}\{\det \mathbf{W}(\mathbf{D},t)\}]$$

(Where $\mathbf{I}_{jj}$ is the Identity Matrix)

$$= \tfrac{n}{2} + \tfrac{1}{2}\ln[\{(2\pi)^n\}\{\det \mathbf{W}(\mathbf{D},t)\}]$$

$$= \tfrac{n}{2} + \tfrac{1}{2}\ln\{(2\pi)^n\} + \tfrac{1}{2}\ln\{\det \mathbf{W}(\mathbf{D},t)\}$$

$$= \tfrac{n}{2} + \tfrac{n}{2}\ln\{(2\pi)\} + \tfrac{1}{2}\ln\{\det \mathbf{W}(\mathbf{D},t)\}$$

$$H(x,t) = \tfrac{1}{2}\ln\{\det \mathbf{W}(\mathbf{D},t)\} + \tfrac{n}{2}(1+\ln 2\pi)$$

Since Controllability Grammian is independent of co-ordinate system and so is the Mutual Information, we try to draw the analogy between the two. Based on the equation (9) we can write the entropy reduction as

$$\Delta H(x,t) = \tfrac{1}{2}\Delta[\ln\{\det \mathbf{W}(\mathbf{D},t)\}]$$

This shows that the entropy reduction which is same as uncertainty reduction is dependent on Controllability Grammian only. Other term being constant for constant *n*, gets canceled.
Therefore, $\Delta H(x,t) = H(x(t_1),t_1) - H(x(t_2),t_2)$

$$= \tfrac{1}{2}\ln\{\det \mathbf{W}_1(\mathbf{D}_1,t_1)\} - \tfrac{1}{2}\ln\{\det \mathbf{W}_2(\mathbf{D}_2,t_2)\}$$

$$\Rightarrow \Delta H(x,t) = \frac{1}{2}\ln\left\{\frac{\det \mathbf{W}_1(\mathbf{D}_1,t_1)}{\det \mathbf{W}_2(\mathbf{D}_2,t_2)}\right\} \tag{12}$$

For simplicity we denote $\Delta H(x,t)$ by ΔH, $\mathbf{W}_1(\mathbf{D}_1,t_1)$ by $\mathbf{W}_1$ and $\mathbf{W}_2(\mathbf{D}_2,t_2)$ by $\mathbf{W}_2$.

Therefore, $\Delta H = \frac{1}{2}\ln\left\{\det(\frac{\mathbf{W}_1}{\mathbf{W}_2})\right\}$

$\Rightarrow \det(\frac{\mathbf{W}_1}{\mathbf{W}_2}) = e^{2(\Delta H)}$

Using the above expression along with the concept of mutual information being the difference of the entropy and the residual conditional entropy i.e. $I(X;U) = H(X) - H(X|U)$ (gain in information is reduction in entropy), we can conclude that Mutual Information $I(X;U)$ between the state X and control input U denoted simply by Shannon Information I_{sh} is given by this ΔH which can be expressed further as

Finally,

$$\det(\frac{\mathbf{W}_1}{\mathbf{W}_2}) = e^{2(\Delta H)} = e^{2I_{sh}} \tag{13}$$

We may conclude that the uncertainty reduction which is directly related to the $\Delta H(x,t)$ will reduce the variance of the state with respect to the steady-state if $\Delta H(x,t)$ converges to zero. The only influence we have on the control signal is related to that of feedback gain, to be chosen such that the norm of grammians, represented by $\det(W(D_i,t))$ converge rapidly to their norm to infinity $\det(W(D_\infty,\infty))$. We will detail the related approach in a future paper.

6 CONCLUSIONS

This paper has addressed some new ideas concerning the relation between control design and information theory. Since the networked control system has communication constraints due to limited bandwidth or noises, we must have to adopt a policy of resource allocation which enhances the information transmitted. This may be done possible if we know the characteristics of the networks, the bandwidth constraints and that of the dynamical system under study.

As demonstrated the grammian of controllability constitute a metric of information theoretic entropy with respect to the noises induced by quantization. Reduction of these noises is equivalent to the design methods proposing a reduction of the controllability grammian norm. In the case of bandwidth constraints it takes its full interest which will be demonstrated in a future paper. Future work in this direction would be also to propose an information-theoretic analysis for enhancing the zooming algorithm proposed (Ben Gaid and Çela, 2006) and optimal allocation of communication bandwidth which maximizes the systems' performances based on Controllability Grammians. Illustration of these results by simulation and / or experimental verification of the theoretical approaches is the objective of our work.

REFERENCES

Sandberg H. and Bernhardsson B.(2005), A Bode Sensitivity Integral for Linear Time-Periodic Systems, *IEEE Trans. Autom. Control*, Vol. 50, No. 12, December.

Antsaklis P. and Baillieul J. (2007), Special Issue on the Technology of Networked Control Systems, *Proc. of the IEEE*, Vol. 95, No. 1.

Nair G. and Evans R.(2004), Stabilizability of stochastic linear systems with finite feedback data rates, *SIAM J. Control Optim.*, Vol. 43, No. 2, p. 413-436.

Tatikonda S. and Mitter S.(2004), Control under communication constraints, *IEEE Trans. Autom. Control*, Vol. 49, No. 7, p. 1056-1068.

Martins N., Dahleh M., and Doyle J.(2007), Fundamental limitations of disturbance attenuation in the presence of side information, *IEEE Trans. Autom. Control*, Vol. 52, No. 1, p. 56-66.

Bode H.(1945), *Network Analysis and Feedback Amplifier Design*, D. Van Nostrand, 1945.

Iglesias P.(2001), Trade-offs in linear time-varying systems : An analogue of Bode's sensitivity integral, *Automatica*, Vol. 37, No. 10, p. 1541-1550.

Middleton D.(1960), *An Introduction to Statistical Communication Theory*, McGraw-Hill Pub., p. 315.

Cover and Thomas(2006), *Elements of Information Theory*, 2nd Ed., John Wiley and Sons.

Shannon C.(1948), The Mathematical Theory of Communication, *Bell Systems Tech. J.* Vol. 27, p. 379-423, p. 623-656, July, October.

Zang G. and Iglesias P.(2003), Nonlinear extension of Bode's integral based on an information theoretic interpretation, *Systems and Control Letters*, Vol. 50, p. 1129.

Mehta P., Vaidya U. and Banaszuk A.(2006), Markov Chains, Entropy, and Fundamental Limitations in Nonlinear Stabilization, *Proc. of the 45th IEEE Conference on Decision & Control*, San Diego, USA, December 13-15.

Freudenberg J. and Looze D.(1988), Frequency Domain Properties of Scalar and Multivariable Feedback Systems, *Lecture Notes in Control and Information Sciences*, Vol. 104, Springer-Verlag, New York.

Zang G.(2004), Bode's integral extensions in linear time-varying and nonlinear systems, *Ph.D. Dissertation*, Dept. Elect. Comput. Eng., Johns Hopkins Univ., USA.

Iglesias P.(2002), Logarithmic integrals and system dynamics : An analogue of Bode's sensitivity integral for continuous-time time-varying systems, *Linear Alg. Appl.*, Vol. 343-344, p. 451-471.

Sung H. and Hara S.(1989), Properties of complementary sensitivity function in SISO digital control systems, *Int. J. Control*, Vol.50, No. 4, p. 1283-1295.

Sung H. and Hara S.(1988), Properties of sensitivity and complementary sensitivity functions in single-input single-output digital control systems, *Int. J. Control*, Vol. 48, No. 6, p. 2429-2439.

Okano K, Hara S. and Ishii H.(2008), Characterization of a complementary sensitivity property in feedback control: An information theoretic approach, *Proc. of the 17th IFAC-World Congress*, Seoul, Korea, July 6-11, p. 5185-5190.

Jialing L.(2006), Fundamental Limits in Gaussian Channels with Feedback: Confluence of Communication, Estimation and Control, *Ph.D. Thesis*, Iowa State University.

Liu J. and Elia N. (2006), Convergence of Fundamental Limitations in Information, Estimation, and Control, *Proceedings of the 45th IEEE Conference on Decision & Control*, San Diego, USA, December.

Kalman R., Ho Y. and Narendra K.(1963), Controllability of linear dynamical systems, *Contributions to Differential Equations*, Vol. 1, No. 2, p. 189-213.

Mitra D.(1969), W-matrix and the Geometry of Model Equivalence and Reduction, *Proc. of the IEE*, Vol. 116, No.6, June, p. 1101-1106.

Ben Gaid M. M., and Çela A.(2006), Trading Quantization Precision for Sampling Rates in Networked Systems with Limited Communication, *Proceedings of the 45th IEEE Conference on Decision & Control*, San Diego, CA, USA, December 13-15.