PROOF GRANULARITY AS AN EMPIRICAL PROBLEM?*

Marvin Schiller
German Research Center for Artificial Intelligence (DFKI), Bremen, Germany

Christoph Benzmüller
International University in Germany, Bruchsal, Germany
Articulate Software, Angwin, CA, U.S.

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Abstract: Even in introductory textbooks on mathematical proof, intermediate proof steps are generally skipped when this seems appropriate. This gives rise to different granularities of proofs, depending on the intended audience and the context in which the proof is presented. We have developed a mechanism to classify whether proof steps of different sizes are appropriate in a tutoring context. The necessary knowledge is learnt from expert tutors via standard machine learning techniques from annotated examples. We discuss the ongoing evaluation of our approach via empirical studies.

1 INTRODUCTION

Our overall motivation is the development of an intelligent tutoring system for mathematics. Our particular interest is in flexible, adaptive mathematical proof tutoring. In order to make progress in this area it is important to reduce the gap between the existing formal domain-reasoning techniques and common mathematical practice. In particular, the step size (granularity) of reasoning employed in proof assistants and automated theorem provers often does not match the step size of human-generated proofs. This hampers their usability within a mathematical tutoring environment. For example, when the theorem prover Otter (McCune, 2003) was used in the EPGY learning environment for checking student-generated proof steps, it sometimes verified seemingly large student steps easily, whereas other, seemingly trivial steps were not verified within an appropriate resource limit (McMath et al., 2001). This criticism applies foremost to machine-oriented theorem proving systems, for example, systems based on fine-grained resolution or tableaux calculi. Techniques and calculi that are apparently better suited in this context include, for example, tactical theorem proving (Gordon et al., 1979), hierarchical proof planning (Bundy et al., 1991; Melis, 1999), assertion level theorem proving (Huang, 1994; Autexier, 2005), (super-)natural deduction (Wack, 2005), and strategic proof search (Sieg, 2007). Such techniques reduce the amount of unnecessary technical details in the generated proofs, support the application of sequences of inference steps as tactics or even the direct application of entire lemmas in single inference steps (assertion level theorem proving). However, the question remains how large a proof step shall or may be within a tutoring context (comprising a particular proof problem, the didactic goal, and the (assumed) prior knowledge of the student) and how many and which intermediate reasoning steps may be performed implicitly.

To investigate this issue we have analyzed a corpus – the DIALOG corpus (Benzmüller et al., 2006) – of tutorial dialogs on proofs. This corpus has been collected in experiments in the DIALOG project (Benzmüller et al., 2007). Exploiting assertion level proof search (where each inference is justified by a mathematical fact, such as a definition, theorem or a lemma) proofs in this corpus have been reconstructed and represented formally in the mathematical assistant system ΩMEGA (Siekmann et al., 2006)[2]. The analyzed students' proof steps generally corre-

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[2]We did not attempt to model erroneous proof steps.

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1 We assume $(y,x) \in (R \circ S)$ and show $\underbrace{(y,x) \in S^{-1} \circ R^{-1}}_{\Gamma}$.
2 Therefore, $(x,y) \in R \circ S$
3 Therefore, $\exists z$ s.t. $(x,z) \in R \wedge (z,y) \in S$

4(a) Therefore, $\exists z$ s.t. $(z,x) \in R^{-1} \wedge (z,y) \in S$
Tutor: "too small"

$$\frac{\exists z : (z,x) \in R^{-1} \wedge (z,y) \in S \vdash \Gamma}{\exists z : (x,z) \in R \wedge (z,y) \in S \vdash \Gamma}\ Def.Inv.$$

4(b) Therefore, $\exists z$ s.t. $(z,x) \in R^{-1} \wedge (y,z) \in S^{-1}$
Tutor: "appropriate"

$$\frac{\dfrac{\exists z : (z,x) \in R^{-1} \wedge (y,z) \in S^{-1} \vdash \Gamma}{\exists z : (z,x) \in R^{-1} \wedge (z,y) \in S \vdash \Gamma}\ Def.Inv.}{\exists z : (x,z) \in R \wedge (z,y) \in S \vdash \Gamma}\ Def.Inv.$$

4(c) This finishes the current part of the proof.
Tutor: "too big"

$$\frac{\dfrac{\dfrac{\Gamma \vdash \Gamma(closed)}{\exists z : (z,x) \in R^{-1} \wedge (y,z) \in S^{-1} \vdash \Gamma}\ Def.\circ}{\exists z : (z,x) \in R^{-1} \wedge (z,y) \in S \vdash \Gamma}\ Def.Inv.}{\exists z : (x,z) \in R \wedge (z,y) \in S \vdash \Gamma}\ Def.Inv.$$

Figure 1: Proof sample (for the proof problem $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$, where $^{-1}$ denotes relation inverse and $\circ$ denotes relation composition) with three alternatives for the fourth step, together with tutor's granularity rating (taken from the ongoing experiments), and (partial) assertion level proof reconstruction (as sequent trees).

sponded to one, two or three assertion level proof steps and very seldomly to four or more. This provides evidence that the step size of assertion level proof comes quite close to the proof step sizes observed in the experiments. However, often combinations of single assertion applications are preferred – even in very elementary proofs.

An example (partial) proof is presented in Figure 1: the student has already performed steps (1)-(3) and three alternatives, 4(a)-4(c), for the next step are investigated. They have been annotated (cf. Section 5) by a mathematician concerning their step size. Below these alternatives we outline parts of the corresponding assertion level proofs. Note that the step consisting of only one assertion level inference has been annotated as too small. Here the different classes of step size coincide with different lengths of the associated assertion level reconstructions. Our hypothesis is that such a coincidence generally exists – clearly, not as simple as here – and that we can learn and represent it and exploit it for intelligent proof tutoring. To confirm this hypothesis we are currently performing an empirical study which we discuss in this paper.

In Section 2 we present our modeling technique for classifying the step size of proof steps in a tutoring context. Our approach uses data mining techniques to generate models from samples of proof steps which have been annotated by human experts. Our system can be used in the diagnosis of student steps (to detect whether a student proceeds with unusually small or unexpectedly large steps (Schiller et al., 2008)), or for the presentation of proofs at a particular level of detail (Schiller and Benzmüller, 2009). In order to facilitate and support the collection of annotated sample proofs, we have developed a dedicated, new system environment, which is motivated in Section 3 and presented in Section 4. We report on an ongoing empirical study using this new environment in Section 5. Key questions of this ongoing study are: (i) How well can we model the judgments of the expert? (ii) How much do judgments differ among various experts? (iii) How well do learned models transfer to other domains? (iv) What are (empirically) relevant properties for classifying the step size of proof steps? We present a summarizing discussion of our approach in Section 6.

2 GRANULARITY AS A CLASSIFICATION PROBLEM

As the basis of our approach to granularity, we hypothesize what properties of compound proof steps[3] may be relevant to judge about their perceived step size. As a result of reviewing our DIALOG corpus of proofs (Benzmüller et al., 2006), we identified the following key features (among others):

- How many different concepts (mathematical facts, such as a particular definition or theorem) are involved within the same compound step? (feature *concepts*)
- How many (assertion level) inferences does a compound step correspond to? (feature *total*)
- Are the employed concepts mentioned explicitly? (feature *verb*)
- Is the student familiar with the employed concepts of the compound step? (feature *mastered/unmastered*)
- What theories do the employed concepts belong to (e.g. naive set theory, algebra, topology)? (features *settheory, algebra, topology, etc...*)

[3]We use the notion *compound proof step* in the following for any proof step, including those that can be decomposed into the application of several individual inferences - which is not the case for (atomic) proof steps such as single natural deduction inference applications.

1. total $\in \{0,1,2\} \Rightarrow$ "appropriate"
2. unmastered $\in \{2,3,4\} \wedge$ relations $\in \{2,3,4\} \Rightarrow$ "step-too-big"
3. total $\in \{3,4\} \wedge$ relations $\in \{0,1\} \Rightarrow$ "step-too-big"
4. unmastered $\in \{0,1\} \Rightarrow$ "appropriate"
5. _ $\Rightarrow$ "appropriate"

Figure 2: Sample rule set generated using the data mining tool C5.0[5] on sample data. Rules are ordered by confidence for conflict resolution.

Respective feature observations can easily be extracted from assertion level proofs[4]. We also employ a simple student model to keep track of the concepts the student is (presumably) already familiar with.

We treat the decision whether a particular step is of appropriate granularity as a classification problem. Given the properties of a particular step (as a collection of its features) we use a classifier (a mapping of feature vectors to class labels) to assign one of the three labels *appropriate*, *step-too-big* and *step-too-small* to it. As classifiers, we use rule sets, which are learned from annotated samples. As an example of such a rule set, consider Figure 2. All our features are numeric, e.g., *unmastered* counts the number of concepts we assume the student is not yet familiar with, *total* counts the number of assertion level inference applications, etc. For example, the proof step 4 (b) in Figure 1, which results in a feature vector (concepts:2, total:1, verb:false,mastered:0,unmastered:1,relations:1, ...) is assigned the label *appropriate* via the rule set in Figure 2 (the first rule fires). Such rule sets can be generated from annotated samples via data mining tools, such as C5.0[5].

3 PREVIOUS EMPIRICAL DATA COLLECTION AND LESSONS LEARNED

The DIALOG corpus collected data from proof-tutoring dialogs. In these dialogs human tutors (mathematicians) were asked to judge the step size of each student proof step, resulting in a corpus with granularity annotations. This was then used for evaluating the classification approach outlined above. Using standard data-mining tools (e.g. C5.0 and Weka[6]), we generated classifiers from the data and estimated their performance (as reported in (Schiller et al., 2008)). However, it became apparent that for an in-depth study of granularity, more focused studies are needed since (i) both the students and the wizards were experimental subjects, and the resulting interactions were more geared towards the identification of specific phenomena rather than a controlled experiment, and (ii) both parties were allowed to use natural language freely, which resulted in a large variety of surface realizations of proof steps, often including comments and questions, which may have had an influence on the judgments of the tutors. Consider for example the dialog fragment:

Student: $(R \circ S)^{-1} = \{(x,y)|(y,x) \in R \circ S\} = \{(x,y)|\exists z(z \in M \wedge (x,z) \in R^{-1} \wedge (z,y) \in S^{-1})\} = R^{-1} \circ S^{-1}$. Can I do it like that?

Tutor: That's a little too fast. Where do you take the second equality from?

By adding the question to the equation, the student reveals uncertainty, which might have effected the tutor's judgment and reaction to some degree.

4 A SYSTEM ENVIRONMENT FOR EMPIRICAL PROOF GRANULARITY STUDIES

The idea of our new environment is to better control the parameters pertaining to the student, in order to more accurately observe their effects on the judgments of the tutor. Therefore, we simulate the student, using: (i) assertion-level proof search in ΩMEGA, (ii) pattern-based generation of simple natural-language output, (iii) randomization of proof step output (producing compound steps of random size, counting assertion level inferences, and randomizing whether concept names are explicitly named, or only the resulting formulae are displayed), (iv) automatic collection of all relevant data, including the proof step output, the names of the employed assertion level inferences, the corresponding granularity features, and the corresponding granularity judgments from the tutor.

The expert providing the granularity judgments uses the interface in Figure 4. It presents the proof step output and collects the expert's judgments. The expert may deny the judgment for a particular step, in which case a different option is presented. When combining several inference steps to a compound

[4]Even though the approach is generally not limited to assertion level proofs, we use this proof representation in our study for convenience.

[5]Data Mining Tools See5 and C5.0: `http://www.rulequest.com/see5-info.html`

[6]`http://www.cs.waikato.ac.nz/ml/weka/`

Proof Step Output	Inferences	Granularity Feature Vector	Judgment
We assume $(y,x) \in (R \circ S)^{-1}$ and show $(y,x) \in S^{-1} \circ R^{-1}$...because of definition of equality and definition of subset	Def.$\subseteq$, Def.=	hypintro:1, total:2, concepts:2, verb:1, ...	appropriate
Therefore, $(x,y) \in R \circ S$	Def. Inv.	hypintro:0, total:1, concepts:1, verb:0, ...	appropriate
Therefore, $\exists z$ s.t. $(x,z) \in R \wedge (z,y) \in S$ ...because of relation composition	Def. $\circ$	hypintro:0 ,total:1, concepts:1, verb:1, ...	appropriate

Figure 3: Sample of the data collected in our study.

step, only inference steps of the same direction (either forward, or backward) are combined, a phenomenon which we clearly observed in the DIALOG corpus.

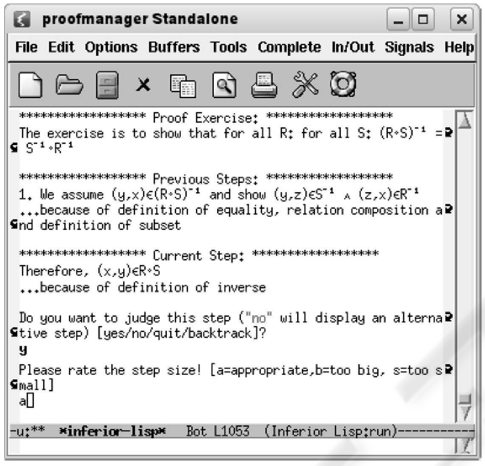

Figure 4: The data collection environment interface. Proofs are presented stepwise. For each step, the display reminds the user of the theorem to be proven and the previous steps in the proof. The user is requested to provide a granularity rating for the step under consideration.

The knowledge and mastery of the simulated student – relative to which the expert has to provide the granularity judgments – is determined by the formal representation of the proof exercise (including relevant definitions and lemmas) provided to ΩMEGA and corresponding entries in the student model. At the start of each exercise (and during the exercise on request), the expert is provided with a list of concepts the (simulated) student is supposed to know, and a list of concepts the student is supposed to learn. Figure 3 shows a sample of collected data.

5 AN EMPIRICAL STUDY ON GRANULARITY

Our approach to granularity relies on two assumptions which we investigate empirically:

- We assume that we need an adaptive approach to granularity, which learns from human experts. The experiments reported in (Benzmüller et al., 2006) hinted at the possibility that experts do not always agree with respect to what step size they consider appropriate. We want to compare samples from different experts with tutoring experience and examine the inter-rater reliability.
- We assume a set of features which we consider relevant for classifying granularity (currently around twenty features plus indicator features for each theory and each concept). Our goal is to evaluate (i) which features are most salient, and (ii) what features are potentially relevant?

Therefore, we perform an empirical study, where several mathematicians with tutoring experience judge proof steps presented to them via our data collection environment. Exercises are taken from the fields of naive set theory, relations (such as our running example), and topology. The recently conducted first experiment session, where a mathematician judged 135 proof steps using our environment, will be followed by sessions with two or three further experts, so that differences in their judgment can be examined. The mathematical experts are not instructed about assertion level proofs and the features we use in our classification before completion of the experiment, to avoid an artificial bias. Afterwards, we discuss our approach with the experts to obtain additional feedback. The annotated proof steps are then used to generate classifiers for granularity, and to evaluate their performance using data mining tools (also concerning the question which features of the proof steps are most useful for the classification task).

6 DISCUSSION

Granularity is a challenging topic in artificial intelligence and education, both from a theoretical viewpoint (e.g. (Hobbs, 1985; Keet, 2008)) but also in several applications, for example in the computer-assisted teaching of programming skills (Mccalla et al., 1992), or in the modeling of biological information systems (Keet, 2008).

In this paper, we have sketched a flexible, adaptive approach for modeling and assessing proof step granularity. It is based on the collection of empirical data from the observed behavior of expert tutors, which is then modeled via artificial intelligence and data mining techniques. These models for granularity can be generated independently of whether the experts are able to introspect or justify their judgments. The learnt classifiers serve to imitate the mathematical practice of the experts (pertaining to granularity) when used within an intelligent tutoring system. An alternative approach would be to establish an explicit best practice of judging proof step granularity by openly engaging tutoring experts in the discussion of the involved cognitive dimensions. It remains debatable which of the two approaches is more adequate for building a granularity-informed proof tutoring system, and we consider our work and our system environment as a fruitful first step in both directions.

Future work will address the questions raised in the introduction. Among other things this is dependent on the successful completion of our ongoing experiments.

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