PERFORMANCE ANALYSIS OF FSK MODULATION WITH LIMITER-DISCRIMINATOR-INTEGRATOR DETECTION OVER HOYT FADING CHANNELS

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Abstract: The focus of this paper is on the performance analysis of frequency shift keying (FSK) modulation with limiter-discriminator-integrator (LDI) detection over frequency-flat Hoyt (Nakagami-q) fading channels. Specifically, a closed-form expression is derived for the probability density function (PDF) of the phase difference of Hoyt faded FSK signals disturbed by additive white Gaussian noise (AWGN). This newly derived PDF is verified to reduce to known results corresponding to the Rayleigh fading channel as a special case of the Hoyt model. The validity of the expression is further demonstrated by simulation for the case of a Hoyt mobile-to-mobile (M2M) fading channel. The analytical PDF of the phase difference is then applied to determine the bit-error probability (BEP) of the LDI receiver taking into consideration the Doppler effects, the click noise as well as the inter-symbol interference (ISI) caused by the intermediate frequency (IF) pre-detection filter. Numerical examples, assuming a Hoyt M2M channel, are given to illustrate the analysis and examine the effects of the FM system parameters and the fading characteristics on the BEP performance.

1 INTRODUCTION

The Hoyt statistical distribution is a general short-term fading model which includes the one-sided Gaussian and the Rayleigh models as special cases (Nakagami, 1960). This multipath propagation model was originally introduced in (Nakagami, 1960) for the study of ionospheric scintillation. Recently, it has been shown in (Youssef et al, 2005) that the model is useful to accurately represent real world mobile satellite channels in heavy shadowing environments, and allows to describe more severe conditions of fading than does the classical Rayleigh distribution. Given the importance of the Hoyt model in statistical modeling of short-term fading, it is of interest to study and analyze its impact on the performance of wireless communication systems. To the best of authors knowledge, there are only few studies that have been reported on this topic so far. For instance, information outage probability of orthogonal space-time block codes over Hoyt fading channels has been investigated in (Ropokis et al, 2007). The BEP of narrow-band digital FSK modulation with LDI detection scheme has been studied in (Hajri and Youssef, 2007). There, the methodology employed relies on the results reported in (Tjhung et al, 1990), and consists on the determination, separately, of the PDF of the phase difference introduced by the receiver noise and that introduced by the Hoyt fading channel. Assuming the statistical independence between the two random phase processes, then the resultant phase PDF, needed for the evaluation of the BEP expression, is calculated by a convolution operation which demands tedious numerical integrations. In addition, the results of (Hajri and Youssef, 2007) are valid only for small values of signal-to-noise ratio.

In this paper, and to avoid the drawback of the method mentioned above, we approach the problem from a different point of view. We derive a closed-form expression for the PDF of the overall phase difference of a Hoyt faded FSK signal corrupted by additive noise. Thereafter, we apply this PDF for the calculation of the desired BEP of FSK systems drawing upon the classical work on error performance analysis of LDI receivers reported in (Pawula, 1981; Ng et al, 1994). Numerical examples are given for various values of the FM system parameters and channel characteristics to study their impact on the BEP performance.

The remainder of the paper is structured as follows. Section II contains a review of known results on the error performance analysis of LDI based digital

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FM receivers. In Section III, we address the derivation of the PDF of the phase difference for a Hoyt faded FSK signal contaminated by receiver noise. In Section IV, the newly derived PDF is applied for the determination of the desired BEP. Illustrations of numerical examples for the case of a Hoyt M2M channel are provided in Section V, and the conclusion is drawn in Section VI.

2 PRELIMINARIES

The link between the transmitter and the receiver is modeled by a narrow-band Hoyt multipath fading channel (Nakagami, 1960). The complex low-pass equivalent Hoyt faded FSK signal, present at the input of the IF pre-detection filter, can be expressed as

$$S_r(t) = (\mu_1(t) + j\mu_2(t)) \exp[j\theta(t)] + w(t) \quad (1)$$

where $\mu_1(t) + j\mu_2(t)$ is a zero-mean complex Gaussian process used to model the Hoyt fading gain (Nakagami, 1960). The variances of $\mu_1(t)$ and $\mu_2(t)$ will be denoted by σ_1^2 and σ_2^2, respectively. Also, $w(t)$ is a zero-mean complex AWGN, and $\theta(t)$ stands for the data phase after FM modulation which is given by

$$\theta(t) = \frac{\pi m}{T} \int_{-\infty}^{t} b(\tau) d\tau. \quad (2)$$

In (2), $b(t)$ is the binary data sequence of bit rate $1/T$, and m is the FSK modulation index. Concerning the IF band-pass filter, it is considered to be of a Gaussian shape with an equivalent low-pass transfer function given by

$$H(f) = \exp\left[-\pi f^2/2B^2\right] \quad (3)$$

where B is the equivalent noise bandwidth. Now, for the determination of the signal resulting at the output of the IF filter, we follow (Tjhung et al, 1990; Ng et al, 1994) by assuming that the Hoyt fading process changes at a rate that is much slower than the data rate $1/T$. This so called "quasi-static" analysis implies that the channel gain $\mu_1(t) + j\mu_2(t)$ is not affected by its passage over the IF filter. In this case, the output of the pre-detection filter can be written as

$$S_0(t) = (\mu_1(t) + j\mu_2(t))\, a(t) \exp[j\phi(t)] + n_1(t) + jn_2(t) \quad (4)$$

where $a(t)$ and $\phi(t)$ are the IF filtered carrier amplitude and information phase, respectively, while $n_1(t)$ and $n_2(t)$ stand for the quadrature components of the IF filtered complex AWGN $w(t)$. The processes $n_1(t)$ and $n_2(t)$ have the same variance σ_n^2 and a commun autocorrelation function (ACF) $\Gamma_n(\tau)$. The output signal of the LDI detector is the phase difference, over the bit time interval $[t-T, t]$, of the IF filtered FSK signal. This phase difference is given by (Pawula, 1981)

$$\Delta\psi = \Delta\phi + \Delta\Omega + 2\pi N(t-T, t) \quad (5)$$

where $\Delta\phi = \phi(t) - \phi(t-T)$ corresponds to the data phase difference, $\Delta\Omega = \Omega(t) - \Omega(t-T)$ is the phase difference introduced by both the fading channel and the additive noise, and $N(t-T,t)$ stands for the number of FM clicks occurring in the time interval $[t - T, t]$. From this and based on (Pawula, 1981), the probability of making an error, when a "+1" symbol is sent, is obtained by computing the quantity $Prob\,(\Delta\psi \leq 0)$, where $Prob\,(\cdot)$ stands for probability, according to

$$Prob\,(\Delta\psi \leq 0) = Prob\,(\Delta\Omega > \Delta\phi) + \overline{N} \quad (6)$$

where $\overline{N}$ is the average number of positive clicks occurring in the time interval $[t-T, t]$. This quantity was shown in (Hajri and Youssef, 2007) to be given by

$$\overline{N} = \frac{1}{2\pi q\gamma} \int_{t-T}^{t} \frac{\dot{\phi}(\tau)}{\sqrt{\left(a^2(\tau) + \frac{1}{q^2\gamma}\right)\left(a^2(\tau) + \frac{1}{\gamma}\right)}} d\tau \quad (7)$$

where $\gamma = \sigma_1^2/\sigma_n^2$ and q is the Hoyt fading parameter defined as $q = \sigma_2/\sigma_1$ (Nakagami, 1960). In (7) also, $\dot{\phi}(\tau)$ defines differentiation of $\phi(\tau)$ with respect to τ. For the determination of $Prob\,(\Delta\psi \leq 0)$ according to (6), we need also to determine the probability $Prob\,(\Delta\Omega > \Delta\phi)$. This quantity can be obtained from the knowledge of the PDF $p(\Delta\Omega)$ of the phase difference $\Delta\Omega$. The derivation of an expression for this PDF will be the subject of the next section.

3 DERIVATION OF THE PDF $p(\Delta\Omega)$

To start with the derivation of the PDF of the phase difference $\Delta\Omega$, over a bit duration T, we consider the FSK complex baseband signals z_1 and z_2 at the two time instants $(t-T)$ and t, respectively, according to

$$z_1 = (a_1\mu_{11} + \zeta_1) + j\,(a_1\mu_{12} + \xi_1) \quad (8)$$

and

$$z_2 = (a_2\mu_{21}\cos(\Delta\phi) - a_2\mu_{22}\sin(\Delta\phi) + \zeta_2) + j\,(a_2\mu_{21}\sin(\Delta\phi) + a_2\mu_{22}\cos(\Delta\phi) + \xi_2) \quad (9)$$

where we have assumed a coordinate system that rotates with an angle ϕ_1, i.e., the modulated phasor at time $(t-T)$ is taken as a reference. In (8) and (9), $\mu_{1i}=\mu_i(t-T)$ $(i=1,2)$, $\mu_{2i}=\mu_i(t)$ $(i=1,2)$, $a_1=a(t-T)$, and $a_2=a(t)$. The noise components ζ_i and ξ_i $(i=1,2)$ are defined relative to the modulated phasor at time $(t-T)$ and are related to $n_{1i}=n_i(t-T)$, and $n_{2i}=n_i(t)$ $(i=1,2)$ according to

$$\zeta_i = n_{i1}\cos(\phi_1)+n_{i2}\sin(\phi_1),(i=1,2), \quad (10)$$

$$\xi_i = n_{i2}\cos(\phi_1)-n_{i1}\sin(\phi_1),(i=1,2). \quad (11)$$

The components ζ_i and ξ_i $(i=1,2)$ can be shown to be zero-mean independent Gaussian random variables having the variance σ_n^2 and the ACF $\Gamma_n(\tau)$. For the sake of convenience, we denote $a_1\mu_{11}+\zeta_1$, $a_1\mu_{12}+\xi_1$, $a_2\mu_{21}\cos(\Delta\phi)-a_2\mu_{22}\sin(\Delta\phi)+\zeta_2$, and $a_2\mu_{21}\sin(\Delta\phi)+a_2\mu_{22}\cos(\Delta\phi)+\xi_2$, respectively, by x_1, y_1, x_2, and y_2. Clearly, for a given information symbol, x_i and y_i $(i=1,2)$ are Gaussian distributed. Since we assume a symmetrical Doppler PSD for the Hoyt fading, uncorrelated noise samples, and that the signal component is independent of the noise component, it can easily be shown that the correlation and cross-correlation quantities of the variables x_1, y_1, x_2, and y_2, are given by

$$\begin{aligned}
c &= E\left(x_1y_2\right)=a_1a_2\Gamma_1(T)\sin(\Delta\phi),\\
d &= E\left(x_2y_1\right)=-a_1a_2\Gamma_2(T)\sin(\Delta\phi),\\
e &= E\left(x_2y_2\right)=a_2^2\sigma_1^2\left(1-q^2\right)\cos(\Delta\phi)\sin(\Delta\phi),\\
g &= E\left(x_1x_2\right)=a_1a_2\Gamma_1(T)\cos(\Delta\phi)+\Gamma_n(T),\\
l &= E\left(y_1y_2\right)=a_1a_2\Gamma_2(T)\cos(\Delta\phi)+\Gamma_n(T), \quad (12)
\end{aligned}$$

where $E(\cdot)$ is the expected value operator, and $\Gamma_i(T)$ $(i=1,2)$ is the ACF of the process $\mu_i(t)$ $(i=1,2)$. Concerning the mean power of the underlying Gaussian processes, it is found to be given by

$$\begin{aligned}
\sigma_{x_1}^2 &= E\left(x_1^2\right)=\sigma_n^2\left(a_1^2\gamma+1\right),\\
\sigma_{y_1}^2 &= E\left(y_1^2\right)=\sigma_n^2\left(a_1^2q^2\gamma+1\right),
\end{aligned}$$

$$\begin{aligned}
\sigma_{x_2}^2 &= E\left(x_2^2\right)=\sigma_n^2\left(a_2^2\gamma\left\{\cos^2(\Delta\phi)+q^2\sin^2(\Delta\phi)\right\}+1\right),\\
\sigma_{y_2}^2 &= E\left(y_2^2\right)=\sigma_n^2\left(a_2^2\gamma\left\{\sin^2(\Delta\phi)+q^2\cos^2(\Delta\phi)\right\}+1\right). \quad (13)
\end{aligned}$$

Finally, we introduce the quantities $R_i=\sqrt{x_i^2+y_i^2}$ and $\varphi_i=\tan^{-1}(y_i/x_i)$ $(i=1,2)$, to denote, respectively, the overall envelope and phase of the Hoyt faded FSK signal contaminated by additive noise. Then, this transformation of random variables results after some lengthy algebraic manipulations (Rice, 1945) in the following expression for the joint PDF $p(\varphi_1,\varphi_2)$ of the random phases φ_1 and φ_2, as

$$p(\varphi_1,\varphi_2)=\frac{A_1}{4A_2^2}\frac{1}{(EF-D^2)}\times\left(1+\frac{D}{(EF-D^2)^{1/2}}\times\left[\frac{\pi}{2}+\tan^{-1}\left(\frac{D}{(EF-D^2)^{1/2}}\right)\right]\right), \quad (14)$$

where A_1 and A_2 are expressed in the Appendix, and the quantities D, E, and F are given by

$$\begin{aligned}
D &= \cos\varphi_1\left\{\chi_{13}\cos\varphi_2+\chi_{14}\sin\varphi_2\right\}\\
&\quad+\sin\varphi_1\left\{\chi_{23}\cos\varphi_2+\chi_{24}\sin\varphi_2\right\},\\
E &= \chi_{11}\cos^2\varphi_1+\chi_{22}\sin^2\varphi_1+2\chi_{12}\cos\varphi_1\sin\varphi_1,\\
F &= \chi_{33}\cos^2\varphi_2+\chi_{44}\sin^2\varphi_2+2\chi_{34}\cos\varphi_2\sin\varphi_2. \quad (15)
\end{aligned}$$

In (15), the quantities χ_{11}, χ_{22}, χ_{12}, χ_{33}, χ_{44}, χ_{34}, χ_{13}, χ_{14}, χ_{23}, and χ_{24} are expressed in terms of the statistical parameters described by (12) and (13), and are given in the Appendix. To the best of our knowledge, (14) is new and it constitutes the basis for the determination of the PDF $p(\Delta\Omega)$. In fact, using (14) and noting that $\varphi_2-\varphi_1=\Delta\phi+\Delta\Omega$, allows us to obtain the PDF $p(\Delta\Omega)$ of the phase difference $\Delta\Omega$ according to

$$p(\Delta\Omega)=\int_{-\pi}^{\pi}p(\varphi_1,\varphi_1+\Delta\phi+\Delta\Omega)d\varphi_1. \quad (16)$$

Unfortunately, the finite range integral involved in (16) is difficult to handle and it can be evaluated only numerically. For the special case given by $q=1$, i.e., the Rayleigh channel case, the underlying integral is easily solved and it is found that (16) is in agreement with the result of (Ng et al, 1994). For further verification of the validity of (16), we have also compared it with corresponding simulation results obtained for the case of a M2M Hoyt fading channel. For this channel type, the ACF $\Gamma_i(\tau)$ $(i=1,2)$ is given by (Akki and Haber, 1986)

$$\Gamma_i(\tau)=\sigma_i^2J_0(2\pi f_{T,max}\tau)J_0(2\pi f_{R,max}\tau) \quad (17)$$

where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind (Gradshteyn and Ryzhik, 1994), and $f_{T,max}$ and $f_{R,max}$ are the maximum Doppler frequencies generated by the motion of the transmitter and the receiver, respectively. Figure 1 shows the theoretical PDF $p(\Delta\Omega)$ of the phase difference $\Delta\Omega$, along with the corresponding simulation data for two values of the Hoyt fading parameter q. It can be observed from this figure that there exists a reasonable mutual agreement between the theoretical and the simulation results.

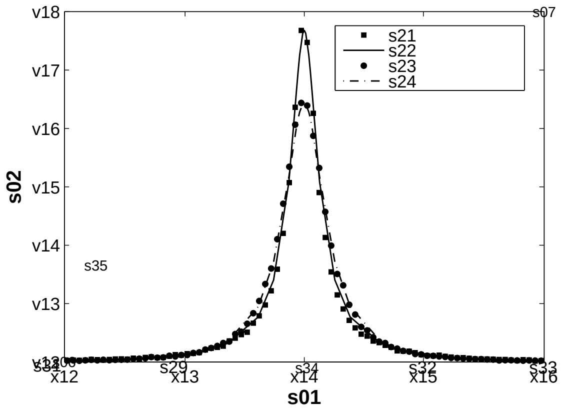

Figure 1: Theoretical and simulated PDF $p(\Delta\Omega)$ for two values of the Hoyt fading parameter q.

4 BIT ERROR PROBABILITY

According to (Pawula, 1981; Tjhung et al, 1990), the BEP for the FSK system under consideration is given by

$$P_e = Prob\,(\Delta\psi \leq 0) = P_{e,1} + P_{e,2} \tag{18}$$

where, for the case of absence of ISI, $P_{e,1} = Prob\,(\Delta\Omega > \Delta\phi)$, which can be computed from (16) according to

$$Prob(\Delta\Omega > \Delta\phi) = \int_{\Delta\phi}^{\pi} p(\Delta\Omega)\,d\Delta\Omega. \tag{19}$$

Also in (19), $P_{e,2} = \overline{N}$, which is given directly by (7). Now, to take into consideration the effect of the ISI caused by the bandwidth limitation of the IF filter, we follow (Pawula, 1981) by assuming a time-bandwidth product $BT \geq 1$. In this case, when a “+1” symbol is sent, only the three bit patterns given by “111”, “010” and “011” are considered in the ISI evaluation. Then, by considering these bit patterns, the quantities $P_{e,1}$ and $P_{e,2}$ can be obtained according to the details reported in (Pawula, 1981; Hajri and Youssef, 2007).

5 NUMERICAL EXAMPLES

In this section, computed numerical results for the closed-form BEP formula given by (18), taking into account the ISI effects, are presented for the Doppler PSD of a M2M Hoyt fading channel. For this channel type, the ACF is given by (17). Concerning the additive Gaussian noise, we assume uncorrelated noise samples, i.e., $\Gamma_n(T) = 0$. The E_b/N_0 parameter, versus which the BEP is plotted, is related to the parameters q and γ according to

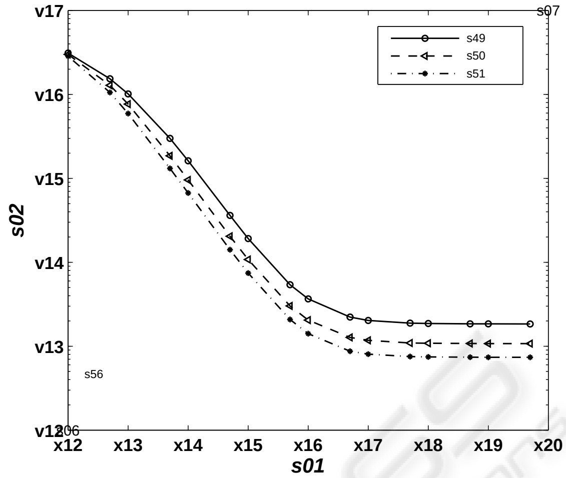

Figure 2: BEP for various values of the Hoyt fading parameter q.

$$E_b/N_0 = \frac{\left(1+q^2\right)BT}{2}\gamma \tag{20}$$

where E_b stands for the average received signal energy per bit at the input of the IF filter. Figure 2 shows the effect of the Hoyt fading parameter q on the BEP, when $BT = 1.0$, $f_{T,max} = f_{R,max} = 40$ Hz, $T = 10^{-4}$ s, and the modulation index $m = 0.7$. As expected, these results indicate that the BEP P_e improves as q increases. The best performance is obtained for $q = 1$, i.e., the case of Rayleigh fading channel.

6 CONCLUSIONS

In this paper, the BEP performance for LDI receivers of narrow-band digital FSK modulation has been analyzed considering Hoyt mobile radio fading channels. Specifically, a closed-form expression for the PDF of the phase difference, over a symbol period, between Hoyt faded FSK signals perturbed by additive Gaussian noise, has been derived. The validity of the derived PDF has been demonstrated based on its comparison against corresponding simulation results obtained for the case of Hoyt M2M fading channels. This newly derived PDF is then applied for the determination of the desired BEP performance. Numerical results of the BEP have been presented for several values of the FM system parameters and the Hoyt M2M fading channel characteristics.

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APPENDIX

In this Appendix, we give the statistical parameters A_1, A_2, χ_{11}, χ_{22}, χ_{12}, χ_{33}, χ_{44}, χ_{34}, χ_{13}, χ_{14}, χ_{23}, and χ_{24}, used in the description of (14) and (15). These quantities are expressed as

$$A_1 = \frac{1}{4\pi^2 \sigma_{x_1}\sigma_{y_1}\sigma_{x_2}\sigma_{y_2} K^{1/2}}, \qquad A_2 = \frac{1}{2K}, \text{ with}$$

$$K = 1 - \left[\frac{g^2}{\sigma_{x_1}^2\sigma_{x_2}^2} + \frac{l^2}{\sigma_{y_1}^2\sigma_{y_2}^2} + \frac{c^2}{\sigma_{x_1}^2\sigma_{y_2}^2} + \frac{d^2}{\sigma_{y_1}^2\sigma_{x_2}^2} + \frac{e^2}{\sigma_{x_2}^2\sigma_{y_2}^2} - \frac{(gl-cd)^2}{\sigma_{x_1}^2\sigma_{y_1}^2\sigma_{x_2}^2\sigma_{y_2}^2} - 2\frac{e}{\sigma_{x_2}^2\sigma_{y_2}^2} \times \left(\frac{ld}{\sigma_{y_1}^2} + \frac{gc}{\sigma_{x_1}^2} \right) \right].$$

$$\chi_{11} = \frac{1}{\sigma_{x_1}^2} \left[1 - \left(\frac{l^2}{\sigma_{y_1}^2\sigma_{y_2}^2} + \frac{d^2}{\sigma_{y_1}^2\sigma_{x_2}^2} + \frac{e^2}{\sigma_{x_2}^2\sigma_{y_2}^2} - 2\frac{lde}{\sigma_{y_1}^2\sigma_{x_2}^2\sigma_{y_2}^2} \right) \right],$$

$$\chi_{22} = \frac{1}{\sigma_{y_1}^2} \left[1 - \left(\frac{g^2}{\sigma_{x_1}^2\sigma_{x_2}^2} + \frac{c^2}{\sigma_{x_1}^2\sigma_{y_2}^2} + \frac{e^2}{\sigma_{x_2}^2\sigma_{y_2}^2} - 2\frac{gce}{\sigma_{x_1}^2\sigma_{x_2}^2\sigma_{y_2}^2} \right) \right],$$

$$\chi_{12} = \frac{1}{\sigma_{x_1}^2\sigma_{y_1}^2} \left[\frac{lc}{\sigma_{y_2}^2} + \frac{gd}{\sigma_{x_2}^2} - \frac{1}{\sigma_{x_2}^2\sigma_{y_2}^2}(cde+gle) \right],$$

$$\chi_{33} = \frac{1}{\sigma_{x_2}^2} \left[1 - \left(\frac{c^2}{\sigma_{x_1}^2\sigma_{y_2}^2} + \frac{l^2}{\sigma_{y_1}^2\sigma_{y_2}^2} \right) \right],$$

$$\chi_{44} = \frac{1}{\sigma_{y_2}^2} \left[1 - \left(\frac{g^2}{\sigma_{x_1}^2\sigma_{x_2}^2} + \frac{d^2}{\sigma_{x_2}^2\sigma_{y_1}^2} \right) \right],$$

$$\chi_{34} = \frac{1}{\sigma_{x_2}^2\sigma_{y_2}^2} \left[\frac{ld}{\sigma_{y_1}^2} + \frac{cg}{\sigma_{x_1}^2} - e \right],$$

$$\chi_{13} = \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2} \left[g - \frac{ce}{\sigma_{y_2}^2} + \frac{1}{\sigma_{y_1}^2\sigma_{y_2}^2}(lcd-gl^2) \right],$$

$$\chi_{14} = \frac{1}{\sigma_{x_1}^2\sigma_{y_2}^2} \left[c + -\frac{ge}{\sigma_{x_2}^2} + \frac{1}{\sigma_{x_2}^2\sigma_{y_1}^2}(gld-cd^2) \right],$$

$$\chi_{23} = \frac{1}{\sigma_{y_1}^2\sigma_{x_2}^2} \left[d + \frac{1}{\sigma_{x_1}^2\sigma_{y_2}^2}(cgl-dc^2) - \frac{le}{\sigma_{y_2}^2} \right],$$

$$\chi_{24} = \frac{1}{\sigma_{y_1}^2\sigma_{y_2}^2} \left[l + \frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2}(gcd-\mathrm{lg}^2) - \frac{de}{\sigma_{x_2}^2} \right]. \qquad \text{(A-1)}$$