MORPHOLOGY-BASED REPRESENTATIONS OF DISCRETE SCALAR FIELDS

Mohammed Mostefa Mesmoudi
Department of Mathematics, Mostaganem University, Route Belhacel Po.Box 227, 27000 Mostaganem, Algeria

Leila De Floriani
Department of Computer Science, University of Genova, Via Dodecaneso n 35, 16146 Genova, Italy

Keywords: Gradient vector field, Morse theory, Geometric modeling, Smale decomposition, Forman theory.

Abstract: Forman introduced in (Forman, 1998) a theory for cell complexes that is a discrete version of the well known Morse theory. Forman theory finds several applications in digital geometry and image processing where the data to be processed are discrete, see for instance (Lewiner et al., 2002a), (Lewiner et al., 2002b). In (DeFloriani et al., 2002b), we have introduced a Smale-like decomposition of a scalar field f defined on a triangulated domain M based on a *discrete gradient field* that simulates well the behavior of the gradient field in the differentiable case. Here, we extend our discrete gradient vector field so that the extended form coincides with a Forman gradient field. The extended gradient field does not change the Smale-like decomposition components and, thus, inherits properties of both smooth Morse and discrete Forman functions.

1 INTRODUCTION

Morse theory is a powerful tool for understanding the topology and the geometry of a manifold on which a C^2-differentiable function is defined. This theory has been developed in the middle of the last century by Thom, Morse, Milnor and Smale. Any C^n-differentiable function (with $n \geq 2$) can be approximated with its derivatives by a Morse function, (Milnor, 1963). Thom (Thom, 1949), followed by Smale (Smale, 1960), has shown that a manifold M endowed with a Morse function admits a *CW* representation composed of cells, called *stable*(or *unstable*) manifolds. Each cell is associated with a critical point of the function. This decomposition is based on the study of the behavior of the gradient vector field of the function. Another decomposition of the manifold into handles can be performed by following the growth of level sets of the function (Milnor, 1963). The topology (i.e., the homotopy type) of the level sets changes when a critical point is reached. Thus, critical points have a crucial role in Morse theory.

For a discrete scalar field, the field values are known only on a discrete set of points scattered over a grid (regular or irregular). To study a scalar field, an interpolation by a differentiable function is usually done, (Watson et al., 1985) and (Nackman, 1984). Then, Morse theory is used to extract critical points and critical lines that bound the cells of a Morse complex. This operation depends on the approximation performed and is expensive in term of computation time and memory space. To reduce that, other authors tried to treat the data of a two-dimensional image (Peucker and Douglas, 1975)and (J.Toriwaki and Fukumura, 1975) by performing a local study around each point. Other authors (Bajaj and Shikore, 1998), (Bajaj et al., 1998) and (Edelsbrunner et al., 2001) have interpolated the discrete data by piecewise linear functions, loosing, thus, the differentiability advantages.

In 1998, Forman introduced for cell complexes a novel theory that is a discrete equivalent to Morse theory (Forman, 1998). He has proven similar results to those proven within the smooth Morse theory. Forman theory handles the data discretely in a new way differently from all the other methods known simulating the differentiable case. Forman succeeded to prove all the main theorems of smooth Morse theory for discrete functions. Forman theory is finding several applications in computer graphics (see, for instance, (Lewiner et al., 2002b) and (Lewiner et al., 2002a)).

Mostefa Mesmoudi M. and De Floriani L. (2007).
MORPHOLOGY-BASED REPRESENTATIONS OF DISCRETE SCALAR FIELDS.
In *Proceedings of the Second International Conference on Computer Graphics Theory and Applications - GM/R*, pages 137-144
DOI: 10.5220/0002080501370144

In (DeFloriani et al., 2002b), we have introduced a Smale-like decomposition a of triangulated n-dimensional domain M associated with a scalar field f. We deduced a discrete gradient vector field $Grad\,f$, which we have been used in (DeFloriani et al., 2002a) to extract, and classify critical points and to extract a discrete Morse decomposition that represents the topology structure of the field. We have shown in (Danovaro et al., 2003) that our discrete gradient vector field simulates well the behavior of the differentiable gradient field case.

Here, we construct an extended form of our discrete gradient vector field that corresponds to the gradient field of a Forman function F whose restriction over vertices of M coincides with the initial scalar field f. We give the explicit formulation of Forman function F that satisfies the above property. As a consequence, we have that:

- All Forman results (specifically, simplification and compression process) can be applied to our discrete scalar field.

- For a triangulated embedded domain, not all the values over the cells of a Forman function are necessary to study the morphology of the domain. Only values at the vertices are required.

- Differentiability simulation of the induced gradient vector field allows us to understand the behavior of the corresponding Morse function well, and hence its decomposition into stable and unstable components.

- The compatibility of our extended gradient field with both smooth Morse and discrete Forman gradient fields provides a powerful tool to handle continuous and discrete properties at the same time.

The remainder of this paper is organized as follows. In the next Section we summarize some results relative to smooth functions, and we recall some combinatorial notions. In Section2, we present the main properties of smooth Morse theory. In Section 4, we report some results from Forman theory that we need for our construction. In Section 5, we review briefly our Smale-like decomposition and we discuss some of its properties. In Section 6, we present the construction of the process that extends our discrete gradient vector field to a Forman one. This proves the compatibility of our Smale-like decomposition with Forman theory. In the last section, we describe our on-going work.

2 BACKGROUND

In this Section we recall some fundamental notions on functions and some combinatorial notions that we need in the remainder of the paper.

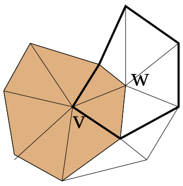
Let f be a differentiable real-valued function defined on a manifold M of dimension n. The gradient of f at a point $P \in M$ is a vector $Grad_P f$ tangent to M at P that is defined by the first derivatives of f at P. We have $Grad_P f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, where $(x_1, \ldots, x_n)$ are local coordinates around P. The set of all gradient vectors in M is called the *gradient vector field* of f and denoted by $Grad\,f$. We say that P is a *critical point* of f if the gradient vector vanishes at P. It is well known that the gradient vector field indicates the steepest direction in which the function is increasing. Curves integrating the gradient vector field (i.e., everywhere tangent to the gradient vector field) are called *integral curves*. Integral curves follow the (gradient) direction in which f has the maximal increasing growth. Hence, integral curves cannot be closed, nor infinite (in a compact manifold), and they do not self-intersect. They are emanating from critical points, or from boundary components of the domain and converge to other critical points, or to boundary components.
Let now recall some combinatorial notions, for details we refer to (Agoston, 1976). Let k be an integer, a *k-simplex* or a *k- dimensional simplex* is the convex hull of $(k+1)$ affinely independent points, called *vertices*. A *face* σ of a k-simplex γ, $\sigma \subseteq \gamma$, is a j-simplex $(0 \leq j \leq k)$ generated by $(j+1)$ vertices of γ. A *simplicial complex* K is a collection of simplexes, called also cells, such that if γ is a simplex in K, then each face $\sigma \subseteq \gamma$ is in K, and, the intersection of two simplexes is either empty or a common face of them. We call a *top* simplex in K a simplex which is not the proper face of any simplex in K.

The *carrier* $\mid K \mid$ of a simplicial complex K is the space of all points in simplexes of K. In this case, K is called a *triangulation* of $\mid K \mid$.

Let K be a simplicial complex and γ be a cell in K. The *star* of γ is the set $St(\gamma)$ of all cells in K which are incident at γ. Thus, $St(\gamma) = \{\sigma \in K : \gamma \subseteq \sigma\}$. The star of γ describes the neighborhood of γ in the complex (see Figure 1(a)). The closure of a set of cells Γ is the smallest subcomplex $\overline{\Gamma}$ of K containing Γ. Clearly, $\overline{\Gamma}$ consists of all cells of Γ plus their faces.

The *link* of cell γ is the subcomplex $Lk(\gamma)$ of K defined as $Lk(\gamma) = \overline{St}(\gamma) - St(\overline{\gamma})$, where $\overline{\gamma}$ is the closure of γ. The link describes the boundary of $St(\gamma)$ (see Figure 1(a)).

A *cone* from a vertex w to a simplex γ is the convex combination of all vertices of γ with w. We denote it by (γ, w). If w is affinely independent of the vertices of γ, then the cone from w to γ is a simplex of dimension $dim(\gamma)+1$, where $dim(\gamma)$ denotes the dimension of γ.



(c)

Figure 1: The shaded region is the star of v. The graph in bold is the link of the vertex w.

3 SMOOTH MORSE THEORY

A *Morse function* on a manifold M is a C^2-differentiable real-valued function f defined on M such that its *critical* points are non-degenerate (Milnor, 1963). This means that the Hessian matrix $Hess_P f$ of the second derivatives of f at any point $P \in R^d$ on which the gradient of f vanishes ($Grad_P f = 0$) is non-degenerate ($Det(Hess_P f \neq 0)$. Morse (Milnor, 1963) has proven that there exists a local coordinate system $(y^1,...,y^n)$ in a neighborhood U of any critical point P, with $y^j(P) = 0$, for all $j = 1,\ldots,n$, such that the identity

$$f = f(P) - (y^1)^2 - ... - (y^\imath)^2 + (y^{\imath+1})^2 + ... + (y^n)^2$$

holds on U, where $\imath$ is the number of negative eigenvalues of $Hess_P f$, and it is called the *index* of f at P. The above formula implies that the critical points of a Morse function are isolated. This allows us to study the behavior of f around them, and to classify their nature according to the signs of the eigenvalues of the Hessian matrix of f. If the eigenvalues are all positives, then the point P is a *strict local minimum* (a pit). If the eigenvalues are all negatives, then P is a *strict local maximum* (a peak). If the index $\imath$ of f at point P is different from 0 and n, then the point P is neither a minimum nor a maximum, and, thus, it is called an $\imath$*-saddle* point (a pass).

The decomposition of the manifold domain associated with f, introduced by Thom (Thom, 1949) and followed by Smale (Smale, 1960), is based on the study of the growth of f along its integral curves. Integral curves originating from a critical point of index $\imath$ form a $\imath$-cell C^s, called a *stable manifold*. In the same way, integral curves converging to a critical point of index $\imath$ form a dual $(n-\imath)$-cell C^u, called an *unstable manifold*. Stable manifolds are pair-wise disjoint and decompose the field domain M into open cells, (see Figure 2). The cells form a complex, as the boundary of every stable manifold is the union of lower dimensional cells. Similarly, the unstable manifolds decompose M into a complex dual to the complex of stable manifolds.

Figure 2 gives an example of a stable decomposition of a two-dimensional scalar field, which is assumed to be a Morse function. It has three minima (shown by $\bullet$), two maxima (shown by $\odot$), and five saddle points (shown by $\bowtie$). Integral curves originate from each minimum in all directions and from the right side of the boundary. Each integral curve converges either to a saddle, to a maximum, or to a boundary component. Two integral curves originate from each saddle point. Integral curves originating from a minimum (or from the right side boundary) sweep a 2D cell, while integral curves emanating from a saddle point form a segment containing the saddle point in its interior. Integral curves connecting saddles to other critical points are called *separatrices*.

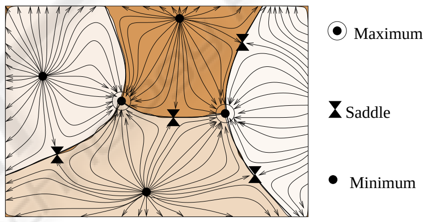


Figure 2: Decomposition of a domain into four stable 2-manifolds.

4 FORMAN THEORY

In this Section, we discuss discrete Morse functions, introduced by Forman, and their main properties as-proved in (Forman, 1998). Let K be simplicial complex, we denote with $\sigma^{(p)}$ a simplex of dimension p. By $\sigma < \tau$ we indicate that σ is a face of the simplex τ.

Definition 1 *Let f be a real valued function defined on K. We say that f is a* discrete Morse function, *or a* Forman function *if and only if, for every simplex $\sigma^{(P)}$,*

$$\#\{\tau^{(p+1)} > \sigma^{(p)} : f(\tau) \leq f(\sigma)\} \leq 1 \qquad (1)$$

$$\#\{v^{(p-1)} < \sigma^{(p)} : f(v) \geq f(\sigma)\} \leq 1 \qquad (2)$$

We observe immediately that, if f is a discrete Morse function on K, then $-f$ is not necessarily a discrete Morse function on K. This fact is not true in the differentiable case.

Definition 2 *We say that a cell $\sigma^{(p)}$ is a critical cell of f if and only if*

$$\begin{array}{l}\#\{\tau^{(p+1)} > \sigma^{(p)} : f(\tau) \leq f(\sigma)\} = 0 \\ \#\{v^{(p-1)} < \sigma^{(p)} : f(v) \geq f(\sigma)\} = 0\end{array} \tag{3}$$

A simple example of a Forman function is given in in Figure 3(a).

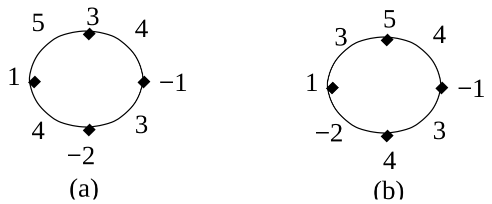


Figure 3: The function defined on the complex in (a) is a Forman function, while the function defined on the complex in (b) is not a Forman function (vertex of image 5 and edge of image -2 violate conditions 2) .

The above definitions extend to a finite CW-complex K. Forman has shown that inequalities $(1$ & $2)$ cannot be equalities in the same time. This means that, for discrete Morse functions, we cannot find simultaneously a face $v^{(p-1)}$ and a co-face $\tau^{(p+1)}$ of a cell σ^p such that $f(\tau) \leq f(\sigma) \leq f(v)$. From the above definitions if K is regular then the absolute minimum of f should occur at a vertex and if the carrier of K has no boundary components then the absolute maximum of f should occur at a maximal dimensional cell, see Figure 3(a).

In the literature the negative gradient vector field is usually used instead of the gradient field. We will stick to this convention and we will call the negative gradient vector field simply the gradient vector field. The (negative) gradient field indicates the steepest directions in which the function decreases so that the gradient flow is uniform. This idea has been used by Forman to define a *discrete gradient vector field* for discrete Morse functions. Forman has shown that critical cells and non critical cells are uniquely characterized by the discrete gradient vector field.

Let $\sigma^{(p)}$ be a cell in a regular complex K. If there exists a cell $\tau^{(p+1)}$ such that $\sigma < \tau$ and $f(\tau) \leq f(\sigma)$, then we draw a vector from σ to τ and we repeat thia for all cells of K. The set of such vectors is the *discrete gradient vector field* corresponding to Forman function f. Obviously, the corresponding functional definition is that, such a cell τ^{p+1} is the image of σ^p by a function ϕ. Relations $(1,2)$ imply that a cell can be the tail or the end of at most one vector. From relation (3), critical cells are not the tail nor the end of a vector, see Figure 4 below. This property allows us to recognize critical cells in a regular complex.

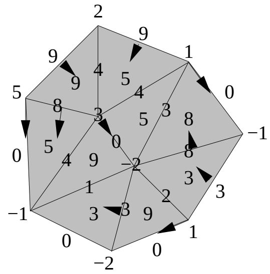


Figure 4: Illustration of a gradient vector field. Critical cells are those which are not the tail nor the end of a vector.

5 SMALE-LIKE DECOMPOSITION PROCESS

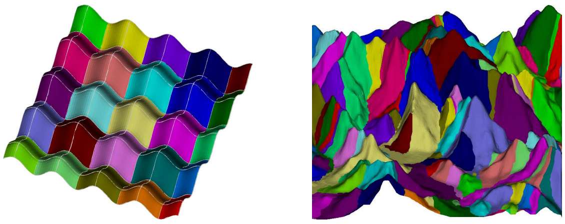
In (DeFloriani et al., 2002b), we have introduced an algorithm that decomposes a d-dimensional triangulated domain K associated wit

a scalar field f into a collection of pair-wise disjoint components. This decomposition is similar to Thom-Smale's decomposition in the differentiable case. We have defined a discrete gradient vector field that behaves on M like a differentiable gradient field. Here, we recall the basic idea of this decomposition and how to construct the corresponding discrete gradient vector field. Without loss of generality, we assume that $f(u) \neq f(v)$ for all vertices $u \neq v$. This can be obtained through a local perturbation of the scalar field f. This condition ensures the uniqueness of the decomposition. We maintain a current complex K' which is initialized to be equal to K. We consider a vertex v in K' corresponding to the global maximum of f. The values of f at the vertices of $St(v)$ are thus less than $f(v)$. In this step, we define the component $C(v)$corresponding to v to be $\overline{St(v)}$. We set $\partial C(v) := Lk(v)$. Then, for each top simplex γ in $\partial C(v)$ that is incident in another simplex (γ, w) in $K' - C(v)$, we compare the values of f at vertices of γ with $f(w)$. If $f(w)$ is less than all of them, then we extend $C(v)$ to be $C(v) \cup \overline{(\gamma, w)}$and we replace γ in $\partial C(v)$ by all faces of cone (γ, w) that contain w.

We thus iteratively extend $C(v)$ at each step to bound a region on which f decreases. The process stops when the region cannot be further extended while maintaining the above property. At this point, we delete such region from K', and we repeat the process.

The result of the above algorithm is, thus, a decomposition $\mathcal{D}$ of M into *unstable* components $C_i = C(v_i)$, each of which corresponds to a local maximum of f. To reduce the number of components we add a merging step that merges two adjacent components $C(v_i)$ and $C(v_j)$ if and only if v_i or v_j belongs to the boundary of the component associated with it. An example of this decomposition is shown in Figure 5 (a) for a synthetic function and in Figure (b) for a real data set.

The decomposition algorithm described above al-



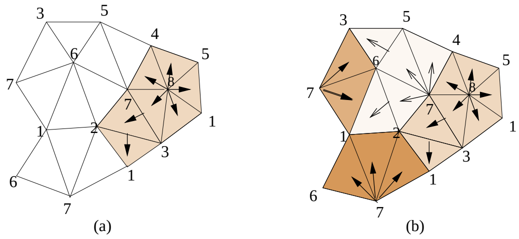
(a) (b)

Figure 5: In (a), Unstable decompositions of a synthetic function $f(x,y) = sinx + siny$ representing the eggs plateau surface. In (b), 119 unstable components produced by our decomposition algorithm applied to a triangulation of the Mount Marcy consisting of 69718 triangles.

lows us to define a discrete form of the gradient vector field for a scalar field f. A *discrete (negative) gradient vector field* is defined by the following two functions. A multi-valued function ϕ which associates each local maximum v, corresponding to a component $C(v)$ of M, with the top cells γ' in $St(v)$, i.e., $\phi(v) = \{\gamma' : \gamma' \text{ is a top cell in } St(v)\}$.

With each cell γ in $C(v) - \{\partial C(v) \cup St(v)\}$, which has been used in the extension process, we associate the added cones (γ, w_i). Equivalently, the vertices (w_i) are sufficient to characterize this single-valued function which we denote by ψ. We have $\psi((\gamma, w_i)) = \{w_i\}$. Functions ϕ and ψ define the discrete (negative) gradient vector field of f. As in the differentiable case, it denotes the directions in which the function decreases, and characterizes the critical cells and points. To obtain a geometric representation of functions ϕ and ψ, we draw vectors from the initial vertex v, to all top cells in $St(v)$, and a vector from γ to the cones (γ, w_i) used in the decomposition process. We obtain a collection of vectors that indicate the directions in which the scalar field is decreasing (cf., Figures 6).

Referring to the example in Figure 6(a), we consider the vertices at which the scalar field reaches its maximum, which is equal to 8. We show the process of growing the component. The shaded regions in Figure 6(a) is the component associated with value 8. In Figure 6(b), we show the final decomposition of the complex in 6(a) with its gradient vector field. Each shaded region correspond to an unstable Smale component.



(a) (b)

Figure 6: In (a), the decomposition process of K: component corresponding to the vertex with the field value equal to 8 with its discrete gradient vector field. In (b), the final components decomposing K and their discrete gradient vector field.

6 EXTENDED GRADIENT FIELD AND COMPATIBILITY WITH FORMAN THEORY

In this Section, we prove that the discrete gradient field obtained from our Smale-like decomposition of a manifold M endowed with a scalar field f can be extended so that a Forman function F is defined over M. From this point of view, the scalar field f becomes the restriction of F over the vertices of M, its gradient vector field $Grad\, f$ becomes a subfield of the gradient vector field of F and the critical points of f are a subset of critical cells of F.

In the construction process of the Smale-like decomposition seen in Section 5, the expansion of components $C(v)$ begins by attaching to $St(v)$, where v is a local maximum, other cones (γ, w) where γ is a top simplex in $Lk(v)$ and $f(w)$ is less than all values of f over vertices of γ. Then function ψ associates, the $(n-1)$-simplex, γ with vertex w. For each pair (γ, w), function ψ can be extended, to a function $\tilde{\psi}$, over all faces σ^i of γ, with $i = 0$ to $dim(\gamma) - 1 = n - 2$ by associating σ^i with vertex w. Geometric equivalent extension consists of emanating vectors from all faces of γ towards vertex w. This is compatible with our decomposition process since $f(w) < f(w')$ for all vertices w' of γ, see Figure 7 below. Note that, according to this construction, all faces of γ are not critical since they are tails of vectors. Since a simplex cannot be the tail and the end of a vector at the same time, then if a simplex $\gamma' = (\sigma^{(n-2)}, w)$ is used to expand $\partial C(v)$ in the Smale-like decomposition process, the vector corresponding to $\tilde{\psi}(\sigma^{n-2})$ has to be removed. This already characterizes a Forman function over all faces of $\overline{C(v)} - St(v)$.

Thus, the extended function $\tilde{\psi}$ can be explicitly defined by

- if γ is a $(n-1)$-simplex expanding $C(v)$ then $\tilde{\psi}(\gamma) := \psi(\gamma) = \{w\}$, such that (γ, w) expands

$C(v)$.

- For all i-simplexes $\sigma^i \in \gamma$, with $i = 0, \ldots, n-3$ we set $\tilde{\psi}(\sigma^i) := \{w\}$ if $\tilde{\psi}(\sigma^i)$ has not been defined before when another $(n-1)$-simplex γ incident is σ^i was considered. Otherwise, σ^i is skipped since it has already an attached value by $\tilde{\psi}$
- For $i = n-2$ and such that (σ^{n-2}, w) does not participate to the expansion process of $C(v)$, we have $\tilde{\psi}(\sigma^{n-2}) = \{w\}$.
- Otherwise $\tilde{\psi}(\sigma^{n-2}) = \emptyset$. In this case, cone (σ^{n-2}, w) represents a new expanding $(n-1)$-simplex of $C(v)$ and $\tilde{\psi}((\sigma^{n-2}, w)) = \psi((\sigma^{n-2}, w))$. Then we return to the first point to define $\tilde{\psi}$ over faces of (σ^{n-2}, w).

For simplicity, we present in Figure 7 the extended gradient vector field for a 2-dimensional scalar field. Simplexes γ are edges and their faces σ^i are vertices (i.e., we have only $i = 0$). Vectors emanating from vertices towards edges are added if the edges do not participate in the expansion process of the component construction. For example, segment $\gamma = [6;7]$ expands $C(8)$ by adding triangle Δ labeled as $6,7$ and 5. Function ψ associates segment $[6;7]$ with vertex $\{5\}$. An arrow from $[6;7]$ towards $\{5\}$ is drawn. End points of segment $[6;7]$ are the $(n-2)$-simplexes described above. Cone (i.e., segment)$(6,5)$ does not participate in the expansion of the new component $C(v) := C(v) \cup \Delta$. Thus, function $\tilde{\psi}(6) = \{5\}$ and a vector emanating from $\{6\}$ towards $\{5\}$ is added to edge $[6;5]$. The other vertex $\{7\}$ of segment $[6;7]$, with vertex 5 forms an edge that expands the updated $C(v)$, then $\tilde{\psi}(7) = \emptyset$ and no vector is emanating from $\{7\}$ towards $\{5\}$ is drawn in triangle Δ. Note that the same vertex $\{7\}$ is revisited again when triangle $(7;3;2)$ is considered. Function $\tilde{\psi}$ associates $\{7\}$ with $\{2\}$ since edge $[7;2]$ does not expand $C(v)$. Vertex $\{7\}$ is revisited again a last time when triangle $(7;5;2)$ is considered. The process skips here vertex $\{7\}$ since it has already a non-empty value by $\tilde{\psi}$. We see clearly that each simplex in the triangulation emanates or receives at most one vector. Hence, the extended gradient vector field is a Forman gradient. Critical cells are those which are not the tail nor the end of vectors. We have here only one (global) minimum $\{0\}$ and the entire star $St(8)$ as a singular cell corresponding to the maximal value 8.

In the general case, function F is not uniaue and can be defined in many ways. In the following, we present an explicit construction of F. Let γ be a $(n-1)$-simplex expanding a component $C(v)$ to $C(v) \cup \{w\}$ and let σ^i be a face of γ where $i \in \{1, \ldots, n-1\}$. Suppose that γ and its faces are visited for the first time.

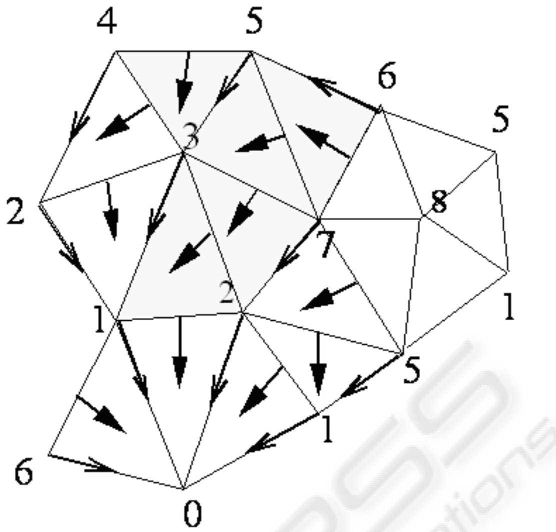


Figure 7: Illustration of the extended gradient vector field for a 2-dimensional scalar field. The extension here acts only on vertices. Arrows are added to edges that do not participate to the expansion process of component $C(v)$ with $f(v) = 8$.

1. We set $F(\gamma) := max\{f(v') : v'$ is a vertex of $\gamma\} + (n-1)\varepsilon$, where ε is positive number chosen so that $F(\gamma) < f(v)$. Then we define $F(\gamma, w) := F(\gamma)$.
   For faces $(\sigma^i)_{i=0}^{n-2}$, we set $F(\sigma^i) := max\{f(v') : v'$ is a vertex of $\sigma^i\} + i\varepsilon$, for all $i \in \{1, \ldots, n-1\}$. Note that for $i = 0$, simplexes σ^0 are simply vertices of γ for which we have $F(\sigma^0) = f(\sigma^0)$.
   Let σ^i be a face of another simplex $\sigma^j \subset \gamma$ (i.e., $i < j$), then vertices of σ^i are included in the set of vertices of σ^j. Hence $max\{f(v') : v'$ is a vertex of $\sigma^i\} \leq max\{f(v') : v'$ is a vertex of $\sigma^j\}$ and consequently $F(\sigma^i) < F(\sigma^j)$. This implies that faces of γ are set to be critical at this definition step except γ from which an arrow in emanated towards cone (γ, w).
2. For cones (σ^i, w), we define $F((\sigma^i, w)) := F(\sigma^i)$, for all $i \in \{0, 1, \ldots, n-2\}$. This means that from each face σ^i we emanate an arrow towards cone (σ^i, w) This definition ensures that Forman relations (1) and (2) are satisfied.
3. Now, for the expansion process of the updated component $C(v)$, $(n-1)$-simplexes of type (σ^{n-2}, w) are considered. Let update γ to be equal to (σ^{n-2}, w). At this moment, γ and each of its faces adjacent to w receive an arrow from a face of σ^{n-2}. We update then w to be the new added point. Since γ is expanding component $C(v)$ to a new cone (γ, w), then new arrows will be em-

anated from faces of γ towards w. Hence, values of faces of γ by F should be re-initialized to make them, first, critical in $C(v)$. To do that, we update, first, value of ε to be equal $\varepsilon - \frac{\varepsilon}{10}$ and then we return to step (1.). Value of ε is updated for the following reason. Simplex γ is adjacent to two $n-$ simplexes, the old cone $(\gamma, w) \subset C(v)$ and the new cone (γ, w) expanding $C(v)$. Then, to preserve Forman relations (1) and (2), value of γ by F should be less then the value of the old cone (γ, w).
Step (1.) defines F over (γ, w), γ and all its faces. Values of vertices (i.e., 0-simplexes) are preserved. We return, then to step (2.) to define F over all faces of type (σ^i, w). New arrows are hence drawn from faces σ^i to (σ^i, w) and we go so on. If a face is re-visited from another expanding simplex then we assign to the simplex a value that preserves Forman relations (1) and (2).

By a such construction Forman relations are satisfied over all the simplexes of the complex.
The simplest way to extend f over $St(v)$ is to consider that all simplexes in the interior of $St(v)$ are critical for F since they are the immediate neighbors of v which is critical for f. We can define F for an i-dimensional face α^i of $St(v)$ to be $f(v) + i\varepsilon$. Relations (3) are, thus, satisfied over all simplexes of $St(v)$.
In Figure 8, we give an example of construction of a Forman function that extends the scalar field f described in Figure 7 with value $\varepsilon = 0.1$. The extended function preserves Forman relations (1) and (2) and corresponds to the above formulas defining F. Thus, the function obtained in the example is a Forman function.
Function $\tilde{\phi}$ describing the gradient vector field over $St(v)$ can be associated with the restriction of F over $St(v)$ and function $\tilde{\psi}$ describing the gradient vector field over $\overline{C(v)} - St(v)$ can be associated with the restriction of F over $\overline{C(v)} - St(v)$.
Outside $St(v)$, the extended gradient vector field follows, naturally, the decreasing growth of the function f over the triangulation simplexes.

To keep the differentiability simulation of the extended gradient vector field over the entire domain, we keep the geometric representation (by vectors) of function ϕ over stars $\{St(v)\}$ of local maxima $\{v\}$. In order to be consistent with the extension $\tilde{\psi}$ of ψ over proper faces of simplexes γ, we extend function ϕ, to a function $\tilde{\phi}$ over all simplexes in $St(v)$ by emanating vectors from v towards all simplexes incident to v. The extended function $\tilde{\phi}$ is defined by $\tilde{\phi}(v) = \{\gamma' : \gamma' \text{ is a simplex in } St(v)\}$.
In Figure 9, we show the representation of both $\tilde{\phi}$ and $\tilde{\psi}$ for the same scalar field represented in Figure7.

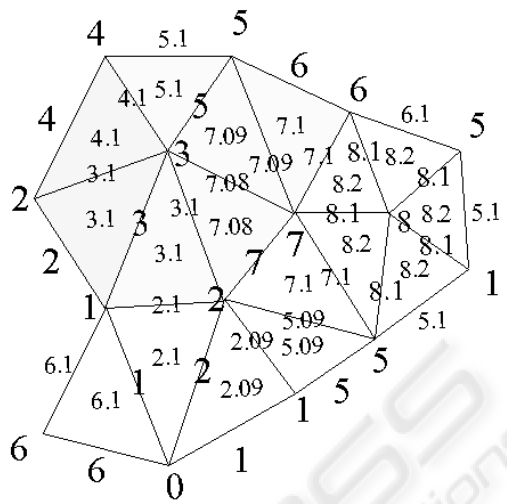


Figure 8: Definition of a Forman function that extends the scalar field over all simplexes and that corresponds to the extended gradient vector field described in Figures 7 and 9 with value $\varepsilon = 0.1$.

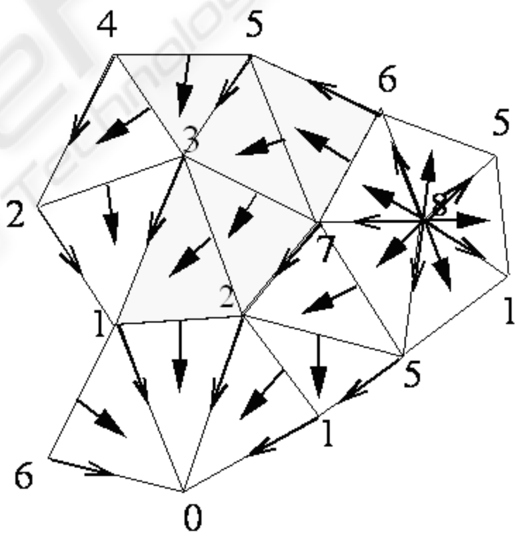


Figure 9: General representation of the extended gradient vector field for a 2-dimensional scalar field over all the triangulated domain.

7 CONCLUDING REMARKS

Here, we have presented an extended form of a discrete gradient vector field associated with a Smale-like decomposition in order to define a Forman function compatible with the decomposition. A Smale-

like decomposition simulates well the differentiable case. Thus, we obtain a good representative of a discrete gradient field that combines properties of both smooth Morse and discrete Forman theories. In our future work, we are planning to implement processes for two- and three-dimensional scalar fields in order to apply them on real image processing data bases. We will apply the Forman simplification meshes and its compression process to our extended gradient field in order to define multi-resolution approach based on both Morse and Forman theory.
Since the algorithm is dimension-independent, a further development of this work consists of applying the approach for clustering .

ACKNOWLEDGEMENTS

This work has been partially supported by a grant of the Polytechnic University of Valencia, Spain (”Programa de Apoyo a la Investigacion y Desarrollo 2006”) and by the National Science Foundation under grant CCF-0541032, by the MIUR-FIRB project SHALOM under contract number RBIN04HWR8, by the MIUR-PRIN project on ”Multi-resolution modeling of scalar fields and digital shapes” and by the European Network of Excellence AIM@SHAPE under contract number 506766.

REFERENCES

Agoston, M. (1976). *Algebraic Topology, A First Course,*. Pure and Applied Mathematics, Marcel Dekker.

Bajaj, C. L., Pascucci, V., and Shikore, D. R. (1998). Visualization of scalar topology for structural enhacement. In *Proceedings of the IEEE Conference on Visualization ’98 1998*, pages 51–58.

Bajaj, C. L. and Shikore, D. R. (1998). Topology preserving data simplification with error bounds. *Journal on Computers and Graphics*, 22(1):3–12.

Danovaro, E., De Floriani, L., and Mesmoudi, M. M. (2003). Topological analysis and characterization of discrete scalar fields. In Asano, T., Klette, R., and Ronse, C., editors, *Theoretical Foundations of Computer Vision, Geometry, Morphology, and Computational Imaging*, volume 2616 of *Lecture Notes on Computer Science*, pages 386–402. Springer Verlag.

DeFloriani, L., Mesmoudi, M. M., and Danovaro, E. (2002a). Extraction of Critical Points and Nets Based on Discrete Scalar Fields. In *To appear In Proceed. of Eurographics conference*.

DeFloriani, L., Mesmoudi, M. M., and Danovaro, E. (2002b). Smale-like Decomposition for Discrete Scalar Fields. In *To appear In Proceed. of Inter. Conf. on Pattern Recognition*.

Edelsbrunner, H., Harer, J., and Zomorodian, A. (2001). Hierarchical morse complexes for piecewise linear 2-manifolds. In *Proc 17th Sympos. Comput. Geom.*, pages 70–79.

Forman, R. (1998). Morse Theory for Cell Complexes. *Advances in Mathematics*, 134:90–145.

J.Toriwaki and Fukumura, T. (1975). Extraction of structural information from grey pictures. *Computer Graphics and Image Processing*, 7:30–51.

Lewiner, T., Lopes, H., and Tavares, G. (2002a). Optimal discrete morse functions for 2-manifolds. Technical report, Pontificia Universidade Catolica do Rio de Janero.

Lewiner, T., Lopes, H., and Tavares, G. (2002b). Visualizing forman’s discrete vector field. In H.-C. Hege, K. P. E., editor, *to appear in Proceed. of Mathematical Visualization III, Springer*.

Milnor, J. (1963). *Morse Theory*. Princeton University Press.

Nackman, L. R. (1984). Two-dimensional critical point configuration graph. *IEEE Transactions on Pattern Analysisand Machine Intelligence*, PAMI-6(4):442–450.

Peucker, T. K. and Douglas, E. G. (1975). Detection of surface-specific points by local paprallel processing of discrete terrain elevation data. *Graphics Image Processing*, 4:475–387.

Smale, S. (1960). Morse inequalities for a dynamical system. *Bulletin of American Mathematical Society*, 66:43–49.

Thom, R. (1949). Sur une partition en cellule associées a une fonction sur une variété. *C.R.A.S.*, 228:973–975.

Watson, L. T., Laffey, T. J., and Haralick, R. (1985). Topographic classification of digital image intensity surfaces using generalised splines and the discrete cosine transformation. *Computer Vision, Graphics and Image Processing*, 29:143–167.