LP FITTING APPROACH FOR RECONSTRUCTING PARAMETRIC SURFACES FROM POINTS CLOUDS

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Abstract: We present a method to reconstruct a surface from a group of points, each provided with two parameters. The kind of reconstructed surface can be a Bezier surface, a B-spline surface or any surface generated by a basis of functions.
The usual method involved in such a reconstruction is the least squares approach. Our original fitting method called LP-fitting uses a linear program for minimizing the uniform error instead of the quadratic error considered in least squares.
Experimental results comparing both approaches show that the surface obtained by LP-fitting is usually closer (from a uniform point of view) to the initial points cloud than the surface obtained by least squares.

1 INTRODUCTION

Surface reconstruction is involved in several domains of applications going from reverse engineering to surface modeling. The problem is the computation of a surface S passing as close as possible to each point of a given subset of $\mathbb{R}^3$. We focus in this paper on reconstruction of parametric surfaces where F belongs to a given linear space of functions (it is the case of Bézier and B-Spline surfaces).

The expression of the nearness between the points cloud and the reconstructed surface requires to introduce the error vector $\delta(S)$ that coordinates $\delta_i(S)$ are the distances between each point of index i in the input and its closest point in surface S. This vector measures the accuracy of the approximation. Thus, the natural surface reconstruction problem is the computation of a function F in the given linear space of functions that error vector has a minimal norm (some variants can be obtained by adding a value measuring the smoothness of the surface, satisfaction of normal constraints...). It is a highly non linear problem of optimization which has been tackled by Newton methods in (Atieg and Watson, 2004).

A more classical approach, developped in the framework of computer graphics, consists in four consecutive steps. (Eck and Hoppe, 1996; Weiss et al., 2002; Sarkar and Menq, 1991; Jüttler, 1997)

1. Mesh generation from the unorganized points cloud (α-shapes, marching cubes, Delaunay triangulations) (Edelsbrunner and Mücke, 1994; Barber et al., 1996)
2. Mesh partitionning in patches homeomorphic to disks by using for instance tools of shape analysis. (Eck et al., 1995)
3. Parametrization (Eck et al., 1995; Floater and Hormann, 2005)
4. Surface fitting. After the parametrization step, a pair of parameters (s_i, t_i) has been assigned to each point (x_i, y_i, z_i). The problem of surface fitting is now to minimize the distances between each point and its corresponding point of the surface $F(s_i, t_i)$ instead of minimizing its real minimal distance to S.

The standard approach of surface fitting is to minimize $\sum_i d_2((x_i, y_i, z_i), F(s_i, t_i))^2$ (where d_2 is the Euclidian distance) in the considered linear space of functions. This objective function is the square of the Euclidian norm $||\delta||_2$ where the coordinate δ_i is the Euclidian distance between the input point (x_i, y_i, z_i) and its corresponding point on the surface $F(s_i, t_i)$.

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This usual routine of the overall reconstruction problem is called ”least-squares fitting” (Farin, 2002; Cohen et al., 2001) because the computation can be done easily by the least squares method.

The task of the paper is to introduce an alternative to least-squares fitting and to compare both approaches. The main idea is to use the uniform norm $||\ .\ ||_\infty$ instead of the Euclidian norm. We minimize $||\delta||_\infty$ where coordinate δ_i is the uniform distance between the input point (x_i, y_i, z_i) and its corresponding point in the surface $F(s_i, t_i)$. More precisely we minimize independently the three infinite norms $||(x_i - F_x(s_i, t_i))||_\infty$, $||(y_i - F_y(s_i, t_i))||_\infty$ and $||(z_i - F_z(s_i, t_i))||_\infty$. These problems of optimization are linear and thus they can be solved by linear programming. We call this approach ”LP-fitting”. Its principle is to control the maximal distance between each point of the input (x_i, y_i, z_i) and its corresponding point in the surface $F(s_i, t_i)$ while the principle of least squares fitting is to search the best solution from a statistical point of view.

We start the paper with the basic definitions and notations (see §2). We introduce in §3 the fitting problems obtained by using Euclidian or uniform norms and their solutions by least-square method or linear programming. We end the paper with the comparison of the experimental results provided by both approaches (see §4).

2 NOTATIONS

We begin with a brief introduction to classical parametric surfaces.

2.1 Parametric Surfaces

Let P_i, $i \in [\![0, n]\!]$ be a subset of points of $\mathbb{R}^3$. Let $f_i(s, t)$, $i \in [\![0, n]\!]$ be a family of functions and let the parametric surface S be the image of interval product $[a, b] \times [c, d]$ by function :

$$F(s,t) = \sum_{i=0}^{n} P_i f_i(s,t)$$

The points P_i are called control points of surface S. The most usual cases are Bézier and B-Spline surfaces, depending on the family f_i we choose.

2.2 Bézier Surfaces

Bézier surfaces are parametric polynomial surfaces with bounded degree. Any basis of polynomials could be used as functions $f_i(s, t)$ but the usual choice is to work with Bernstein polynomials $B_{i,n}(t) = \binom{n}{i} * t^i * (1-t)^{n-i}$ or more precisely with their tensor product (their sum being 1, the surface belongs to the the convex hull of the control points) :

$$F(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i,m}(s) B_{j,n}(t)$$

2.3 B-Spline Surfaces

The B-Spline surfaces require other parameters. We introduce two integers k and l and two vectors $S = \{s_0, \ldots, s_{m+k-1}\}$ and $T = \{t_0, \ldots, t_{n+l-1}\}$, with $s_0 \le s_1 \le \ldots \le s_{m+k-1}$ and $t_0 \le t_1 \le \ldots \le t_{n+l-1}$ called knot vectors. Then the considered family of functions is obtained with tensor product :

$$F(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} N_{i,k}(s) N_{j,l}(t)$$

where the $N_{i,k}$ is the basis function defined by Cox de Boor formula (Farin, 2002).

Note that by adjusting the number of parameters by tensor product and by changing the dimension by cartesian product, previous definitions and next results hold in a general framework.

3 SURFACE RECONSTRUCTION

3.1 General Problem

Let $\{M_k\}_{1 \le k \le p}$, with $M_k = (\ x_k,\ \ y_k,\ \ z_k\ )^T$, be a subset of p points in $\mathbb{R}^3$. We consider that each point M_k is provided in the input with a pair of parameters s_k and t_k. The purpose of this section is to find a parametric surface S, defined by a function $F(s, t)$, which is close (in a sense to be defined) to the given set of points. We introduce the error δ_k of the approximation on the point M_k:

$$\delta_k = F(s_k, t_k) - M_k \text{ for } 1 \le k \le p \qquad (1)$$

with a function F chosen in the linear space generated by functions $(f_i)_{0 \le i \le n}$ defined in Section 2.1.

3.2 Interpolation

If the error is constrained to be null, equalities $\delta_k = 0$ for $k \in [\![1, p]\!]$ leads to the linear system of equations $M_k = \sum_{i=0}^{n} P_i f_i(s, t)$ that unknowns are the three coordinates of each control point P_i^x, P_i^y and P_i^z.

We can be sure of the existence of a solution if the dimension of the basis of functions f_i is at least equal to the number of points p. Otherwise, the linear system is much often not feasible with the consequence that usually no interpolation exists. This usual case is the framework of fitting methods. Their task is to find approximate solutions of unfeasible systems. It allows to work with linear spaces of restricted dimensions while the huge dimensions and degrees necessary for interpolation introduce detrimental variations (Figure 1).

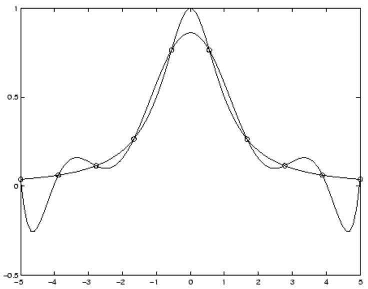

Figure 1: Consequences of interpolation with a high degree polynomial.

3.3 Approximate Reconstruction

The function $F(s,t)$ of the linear space generated by functions f_i is denoted $F(s,t) = (x(s,t), y(s,t), z(s,t))$ with $\begin{cases} x(s,t) = \sum_{i=0}^{n} P_i^x * f_i(s,t) \\ y(s,t) = \sum_{i=0}^{n} P_i^y * f_i(s,t) \\ z(s,t) = \sum_{i=0}^{n} P_i^z * f_i(s,t) \end{cases}$ and where $P_i = \begin{pmatrix} P_i^x, & P_i^y, & P_i^z \end{pmatrix}^T$, $i \in [\![0,n]\!]$ are the control points of surface F.

The principle of fitting methods is to minimize the error expressed by Equation 1:

$$\begin{cases} \delta_k^x = \sum_{i=0}^{n} P_i^x * f_i(s_k,t_k) - x_k & \forall k \in [\![1,p]\!] \\ \delta_k^y = \sum_{i=0}^{n} P_i^y * f_i(s_k,t_k) - y_k & \forall k \in [\![1,p]\!] \\ \delta_k^z = \sum_{i=0}^{n} P_i^z * f_i(s_k,t_k) - z_k & \forall k \in [\![1,p]\!] \end{cases}$$

Let us note that the three systems are independent (their unknowns are different). Equation on the x-coordinates leads to system 2 :

$$A * X = Y + \delta \tag{2}$$

where A is matrix

$$\begin{pmatrix} f_0(s_1,t_1) & \dots & f_n(s_1,t_1) \\ f_0(s_2,t_2) & & f_n(s_2,t_2) \\ \vdots & & \vdots \\ f_0(s_p,t_p) & \dots & f_n(s_p,t_p) \end{pmatrix}$$

where vector $X = \begin{pmatrix} P_0^x, & \dots, & P_n^x \end{pmatrix}^T$ contains the x-coordinates of the control points, $Y = \begin{pmatrix} x_1, & \dots, & x_p \end{pmatrix}^T$ contains the x-coordinates of the real points, and $\delta = \begin{pmatrix} \delta_1^x, & \dots, & \delta_p^x \end{pmatrix}^T$ is the error on the x-coordinates.

In the framework of interpolation, δ is constrained to be null but we have seen in previous section that in many cases, the linear rank of matrix A is not sufficient to guarantee the existence of an exact solution. Thus the idea of fitting methods is to optimize the vector δ according to some criteria.

3.4 Least Squares Fitting

The criteria of least squares fitting is the Euclidian norm of the error $\|\delta\|_2$ (Cohen et al., 2001; Farin, 2002). Equation 2 leads to the following problem of minimization:

$$\underset{X}{Min} \|A * X - Y\|_2$$

$$\iff \begin{cases} \underset{X}{Min} \sum_{i=1}^{p} {\delta_i}^2 \\ \delta = A * X - Y = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} \end{cases}$$

Thus the problem is to minimize the distance between the given point Y and a point $A * X$ belonging to linear space $Im(A)$. It is well known (Whittaker and Robinson, 1967) that the the minimum is obtained with the orthogonal projection of Y on $Im(A)$ namely with $Y = A * X + \delta$ and $\delta \in Im(A)^\perp$.

This decomposition of Y (corresponding to the direct sum $\mathbb{R}^p = Im(A) \oplus Im(A)^\perp$) is denoted :

$$\underbrace{A * X_{LSF}}_{\in Im(A)} - \underbrace{\delta_{LSF}}_{\in Im(A)^\perp} = Y$$

By recalling that $Im(A)^\perp$ is also $KerA^T$, we have :

$$\underbrace{(A^T * A)}_{\text{square matrix, symmetric}} * X_{LSF} - \underbrace{A^T * \delta_{LSF}}_{=0} = A^T * Y$$

and by multiplying on the left by the inverse of $A * A^T$ (we assume that A is a full rank matrix), it provides the following expression of the least squares solution :

$$X_{LSF} = (A^T * A)^{-1} * A^T * Y \tag{3}$$

The $n+1$ coordinates of X_{LSF} are the coordinates (in present case : the x-coordinates) of the $n+1$ control points of the least squares fitting reconstructed surface.

3.5 LP Fitting

Our original approach consists in working with a uniform criterion on the error. We minimize the uniform norm of δ instead of the Euclidian norm used in least squares fitting. This method leads us to a linear program (Chvatal and Vasek, 1983). According to Equation 2 we obtain the following problem of minimization :

$$\underset{X}{Min}\|A*X-Y\|_{\infty}$$
$$\Longleftrightarrow \begin{cases} \underset{X}{Min}\left(\underset{1\le i\le p}{Max}|\delta_i|\right) \\ \delta = A*X-Y = \begin{pmatrix}\delta_1\\ \vdots \\ \delta_p\end{pmatrix}\end{cases}$$

The problem can be modified by introducing an auxiliary real variable that we denote h. It plays the role of a bound on the each coordinate of $A*X-Y$.

$$\begin{cases} \underset{X}{Min}\left(\underset{1\le i\le p}{Max}|\delta_i|\right) \\ \delta = A*X-Y \end{cases}$$
$$\Longleftrightarrow \begin{cases} \underset{X,h}{Min}(h) \\ \delta = A*X-Y \\ -h\le\delta_i\le+h \quad \forall i, 1\le i\le p\end{cases}$$
$$\Longleftrightarrow \begin{cases} \underset{X,h}{Min}(h) \\ -h*\mathbb{1}\le A*X-Y\le h*\mathbb{1}\end{cases}$$
$$\Longleftrightarrow \begin{cases} \underset{X,h}{Min}(h) \\ A*X+h*\mathbb{1}\ge Y \\ A*X-h*\mathbb{1}\le Y\end{cases}$$

It leads to the following linear program 4:

$$\begin{cases} \underset{X}{Min}(h) \\ \begin{pmatrix} & 1 \\ A & \vdots \\ & 1\end{pmatrix} * \begin{pmatrix} X \\ h \end{pmatrix} \ge Y \\ \begin{pmatrix} & -1 \\ A & \vdots \\ & -1\end{pmatrix} * \begin{pmatrix} X \\ h \end{pmatrix} \le Y \end{cases} \tag{4}$$

where the variables are $X\in\mathbb{R}^{n+1}$ (or each coordinate of X) and $h\in\mathbb{R}$ and where the objective function is the linear form h. By construction, the linear program is feasible. Its solution is denoted X_{LPF}. Its $n+1$ coordinates are again the coordinates (in our case the x-coordinates) of the $n+1$ control points of the LP-fitting reconstructed surface.

4 RESULTS

4.1 Protocole

We have generated point clouds from existing parametric surfaces: Bézier surfaces, B-spline surfaces and sphere surfaces with spherical angles as parameters. The coordinates of the points and the corresponding parameters have been perturbed by gaussian noise to obtain relatively realistic points clouds. Then we proceeded with the reconstruction according to least squares and LP fitting methods. The results have been recorded in histograms in order to understand the relative merit of our new fitting method with respect to classical least squares approach.

4.2 Interpretation of Results

For each surface that we reconstructed, we computed the errors on each point (see Figure 2) .

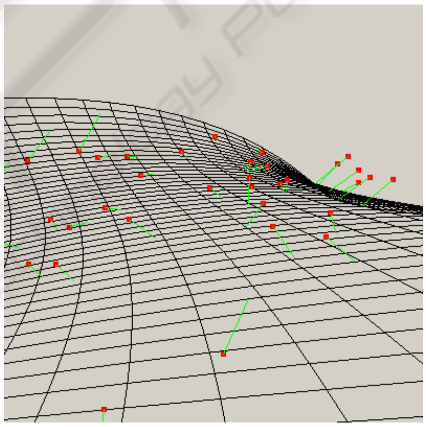

Figure 2: Result of reconstruction with the original 3D coordinates M_k in red, the surface $F(s,t)$ in black, and the error vectors $\delta_k = \overrightarrow{M_kF(s_k,t_k)}$ in green.

Figures 3 and 4 show the reconstruction of the same perturbed sphere by Bézier surface, with the two methods. We can notice that the least squares fitting result seems to be a practically perfect sphere while the LP fitting result is bumpy. It is the consequence of an expectable behavior. The noise that we have introduced on the sphere has a deeper impact on the LP-approach than on least squares fitting. This feature of both methods is a great difference. Least squares would be better to deal with noisy data but there is the risk that some important details represented only by few points or a lower density of points in the input could be treated as noise and disappear from the reconstructed surface. With least squares fitting, the reconstructed surface could be very far from one point because the quadratic criteria has favored a solution passing close to another region with a lot of points. With LP-fitting, the uniform criterion constrains the solution to be close to any point without any consideration on its statistical weight. Thus LP-fitting is better if all the points have an equivalent signification and is probably less satisfying than least squares fitting if the input is noisy (with non significant points in the input).

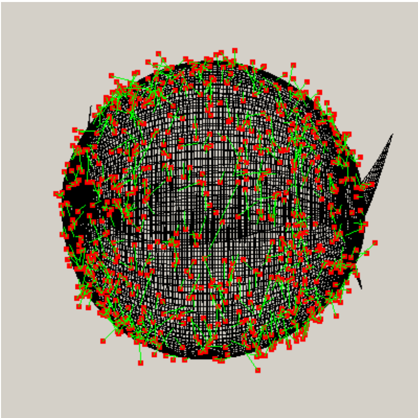

Figure 3: Reconstruction of sphere with Least Squares Fitting (LSF).

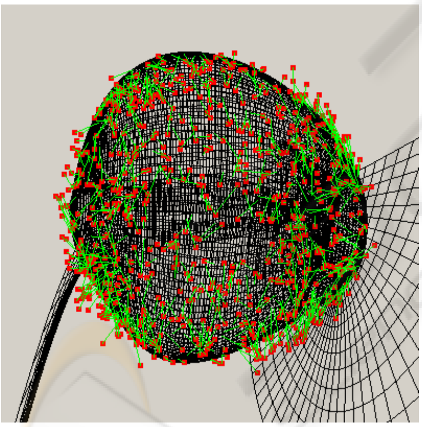

Figure 4: Reconstruction of sphere with LP Fitting (LPF).

4.3 Histograms

For each reconstructed surface, the Euclidian norms of the error vectors (in green in the figures) have been recorded in histograms.

In most cases the maximal error between a point and its corresponding point belonging to the reconstructed surface is much smaller with LP-fitting than with least squares fitting (see for instance Figure 5). There exist actualy some cases where the tendency is inverted (see Figure 6). It is due to the fact that the maximum of the error on the x-coordinate, on the y-coordinate and on the z-coordinate are obtained with points of different indices with the consequence that an independent minimization of each of them (what is done in LP-fitting) does not guarantee to minimize the maximum of their Euclidian norm. Some more standard histograms are drawn in Figure 7 and 8. They show results with little disturbed data and much more disturbed data. In general, on all computed examples, with few and much more disturbances, the LP-fitting maximum error is about 80% of the least squares maximum error.

The main result of our experiment is that LP-fitting provides surfaces which are significantly closer from the points cloud in terms of maximal distance. We can also notice on the shape of the corresponding histograms that by using LP-fitting the cost on the mean error (which is minimum by using least squares fitting) remains quite low.

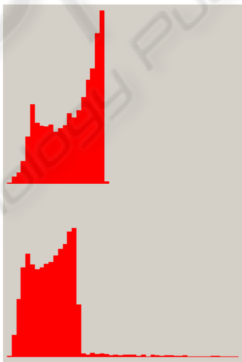

Figure 5: Histograms of reconstruction of a perturbed sphere by Bézier surface 5*5 : LP-fitting (top) least squares fitting (bottom).

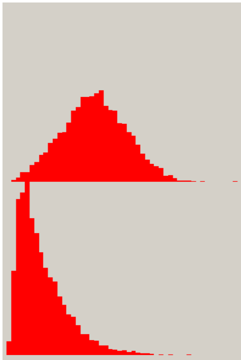

Figure 6: Histograms of reconstruction of another perturbed sphere by Bézier 5*5 : LP-fitting (top) least squares fitting (bottom).

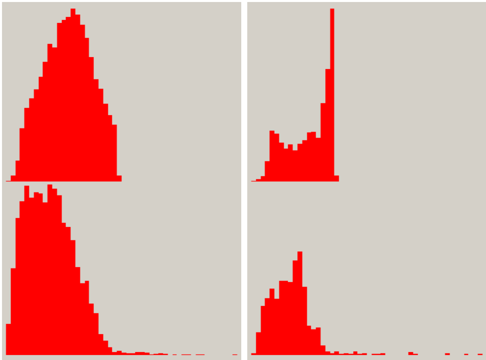

Figure 7: Two histograms of reconstructions : LP-fitting (top) least squares fitting (bottom).

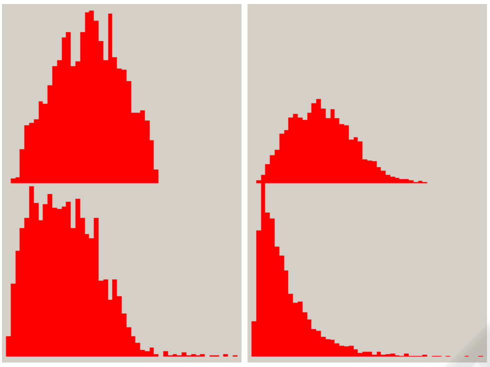

Figure 8: Two histograms of reconstructions of highly perturbed input : LP-fitting (top) least squares fitting (bottom).

5 CONCLUSION

Least squares fitting is a subroutine of the general problem of surface reconstruction. The input is a finite subset of $\mathbb{R}^3$ provided with a pair of parameters for each point. The output is a grid of control points of a Bézier, B-spline surface (or any surface of the same kind). We propose in this paper an alternative method based on the idea to minimize the uniform error on the input instead of usual quadratic Euclidian error. From a computational point of view, it leads to a linear program which can be solved by any solver while classical least squares approach only requires to compute an orthogonal projection on a linear space.

The different features of the two methods are related to the choice to minimize the uniform or Euclidian error. With least squares approach the statistical weight of a subset of points concentrated in a given region enforces the reconstructed surface to be close to it while an isolated point can be considered as noise with the consequence that the surface can be far from it. With LP-fitting the reconstructed surface is close to all the points of the input independently of their number in each region. This different behavior is the main point allowing to choose one or the other fitting method.

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