SUFFICIENT CONDITION OF MAX-PLUS ELLIPSOIDAL INVARIANT SET AND COMPUTATION OF FEEDBACK CONTROL OF DISCRETE EVENT SYSTEMS

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Abstract: Haar's Lemma (1918) provides the algebraic characterization of the inclusion of polyhedral sets. This Lemma has been involved many times in automatic control of linear dynamical systems when the constraint domains (state and/or control) are polyhedrons. More recently, this Lemma has been used to characterize stochastic comparison w.r.t linear orderings of Markov chains with different state spaces. Stochastic comparison is involved in the simplification of complex stochastic systems in order to control the approximation error made. In this paper we study the positive invariance of a max-plus ellipsoid by a max-plus linear dynamical system. We remark that positive invariance of max-plus ellipsoid is a particular case of polyhedron inclusion and we use Haar's Lemma to derive sufficient condition for the positive invariance. As an application we propose a method to compute a static state feedback control.

1 INTRODUCTION

In 1918, Haar demonstrated the following result which provides an algebraic characterization of the inclusion of two polyhedral sets. For more details see e.g. (Haar, 1918). This result can be found in (Hennet, 1989) where it is called extended Farkas'Lemma. Let $\leq_n$ denotes the component-wise order on $\mathbb{R}^n$ (i.e. $x \leq_n y \Longleftrightarrow \forall i, x_i \leq y_i$).

Result 1 (Haar's Lemma) *Let Φ (resp. Ψ) be a $m \times d$ (resp. $m' \times d$) matrix. Let p (resp. q) be a m (resp. m') dimensional column vector. The following assertion*

$$\emptyset \neq \mathcal{P}(\Phi, p) \subseteq \mathcal{P}(\Psi, q), \tag{1}$$

which is can be written as

$$\emptyset \neq \{x \in \mathbb{R}^d | \Phi x \leq_m p\} \subseteq \{x \in \mathbb{R}^d | \Psi x \leq_m q\},$$

is true iff

$$\exists H \in \mathbb{R}^{m' \times m}, \quad \begin{array}{l} (a).H \geq \mathbf{0}_{m' \times m}, \\ (b).\Psi = H\Phi, \\ (c).Hp \leq_d q. \end{array}$$

In the case where p and q are equal to the null vector (i.e. the homogeneous case), Haar's Lemma reduces to Farkas' Lemma (Farkas, 1902). A recent reference for such material is ((Urruty and al., 2001), pp. 58-61).

Max-plus algebraic approach to modelling Discrete Event Systems (DES) such as manufacturing systems, communication protocols (TCP), computer networks, dynamic programming, transportation networks (see e.g. (Baccelli and al., 1992)) is now almost classical. The geometric approach for the control of DES is one of the main topics of research already mentioned by (Cohen and al.,1999) which has not yet been so developed than in linear algebra.

In linear algebra, the geometric approach for the positive invariance has been mainly developed by (Wonham, 1985) in the case of vectorial subspaces (see also (Basile and al., 1992)). The positive invariance of a subset of the state space of a given dynamical system is characterized by the following property: if at some time a positively invariant set contains the state system, then it will contains it also in the future. (See e.g. the survey paper (Blanchini, 1999) on this subject and references therein). It is known at least since (Kalman and al., 1960) that stability in the sense of Lyapuonv of classical linear systems has links with the existence of a positively invariant ellipsoid

$$\{x | \; x^T P x \leq c\}.$$

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At the end of the 1980's, (Bitsoris, 1988), (Hennet, 1989), and also more recently (Farina and al., 1998) have explored and characterized other kind of positively invariant sets.

In (Ahmane and al., 2004) it has been shown that the algebraic characterization of the inclusion of two polyhedral sets provided by Haar's Lemma is the fundamental notion for the characterization of the positive invariance of discrete-time Markov chains in particular and linear systems in general. This idea is adapted here in the context of the control of Discrete Event Systems (DES) using the properties of A-invariance and $(A \oplus B \otimes F)$-invariance.

Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e) = (\mathbb{R} \cup \{-\infty, +\infty\}, \max, +, -\infty, 0)$ be a complete idempotent semiring (see definitions in section 2.3). Let us now consider the linear systems over $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ specified by $(d, \otimes, A)$ and defined by:

$$(d, \otimes, A) : \begin{cases} x(0) \in \mathbb{S}^d, \\ x(n) = A \otimes x(n-1), \quad n \geq 1, \end{cases} \tag{2}$$

with $d \in \mathbb{N}$, $x(n)$ is the state vector at time instant n and $A \in \mathbb{S}^{d \times d}$ is the state matrix. The vectorial equation (2) means that

$$\forall i, x_i(n) = \max_{j=1,\ldots,d}(A_{i,j} + x_j(n-1)).$$

Define now the Max-Plus ellipsoidal set as follows:

$$\mathcal{E}(P, w, \alpha) = \{x \in \mathbb{S}^d | \; x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w\}, \tag{3}$$

where matrix $P \in \mathbb{S}^{d \times d}$, $w \in \mathbb{S} \setminus \{\varepsilon\}$, $(\cdot)^T$ denotes the transpose operator and

$$\begin{aligned} \forall x \in \mathbb{S}^d, \; x^{\otimes(\alpha)} &\overset{\text{def}}{=} (x_1^{\otimes\alpha}, \ldots, x_d^{\otimes\alpha})^T \\ &= (x_j^{\otimes\alpha})^T_{j=1,\ldots,d}, \end{aligned}$$

with

$$\forall a \in \mathbb{S}, \; \forall \alpha \in \mathbb{R}, \; a^{\otimes\alpha} \overset{\text{def}}{=} \alpha a,$$

with αa denotes the usual multiplication $\alpha \times a$.
In linear algebra, $\mathcal{E}(P, w, \alpha)$ coincides with the notion of ellipsoid.

The main results of this paper are as follows. First, Using Haar's Lemma (see Result 1), we obtain sufficient condition under which the following assertion is true (cf. Theorem 1):

$$\forall x(0) \in \mathbb{S}^d, [x(0) \in \mathcal{E}(P, w, \alpha) \Rightarrow \forall n, x(n) \in \mathcal{E}(P, w, \alpha)]. \tag{4}$$

We will then introduce the concept of the positive invariance by the max-plus linear mapping $x \mapsto A \otimes x$ or the A-invariance of the set $\mathcal{E}(P, w, \alpha)$. It means that if the state of the system at time instant 0 is on the set $\mathcal{E}(P, w, \alpha)$, one is sure that this state will remain within the set $\mathcal{E}(P, w, \alpha)$ at every time instant n. Second, we show that this sufficient condition for the positive invariance of such set allow us to determine a max-plus linear state feedback control law. It means to find a feedback $F \in \mathbb{S}^{m \times d}$ such that the following linear system over a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined by:

$$\begin{cases} x(0) \in \mathbb{S}^d, A \in \mathbb{S}^{d \times d}, B \in \mathbb{S}^{d \times m} \\ x(n) = A \otimes x(n-1) \oplus B \otimes u(n), \\ u(n) = F \otimes x(n-1), \; n = 1, 2, \ldots, \end{cases}$$

verifies the assertion (4). We will also then introduce the concept of the $(A \oplus B \otimes F)$-invariance of the set $\mathcal{E}(P, w, \alpha)$. It means that if the state of the system at time instant 0 is in the set $\mathcal{E}(P, w, \alpha)$, one is sure with applying an adequate control u that this state will remain within the set $\mathcal{E}(P, w, \alpha)$ at time instant n. Otherwise says, the state $x(n)$ can be forced to remain inside $\mathcal{E}(P, w, \alpha)$ by an adequate choice of the control u. To obtain such feedback on the state system, the residuation theory is one of the main tool.

The paper is organized as follows. In Section 2, we introduce the main notations used in this paper and we present the main definitions of max-plus algebra. The main references are e.g. ((Baccelli and al., 1992), (Blyth and al. 1972), (Golan, 1992)). In Section 3, we identify and characterize the positive invariance of a max-plus ellipsoidal set. Sufficient conditions for A-invariance and $(A \oplus B \otimes F)$-invariance are provided using Haar's Lemma (Result 1). As an application, Subsection 3.1 and 3.2 are respectively devoted to the existence of a max-plus linear state feedback controller and to a method for computing this max-plus linear state feedback control law. Finally, some conclusions are given in Section 4.

2 BACKGROUND

2.1 Notations and Definitions

- $(\cdot)^T$ denotes the transpose operator.
- $\leq_m$ denotes the component-wise ordering of $\mathbb{S}^m$, for all $m \in \mathbb{N}$.
- All vectors are column vectors.
- If $A, B \in \mathbb{S}^{m \times n}$ then $A \leq B$ denotes the entry-wise comparison of the matrices A and B.

2.2 Ordered Sets and Elements of Residuation Theory

Let $(\Omega, \leq)$ be a partially ordered set. $(\Omega, \leq)$ is a sup semilattice (resp. inf semilattice) iff any set $\{\omega_1, \omega_2\} \subset \Omega$ has a supremum $\bigvee\{\omega_1, \omega_2\}$ (resp. an infimum $\bigwedge\{\omega_1, \omega_2\}$). $(\Omega, \leq)$ is a lattice iff $(\Omega, \leq)$ is a sup and inf semilattice. $(\Omega, \leq)$ is complete iff any set $A \subset \Omega$ has a supremum $\bigvee A$. A complete ordered set is also a complete lattice because

$\bigwedge A \overset{\text{def}}{=} \bigvee\{\omega \in \Omega:\ \forall a \in A,\ \omega \leq a\}$. A lattice is distributive iff $\wedge$ and $\vee$ are distributive with respect to (w.r.t) one another.

A map $f : (\Omega, \leq) \to (\Omega', \preceq)$, where $(\Omega, \leq)$ and $(\Omega', \preceq)$ are two ordered sets, is $(\leq, \preceq)$-monotone if it is a compatible morphism with respect to $\leq$ and $\preceq$. The map $f : (\Omega, \leq) \to (\Omega', \preceq)$ is residuated iff there exists a map $f^{\natural} : (\Omega', \preceq) \to (\Omega, \leq)$ such that: $\forall \omega \in \Omega, \forall \omega' \in \Omega',\ f(\omega) \preceq \omega' \Leftrightarrow \omega \leq f^{\natural}(\omega')$.
This relation can be defined as follows:

$$f^{\natural}(\cdot) \overset{\text{def}}{=} \bigvee\{\omega \in \Omega:\ f(\omega) \leq\ \cdot\ \}.$$

A monotone map $f : (\Omega, \leq) \to (\Omega', \preceq)$, where $(\Omega, \leq)$ and $(\Omega', \preceq)$ are complete sets, is said to be continuous iff $\forall A \subset \Omega$, $f(\bigvee_{\leq} A) = \bigvee_{\preceq} f(A)$, $\bigvee_{\leq}$ (resp. $\bigvee_{\preceq}$) denotes the supremum w.r.t $\leq$ (resp. $\preceq$); $f(A) \overset{\text{def}}{=} \{f(a):\ a \in A\}$.

the following result (see e.g. (Blyth and al. 1972, Th 5.2) or (Baccelli and al., 1992, Th 4.50)) provides a characterization for a residuable function over two complete ordered sets.

Result 2 *Let $(\Omega, \leq)$ and $(\Omega', \preceq)$ two complete sets. A map $f : (\Omega, \leq) \to (\Omega', \preceq)$ is residuated iff f is continuous and $f(\bigwedge \Omega) = \bigwedge \Omega'$.*

2.3 Basic Algebraic Structures

For any set $\mathbb{S}$, $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a semiring if $(\mathbb{S}, \oplus, \varepsilon)$ is a commutative monoid, $(\mathbb{S}, \otimes, e)$ is a monoid, $\otimes$ distributes over $\oplus$, the neutral element ε for $\oplus$ is also absorbing element for $\otimes$, i.e. $\forall a \in \mathbb{S},\ \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$, and e is the neutral element for $\otimes$.

$(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is an idempotent semiring (called also dioid) if $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a semiring, the internal law $\oplus$ is idempotent, i.e. $\forall a \in \mathbb{S},\ a \oplus a = a$. If $(\mathbb{S}, \otimes, e)$ is a commutative monoid, then the idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is said commutative.

$(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a an idempotent semifield if $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is an idempotent semiring and $(\mathbb{S} \backslash \{\varepsilon\}, \otimes, e)$ is a group, i.e. $(\mathbb{S} \backslash \{\varepsilon\}, \otimes, e)$ is a monoid such that all its elements are invertible ($\forall a \in \mathbb{S} \backslash \{\varepsilon\},\ \exists a^{-1} : a \otimes a^{-1} = a^{-1} \otimes a = e$). Also if $(\mathbb{S} \backslash \{\varepsilon\}, \otimes, e)$ is a commutative monoid, then the idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is said commutative.

Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be an idempotent semiring. Each element of $\mathbb{S}^n$ is a n-dimensional column vector. We equip $\mathbb{S}^n$ with the two laws $\oplus$ and $\cdot$ as follows: $\forall x, y \in \mathbb{S}^n,\ (x \oplus y)_i = x_i \oplus y_i, \forall s \in \mathbb{S},\ (s \otimes x)_i \overset{\text{def}}{=} s \otimes x_i,\ i = 1, \ldots, n$. The addition $\oplus$ and the multiplication $\otimes$ are naturally extended to matrices with compatible dimension. Any $n \times p$ matrix A is associated with a $(\oplus, \otimes)$-linear map $A : \mathbb{S}^p \to \mathbb{S}^n$. The (i, j) entry, the l^{th} row-vector and the k^{th} column-vector of matrix A, are respectively denoted $a_{i,j}$, $a_{l,\cdot}$ and $a_{\cdot,k}$. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be an idempotent semiring or an idempotent semifield, then $(\mathbb{S}, \oplus, \varepsilon)$ is an idempotent monoid, which can be equiped with the natural order relation $\leq$ defined by:

$$\forall a, b \in \mathbb{S},\ a \leq b \overset{\text{def}}{\Leftrightarrow} a \oplus b = b. \tag{5}$$

We say that $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is complete if it is complete as a naturally ordered set and if the respective left and right multiplications, $\lambda_a, \rho_a : \mathbb{S} \to \mathbb{S}$, $\lambda_a(x) = a \otimes x$, $\rho_a(x) = x \otimes a$ are continuous for all $a \in \mathbb{S}$. In such case we adopt the following notations for $\forall a, b \in \mathbb{S}$:

$$\lambda_a^{\natural}(b) \overset{\text{not.}}{=} a \backslash b \overset{\text{def}}{=} \bigvee\{x \in \mathbb{S}:\ x \otimes a \leq b\},$$
$$\rho_a^{\natural}(b) \overset{\text{not.}}{=} b / a \overset{\text{def}}{=} \bigvee\{x \in \mathbb{S}:\ a \otimes x \leq b\}.$$

A typical example of complete dioid is the top completion of an idempotent semifield. Let us note that if $a \in \mathbb{S}$ is invertible then: $a \backslash b = a^{-1} \otimes b$ and $b / a = b \otimes a^{-1}$. Let us note also that as $\mathbb{S}$ is complete it possesses a top element $\bigvee \mathbb{S} \overset{\text{not.}}{=} \top = +\infty$. We have by convention the following identities:

$$\begin{array}{l} \varepsilon \otimes \top = \top \otimes \varepsilon = \varepsilon, \\ \forall a \in \mathbb{S},\ a \oplus \top = \top, a \wedge \top = \top \wedge a = a. \end{array} \tag{6}$$

We suppose besides that (for a discussion to this topic, see e.g. ((Baccelli and al., 1992, p. 163-164)):

$$\forall a \neq \varepsilon,\ a \otimes \top = \top \otimes a = \top. \tag{7}$$

By definition of / (idem for \) and properties of $\top$, ε and of [7] we have for all $a \in \mathbb{S}$:

$$a / \varepsilon = \top,\ \top / a = \top, \tag{8a}$$

$$a / \top = \begin{cases} \varepsilon & \text{if } a \neq \top \\ \top & \text{if } a = \top \end{cases} \quad \varepsilon / a = \begin{cases} \varepsilon & \text{si } a \neq \varepsilon \\ \top & \text{if } a = \varepsilon \end{cases}. \tag{8b}$$

The operations $\cdot / \cdot$, $\cdot \backslash \cdot$ are extended to matrices and vectors with compatible dimensions assuming that all the elements of these matrices and vectors are in a complete set $\mathbb{S}$:

$$(A \backslash y)_i = \underset{j}{\wedge}\ (a_{j,i} \backslash y_j); \tag{9a}$$

$$(A \backslash B)_{i,j} \overset{\text{def}}{=} (\bigvee\{X:\ A \otimes X \leq B\})_{i,j} = \underset{k}{\wedge}\ (a_{k,i} \backslash b_{k,j}); \tag{9b}$$

$$(D / C)_{i,j} \overset{\text{def}}{=} (\bigvee\{X:\ X \otimes C \leq D\})_{i,j} = \underset{l}{\wedge}\ (d_{i,l} / c_{j,l}). \tag{9c}$$

3 APPLICATIONS FOR THE CONTROL OF DISCRETE EVENT SYSTEMS

In this section, we identify and characterize the positive invariance of the max-plus ellipsoidal set previously defined by (3). Using Haar's Lemma (cf.

Result 1), we give sufficient conditions for the A-invariance (cf. Theorem 1) and $(A \oplus B \otimes F)$-invariance (cf. Theorem 2). (Truffet, 2004) treated the case where the variable α is equal to 1. He gave necessary and sufficient conditions for the positive invariance of the max-plus ellipsoidal set previously defined in the case where $\alpha = 1$. As an application we show that this sufficient characterization for the positive invariance of such sets allow us to determine a max-plus linear state feedback control law (F to be determined).

3.1 A-invariance

Let us consider the following linear system $(d, \otimes, A)$ over a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined as follows:

$$(d, \otimes, A) : \begin{cases} x(0) \in \mathbb{S}^d, \\ x(n) = A \otimes x(n-1), \quad n \geq 1, \end{cases}$$

where $x(n)$ is the state vector at time instant n and $A \in \mathbb{S}^{d \times d}$ is the state matrix.

Recall also the max-plus ellipsoidal set defined by (3):

$$\begin{cases} \mathcal{E}(P, w, \alpha) = \{x \in \mathbb{S}^d | \; x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w\}, \\ P \in \mathbb{S}^{d \times d}, \; w \in \mathbb{S} \setminus \{\varepsilon\}. \end{cases} \tag{10}$$

where

$$\forall x \in \mathbb{S}^d, \; x^{\otimes(\alpha)} \overset{\text{def}}{=} (x_1^{\otimes\alpha}, \ldots, x_d^{\otimes\alpha})^T = (x_j^{\otimes\alpha})^T_{j=1,\ldots,d},$$

We are interested now in conditions under which the following sets inclusion is true:

$$A \otimes \mathcal{E}(P, w, \alpha) \subseteq \mathcal{E}(P, w, \alpha), \tag{11}$$

where, by definition,

$$A \otimes \mathcal{E}(P, w, \alpha) \overset{\text{def}}{=} \{A \otimes x : \; x \in \mathcal{E}(P, w, \alpha)\}.$$

We will then introduce the concept of the A-invariance of the set $\mathcal{E}(P, w, \alpha)$.

Definition 1 (A-invariant set) *The set $\mathcal{E}(P, w, \alpha)$ is positively invariant by the linear mapping $x \mapsto A \otimes x$ or A-invaraint if the last assertion defined by (11) is verified.*

This last definition means that, if the state vector of the system $(d, \otimes, A)$ at time instant 0 is in the set $\mathcal{E}(P, w, \alpha)$, one is sure that this state vector will remain it at time instant n. Otherwise says, the equation (11) means that:

$$\forall x(0) \in \mathbb{S}^d, \quad \begin{array}{l} [x(0) \in \mathcal{E}(P, w, \alpha) \Rightarrow \\ \forall n \in \mathbb{N}, x(n) \in \mathcal{E}(P, w, \alpha)] \end{array}. \tag{12}$$

The last assertion (12) can be rewritten at each time instant by:

$$\forall x \in \mathbb{S}^d, [x \in \mathcal{E}(P, w, \alpha) \Rightarrow A \otimes x \in \mathcal{E}(P, w, \alpha)].$$

Using the definition of $\mathcal{E}(P, w, \alpha)$ given by (10), this last implication is equivalent to:

$$\forall x \in \mathbb{S}^d, \quad \begin{array}{l} [x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w \Rightarrow \\ (A \otimes x)^T \otimes P \otimes (A \otimes x)^{\otimes(\alpha)} \leq w] \end{array}. \tag{13}$$

Lemma 1 *For any matrix $P \in \mathbb{S}^{d \times d}$ we have:*

$$\forall x \in \mathbb{S}^d, (x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w) \Leftrightarrow (x^{\otimes(\alpha)} \otimes x^T \leq w/P) \tag{14}$$

Proof.
We have:

$$x^T \otimes P \otimes x^{\otimes(\alpha)} = \bigoplus_{i,j} (x_i \otimes P_{i,j} \otimes x_j^{\otimes\alpha}).$$

Since $\otimes$ is commutative and $\oplus = \vee$, the following inequality

$$x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w,$$

is logically equivalent to:

$$\forall i, j = 1, ..., d, \; x_j^{\otimes\alpha} \otimes x_i \otimes P_{i,j} \leq w.$$

what implies, by residuation,

$$\forall i, j = 1, ..., d, \; x_j^{\otimes\alpha} \otimes x_i \leq w/P_{i,j},$$

which ends the proof. ■

Notice that $\alpha \in \mathbb{R}$, then it can be take some positive or negative values, therefore in what follows we treat the two cases: $\alpha \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_-$.

- **α is a negative real number ($\alpha \in \mathbb{R}_-$)**

Proposition 1 *The assertion*

$$\forall x \in \mathbb{S}^d, \quad \begin{array}{l} [x^T \otimes P \otimes x^{\otimes(\alpha)} \leq w \Rightarrow \\ x^T \otimes A^T \otimes P \otimes A^{\otimes(\alpha)} \otimes x^{\otimes(\alpha)} \leq w], \end{array} \tag{15}$$

implies the assertion defined above by (13).

Proof.
It is sufficient to remark that :

$$\forall x \in \mathbb{S}^d, \forall \alpha \in \mathbb{R}_-; \; (A \otimes x)^{\otimes(\alpha)} \leq A^{\otimes(\alpha)} \otimes x^{\otimes(\alpha)},$$

where the matrix $A^{\otimes(\alpha)}$ is defined by:

$$A^{\otimes(\alpha)} = \begin{bmatrix} A_{1,1}^{\otimes\alpha} \ldots A_{1,d}^{\otimes\alpha} \\ \vdots \ddots \vdots \\ A_{d,1}^{\otimes\alpha} \ldots A_{d,d}^{\otimes\alpha} \end{bmatrix}.$$

and

$$(A \otimes x)^T = x^T \otimes A^T.$$

The result is now achieved by noticing that $\oplus$ and $\otimes$ are non-decreasing w.r.t. $\leq$. ■

In the following theorem, using Haar's Lemma (see Result 1) we give sufficient condition for the A-invariance of the set $\mathcal{E}(P, w, \alpha)$.

Theorem 1 *Let us assume that the set $\mathcal{E}(P,w,\alpha)$ is not empty. Let also $A \in \mathbb{S}^{d\times d}$. The set $\mathcal{E}(P,w,\alpha)$ defined by (10) is positively invariant by the linear mapping $x \mapsto A\otimes x$ or A-invariant if there exists a $d^2\times d^2$-matrix H such that the following conditions hold:*

$$\begin{array}{l}(a).\ H \geq \mathbf{0}_{d^2\times d^2} \text{ (elementwise)},\\ (b).\ Q = HQ \text{ (linear algebra)},\end{array} \tag{16}$$

$$(c).\ \forall i,j = 1,\dots,d;\quad \bigotimes_{k,l\in\{1,\dots,d\}} (w/P_{k,l})^{\otimes(H_{(i-1)d+j,(k-1)d+l})} \leq w/(A^T\otimes P\otimes A^{\otimes(\alpha)})_{i,j} \tag{17}$$

where Q denotes the $d^2\times d$-matrix defined by:

$$\forall i,j = 1,\dots d,\ Q_{(i-1)d+j,\cdot} = b_i^T + \alpha b_j^T, \tag{18}$$

with $(b_i)_{i=1,\dots,d}$ senotes the canonical basis of $\mathbb{R}^d$: $b_i = (\delta_{\{k=i\}})_{k=1,\dots,n}$; $\delta_{\{\cdot\}} = 1$ if assertion $\{\cdot\}$ is true and 0 otherwise.

Proof.
Based on Lemma 1, The assertion (15) given in Proposition 1 can be rewritten as the following sets inclusion:

$$\mathcal{P}(Q,p) \subseteq \mathcal{P}(Q,q),$$

where, by definition,

$$\mathcal{P}(Q,\cdot) = \{x\in\mathbb{R}^d | Qx \leq \cdot\}.$$

The matrix Q is defined by (18), and the d^2-dimensional vectors p and q are respectively defined by:
$\forall i,j = 1,\dots,d.$

$$p_{(i-1)d+j} = w/P_{i,j}, \tag{19}$$

$$q_{(i-1)d+j} = w/(A^T\otimes P\otimes A^{\otimes(\alpha)})_{i,j}. \tag{20}$$

Using Haar's Lemma (cf. Result 1), with in our case $\Phi = \Psi = Q$ in assertion (1), we obtain the conditions $(a),(b)$ and (c), and the proof is now achieved. ■

- α **is a positive real number (**$\alpha\in\mathbb{R}_+$**)**

Remark 1 *In the case where α is a non-negative real number, the sufficient condition of Theorem 1 is also necessary because :*

$$\forall x\in\mathbb{S}^d, \forall\alpha\in\mathbb{R}_+;\ (A\otimes x)^{\otimes(\alpha)} = A^{\otimes(\alpha)}\otimes x^{\otimes(\alpha)}.$$

3.2 $(A\oplus B\otimes F)$-invariance

Let us consider now the following linear system over a complete idempotent semiring $(\mathbb{S},\oplus,\otimes,\varepsilon,e)$ defined by:

$$\begin{cases} x(0)\in\mathbb{S}^d,\\ x(n) = A\otimes x(n-1)\oplus B\otimes u(n), \quad \forall n\geq 1\\ u(n) = F\otimes x(n-1),\end{cases} \tag{21}$$

with $A\in\mathbb{S}^{d\times d}$, $B\in\mathbb{S}^{d\times m}$ and $F\in\mathbb{S}^{m\times d}$ to be determined (Any F is called a feedback) such that the following assertion to be true:

$$\forall x(0)\in\mathbb{S}^d,\quad \begin{array}{l}[x(0)\in\mathcal{E}(P,w,\alpha)\Rightarrow\\ \forall n\in\mathbb{N}, x(n)\in\mathcal{E}(P,w,\alpha)]\end{array}, \tag{22}$$

where $\mathcal{E}(P,w,\alpha)$ is the set previously defined by (10). It means that if the state of the system at time instant 0 is in the set $\mathcal{E}(P,w,\alpha)$, one is sure with applying an adequate control u that this state will remain within the set $\mathcal{E}(P,w,\alpha)$ at time instant n.

Notice that (21) can be rewritten as follows:

$$\begin{cases} x(0)\in\mathbb{S}^d,\\ x(n) = (A\oplus B\otimes F)\otimes x(n-1),\end{cases} \tag{23}$$

If the system (23) verifies the condition (22), we will then introduce the concept of the $(A\oplus B\otimes F)$-invariance of the set $\mathcal{E}(P,w,\alpha)$. For more details, see e.g. (Dorea, 1997), (Lhommeau, 2003), (Castelan and al., 1993).

3.2.1 Existence Condition of Max-plus Linear State Feedback Control

By definition of the system (21), the condition (22) is equivalent to say that there exists a feedback $F\in\mathbb{S}^{m\times d}$ such that the set $\mathcal{E}(P,w,\alpha)$ is $(A\oplus B\otimes F)$-invariant. In the following theorem, we give a sufficient condition for the $(A\oplus B\otimes F)$-invariance of the set $\mathcal{E}(P,w,\alpha)$.

Theorem 2 *Let us assume that the set $\mathcal{E}(P,w,\alpha)$ is not empty. The set $\mathcal{E}(P,w,\alpha)$ is $(A\oplus B\otimes F)$-invariant if there exists a $d^2\times d^2$-matrix H such that the following conditions hold:*

$$\begin{array}{l}(a).\ H \geq \mathbf{0}_{d^2\times d^2} \text{ (elementwise)},\\ (b).\ Q = HQ \text{ (linear algebra)},\end{array} \tag{24}$$

$$(c).\ \forall i,j = 1,\dots,d;\quad \bigotimes_{k,l\in\{1,\dots,d\}} (w/P_{k,l})^{\otimes(H_{(i-1)d+j,(k-1)d+l})} \leq w/((A\oplus B\otimes F)^T\otimes P\otimes(A\oplus B\otimes F)^{\otimes(\alpha)})_{i,j} \tag{25}$$

where matrix Q is defined by (18).

Proof.
We just have to apply the result of Theorem 1 with matrix A replaced by matrix $(A\oplus B\otimes F)$. ■

From this sufficient condition of the existence of static state feedback control law we can elaborate a methodology based on linear programming to compute F.

3.2.2 Computation of the Max-plus Linear State Feedback Control

In this subsection, we give a method divided in three main steps to compute all possible F such that the following condition holds true:

$$\forall x(0) \in \mathbb{S}^d, \quad [x(0) \in \mathcal{E}(P,w,\alpha) \Rightarrow \forall n, \\ (A \oplus B \otimes F) \otimes x(n) \in \mathcal{E}(P,w,\alpha)].$$

Step I:
We first compute the d^2-dimensional vector $\hat{q}$ by solving (see Theorem 1):

$$\begin{cases} \forall i,j = 1,\ldots,d; \;\; \hat{q}_{(i-1)d+j} \geq (H_{(i-1)d+j,.})\, p \\ \text{under:} \\ \quad (H_{(i-1)d+j,.})\, Q = Q_{(i-1)d+j,\cdot}, \\ \quad (H_{(i-1)d+j,\cdot} \geq \mathbf{0} \text{ (elementwise)}, \end{cases} \tag{26}$$

because the left term of inequality (25):

$$\bigotimes_{k,l \in \{1,\ldots,d\}} (w/P_{k,l})^{\otimes(H_{(i-1)d+j,(k-1)d+l})}$$

can be rewritten in linear algebra as:

$$\forall i,j = 1,\ldots,d; (H_{(i-1)d+j,\cdot})\, p,$$

with p and Q are respectively defined by (19) and (18).

The resolution of the system (26) can be done using for example the simplex algorithm proposed by Dantzig in 1947 (see e.g. (Schrijver, 1986)).

Step II:
Let us consider now an unknown $d \times d$-dimensional matrix G. We denote $q(G)$ the d^2-dimensional vector defined by (20, with matrix A replaced by matrix G). Then, we have to find the solutions in G of the following inequality:

$$\hat{q} \leq q(G),$$

where $\hat{q}$ is defined by the systems of inequalities (26). This is equivalent to find the solutions of the following system of d^4-inequalities:

$$(i).\ \forall i,j,k,l = 1,\ldots,d, \\ G_{k,i} \otimes G_{l,j}^{\otimes\alpha} \leq (w/P_{k,l})/\hat{q}_{(i-1)d+j}.$$

This last system of inequalities can be rewritten in linear algebra as:

$$Q^{'} g \leq \psi \tag{27}$$

with ψ is the d^4-dimensional vector defined by:

$$(ii).\ \forall i,j,k,l = 1,\ldots d, \\ \psi_{(i-1)d^3+(j-1)d^2+(k-1)d+l} = (w/P_{k,l})/\hat{q}_{(i-1)d+j}, \tag{28}$$

and g is the d^2-dimensional vector associated to the unknown matrix G and defined by:

$$(iii).\ \forall i,j = 1,\ldots d,\; g_{(i-1)d+j} = G_{i,j}, \tag{29}$$

and $Q^{'}$ is the $d^4 \times d^2$-dimensional matrix defined by:

$$(iv).\ \forall i,j,k,l = 1,\ldots d, \\ Q^{'}_{(i-1)d^3+(j-1)d^2+(k-1)d+l,\cdot} = b^T_{(k-1)d+i} + \alpha b^T_{(l-1)d+j}\;, \tag{30}$$

where $(b_m)_{m=1,\ldots,d^2}$ denotes the canonical basis of $\mathbb{R}^{d^2}$.

The problem is now to enumerate all possible solutions in g of the system of inequalities (27). This, can be done by using the Γ-algorithm (Castillo and al., 1999) pp. 161-162.

Step III:
The final step is to find all possible feedback F such that:

$$A \oplus B \otimes F = G, \tag{31}$$

where A, B are given matrices. The matrix $G \in \mathbb{S}^{d\times d}$ is computed by using of the formulæ(29) from a solution g of the system of inequality (27).

In the case of the assertion (31), using elements of residuation theory (see for example (Baccelli and al., 1992)), it can be shown that F is a solution of (31) iff

$$\begin{cases} A \leq G \\ B \otimes (B \backslash G) = G \end{cases}.$$

Example 1 *Let us consider the following linear system given over a complete idempotent semiring* $(\mathbb{S},\oplus,\otimes,\varepsilon,e) = (\mathbb{R}\cup\{-\infty,+\infty\},\max,+,-\infty,0)$*:*

$$\begin{cases} x(0) \in \mathbb{S}^2, \\ x(n) = A \otimes x(n-1) \oplus \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \otimes u(n), \\ u(n) = F \otimes x(n-1), \end{cases}$$

Let us consider also the following set:

$$\begin{cases} \mathcal{E}(P,w,\alpha) = \{x \in \mathbb{S}^d |\, x^T \otimes \underbrace{\begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}}_{P} \otimes x^{\otimes(\alpha)} \leq w\}, \end{cases}$$

with $\alpha = -1$ *and* $w = 9$.

Now we try to find the feedback F such the set $\mathcal{E}(P,w,\alpha)$ is $(A \oplus B \otimes F)$-invariant.

STEP I:
Using (19) and (18) we obtain:

$$Q = \begin{bmatrix} 1+\alpha & 0 \\ 1 & \alpha \\ \alpha & 1 \\ 0 & 1+\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$p = \begin{bmatrix} 9 \\ 1 \\ 9 \\ 9 \end{bmatrix}.$$

Using condition (b) of Theorem 1, we obtain for example:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and by using (26) we obtain:

$$\hat{q} = \begin{bmatrix} 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}.$$

STEP II:
Using (30):

$$Q' = \begin{bmatrix} 1+\alpha & 0 & 0 & 0 \\ 1 & 0 & \alpha & 0 \\ \alpha & 0 & 1 & 0 \\ 0 & 0 & 1+\alpha & 0 \\ 1 & \alpha & 0 & 0 \\ 1 & 0 & 0 & \alpha \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & 1 & \alpha \\ \alpha & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ \alpha & 0 & 0 & 1 \\ 0 & 0 & \alpha & 1 \\ 0 & 1+\alpha & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & \alpha & 0 & 1 \\ 0 & 0 & 0 & 1+\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Using (28), the column vector ψ is

$$\psi = \begin{bmatrix} 9 \\ 1 \\ 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 8 \\ 0 \\ -8 \\ 0 \\ 0 \\ 9 \\ 1 \\ 9 \\ 9 \end{bmatrix}.$$

Using (27):
The set of all solutions in g is:

$$g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 13 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 12 \\ 8 \end{bmatrix}, \ldots \right\}$$

For example, taking

$$g = \begin{bmatrix} 3 \\ 1 \\ 9 \\ 3 \end{bmatrix},$$

and then by (29):

$$G = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}.$$

STEP III:
If the matrix A of the system satisfied the condition $A \leq G$, we can compute the matrix F using residuation theory, then the equation (31) for example admits the solution

$$F = B \backslash G = [\ 3 \quad 1\ ].$$

4 CONCLUSION

In this paper we identified and characterized the positive invariance of max-plus ellipsoidal set. Based on Haar's lemma (cf. Result 1)(also called an extension of Farkas' Lemma due to (Hennet, 1989)), sufficient condition for the A-invariance (cf. Theorem 1) and the $(A \oplus B \otimes F)$-invariance (cf. Theorem 2) are provided. This sufficient condition leads us to provide a methodology for computing and enumerating all possible max-plus linear state feedback control laws.

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