STABILITY OF TAKAGI-SUGENO FUZZY SYSTEMS

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Abstract: Takagi-Sugeno (T-S) fuzzy models are usually used to describe nonlinear systems by a set of IF-THEN rules that gives local linear representations of subsystems. The overall model of the system is then formed as a fuzzy blending of these subsystems. It is important to study their stability or the synthesis of stabilizing controllers. The stability of TS models has been derived by means of several methods: Lyapunov approach, switching systems theory, linear system with modeling uncertainties, etc. In this study, the uniform stability, and uniform exponential stability of a discrete time T-S model is examined. Moreover, a perturbation result and an instability condition are given. The subsystems of T-S models that is studied here are time varying and a new exponential stability theorem is given for these types of TS models by examining the existence of a common matrix sequence.

1 INTRODUCTION

Fuzzy systems can approximate a wide class of nonlinear systems as accurately as required with some number of fuzzy IF-THEN rules. They are known as universal approximators, and their use offers many advantages (L.X.Wang, 1996). Stability is the most important concept for analysis and design of a control system. Stability analysis of fuzzy systems has been difficult because fuzzy systems are essentially nonlinear systems (Tanaka 1996, Calcev, 1998, Kim, 2001). The issue of the stability of fuzzy control systems has been studied using nonlinear stability frameworks (Tanaka, 1990).

Takagi-Sugeno (T-S) fuzzy models (Takagi, 1985) are nonlinear systems in nature. In this type of fuzzy model the consequent part of a fuzzy rule is a mathematical formula, representing local dynamics in different state space regions (subsystems) as linear input-output relations (Tanaka, 1996). Thus, T-S fuzzy systems are considered as a weighted average of the values in the consequent parts of the fuzzy rules. The overall model of the system is consequently a fuzzy blending of these subsystems.

Recently, fuzzy control and modeling is being used in many practical industrial applications. One of the first questions to be answered is the stability of the fuzzy system. Tanaka and Sugeno (Tanaka, 1992), have provided a sufficient condition for the asymptotic stability of a fuzzy system in the sense of Lyapunov through the existence of a common Lyapunov function for all the subsystems.

A system is said to be stable in the sense of Lyapunov if its trajectories can be made arbitrarily close to the origin for any initial starting state. When a system is stable and initial states that are close to the region of origin converge to the origin, the system has asymptotic stability. A stable system in Lyapunov sense does not guarantee asymptotic stability because asymptotic stability is stricter than Lyapunov stability.

Additionally, one needs to know how fast the system converges to the equilibrium point. On the other hand, exponential stability is used to estimate how fast the system trajectory approaches and converges to the equilibrium point as time goes to infinity. Since exponential stability is stricter than asymptotic stability it guarantees both Lyapunov stability and asymptotic stability but not vice versa.

The preliminaries were presented in Section 2. Section 3 discusses the main results on the uniform stability, uniform exponential stability and instability. Moreover, a perturbation result is presented. Finally, Section 4 contains some concluding remarks.

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2 PRELIMINARIES

A T-S fuzzy model of a plant with r rules can be represented as

Plant Rule i:
IF $x_1(t)$ is M_{i1} *AND* ... *AND* $x_g(t)$ is M_{ig}
THEN $\delta x(t) = A_i x(t) + B_i u(t), \quad i = 1,...,r$

where x(t)$\in \mathbb{R}^n$ is the state vector, u(t)$\in \mathbb{R}^m$ is the control input, the matrices A_i and B_i are of appropriate dimensions, M_{ij} (j=1,2,…, n) is the j^{th} fuzzy set of the i^{th} rule, and, $x_k(t)$ (k=1,2,…,g) are the premise variables. It should be noted that $\delta x(t) = \dot{x}(t)$ for the continuous-time T-S fuzzy model and $\delta x(t) = x(t+1)$ for the discrete time T-S fuzzy model.

Given a pair of (x(t), u(t)), the resulting fuzzy system model is inferred as follows:

$$\delta \mathrm{x(t)} = \sum_{i=1}^{r} h_i(x(t))\{A_i x(t) + B_i u(t)\} \qquad (1)$$

where

$$w_i(x(t)) = \prod_{i=1}^{g} M_{ij}(x(t)), \qquad h_i(x(t)) = \frac{w_i(x(t))}{\sum_{i=1}^{r} w_i(x(t))},$$

$$x(t) = \left[x_1(t), \ ..., x_g(t)\right].$$

$M_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in M_{ij}. It is assumed that $w_i(x(t)) \geq 0$, i=1,…, r, $\sum_{i=1}^{r} w_i(x(t)) > 0$ for all t. Therefore, $h_i(x(t)) \geq 0$, $\sum_{i=1}^{r} h_i(x(t)) = 1$ for all t. Each linear consequent equation given by $A_i x(t) + B_i u(t)$ is called a subsystem. The free system of Eq.(1) is defined as

$$\delta x(t) = \frac{\sum_{i=1}^{r} w_i(x(t)) A_i x(t)}{\sum_{i=1}^{r} w_i(x(t))} \qquad (2)$$

In this paper, it is also assumed that A_i matrices are time varying, where the coefficients are real matrix sequences defined for all integer t, from $-\infty$ to $+\infty$. Therefore, the consequent part of each IF-THEN rule has a linear time varying state equation.

Tanaka and Sugeno (1992) proposed a theorem on the stability analysis of a T-S fuzzy model, which was an important breakthrough in the field of fuzzy control. They proved that finding a common symmetric positive definite matrix P for all the subsystems could show the stability of a T-S fuzzy model. This sufficient condition for ensuring stability of Eq.(2) is given as follows.

Theorem: The equilibrium of the continuous-time (discrete-time) T-S fuzzy model (namely, x=0) is globally asymptotically stable if there exists a common symmetric positive definite matrix P such that ($\forall$ i = 1, ..., r)

$$A_i^T P + P A_i < 0 \qquad (3)$$

$$(A_i^T P A_i - P < 0) \qquad (4)$$

Proof: See (Tanaka, 1992).

Note that Eq.(3) and Eq.(4) depends only on A_i. In other words, it does not depend on $w_i(x(t))$. It is clear that this theorem reduces to Lyapunov stability theorem for continuous-time (discrete-time) linear systems when r =1. It should be noted that the total system might not be stable even if every subsystem is stable. Eq.(3) and Eq.(4) are sufficient conditions for stability, but are not necessary conditions. To satisfy these conditions most of the time a trial and error type procedure has been used.

In recent years, the stability analysis and control design problems of fuzzy systems are reduced to linear matrix inequality (LMI) problems Numerically, LMI problems can be solved very efficiently using the interior point algorithms (Boyd, 1994). However, the problem of finding a common P matrix becomes a very difficult job even by the LMI method as the number of fuzzy rules increases.

Definition: The discrete-time linear state equation

$$x(t+1) = A(t)x(t), \quad x(t_o) = x_o \qquad (5)$$

is called uniformly stable if there exists a finite positive constant γ such that for any t_o and x_o the corresponding solution satisfies

$$\|x(t)\| \leq \gamma \|x_o\|, \quad t > t_o$$

Definition: Eq.(5) is called uniformly exponentially stable if there exists a finite positive constant γ and a constant 0≤λ<1 such that for any t_o and x_o the corresponding solution satisfies

$$\|x(t)\| \leq \gamma \lambda^{t-t_o} \|x_o\|, \quad t > t_o$$

It is called *uniform* when γ does not depend on the choice of initial time (Rugh, 1996).

Lemma: If P is a positive definite matrix such that $A^TPA-P<0$ and $B^TPB-P<0$ where A, B, and P are n×n matrices, then $A^TPB+B^TPA-2P<0$ (Jamshidi, 1997) .

3 MAIN RESULTS

Theorem: The equilibrium point of the fuzzy system

$$x(t+1) = \sum_{i=1}^{r} h_i(x(t))A_i(t)x(t) \qquad (6)$$

is uniformly exponentially stable if there exists a common n×n matrix sequence P(t) that for all t is symmetric and such that

$$\eta I \le P(t) \le \rho I \qquad (7)$$

$$A_i^T(t)P(t+1)A_i(t) - P(t) \le -\nu I, \quad i = 1,...,r \qquad (8)$$

where η, ρ and ν are positive constants.

Proof: Suppose P(t) satisfies the requirements of the theorem. Multiplying both sides of Eq. (7) and Eq. (8) by $x^T(t)$ and $x(t)$ for any t_o and x_o, one obtains the following relations for $t \ge t_0$

$$-\|x(t)\|^2 \le -\frac{1}{\rho}x^T(t)P(t)x(t) \qquad (9)$$

$$x^T(t+1)P(t+1)x(t+1) - x^T(t)P(t)x(t) \le -\nu\|x(t)\|^2 \qquad (10)$$

Furthermore, by the combination of Eq. (9) and Eq.(10) one gets

$$x^T(t+1)P(t+1)x(t+1) \le (1-\frac{\nu}{\rho})x^T(t)P(t)x(t) \qquad (11)$$

Then, substituting Eq.(6) in Eq.(11) one obtains

$$\frac{\sum_{i,j=1}^{r}\omega_i(x)\omega_j(x)x^T[A_i^T(t)P(t+1)A_j(t) - (1-\frac{\nu}{\rho})P(t)]\ x}{\sum_{i,j=1}^{r}\omega_i(x)\omega_j(x)} \le 0 \qquad (12)$$

Eq. (12) implies

$$\frac{\left[\sum_{i=1}^{r}\omega_i^2 x^T(A_i^T P(t+1)A_i - (1-\frac{\nu}{\rho})P(t))x\right]}{\sum_{i,j=1}^{r}\omega_i\omega_j} + \frac{\left[\sum_{i<j}^{r}\omega_i\omega_j x^T[A_i^T P(t+1)A_j + A_j^T P(t+1)A_i - 2(1-\frac{\nu}{\rho})P(t)]x\right]}{\sum_{i,j=1}^{r}\omega_i\omega_j} \le 0$$

(13)

Using Eq.(7) and Eq.(8) one obtains the following inequalities:

$$x^T[A_i^T(t)P(t+1)A_i(t) - (1-\frac{\nu}{\rho})P(t)]x(t) \le 0 \qquad (14)$$

$$x^T[A_j^T(t)P(t+1)A_j(t) - (1-\frac{\nu}{\rho})P(t)]x(t) \le 0 \qquad (15)$$

Using Lemma (Jamshidi, 1997), Eq.(14) and Eq.(15) the following relation

$$A_i^T(t)P(t+1)A_j(t) + A_j^T(t)P(t+1)A_i(t) - 2(1-\frac{\nu}{\rho})P(t) \qquad (16)$$

can be rewritten as

$$\begin{aligned} &-(A_i(t)-A_j(t))^T P(t+1)(A_i(t)-A_j(t)) + \\ &A_i^T(t)P(t+1)A_i(t) - (1-\frac{\nu}{\rho})P(t) + \\ &A_j^T(t)P(t+1)A_j(t) - (1-\frac{\nu}{\rho})P(t) \end{aligned}$$

(17)

Since P(t+1) is positive definite, it is obvious that

$$-(A_i(t)-A_j(t))^T P(t+1)(A_i(t)-A_j(t)) \le 0$$

and

$$A_i^T(t)P(t+1)A_i(t) - (1-\frac{\nu}{\rho})P(t) \le 0,\ \forall i,j\ .$$

It follows from Eq. (17)

$$A_i^T(t)P(t+1)A_j(t) + A_j^T(t)P(t+1)A_i(t) - 2(1-\frac{\nu}{\rho})P(t) \le 0$$

(18)

This proves that Eq. (13) is valid. It can be easily seen from Eq.(7) and Eq.(8) that ρ≥ν, so the following inequality can be stated

$$0 \le (1-\frac{\nu}{\rho}) < 1 \qquad (19)$$

Setting $\lambda^2 = 1 - \frac{\nu}{\rho}$ in Eq.(11) and iterating it for t≥t_o one obtains for t≥t_o,

$$x^T(t)P(t)x(t) \le \lambda^{2(t-t_o)} x_o^T P(t_o) x_o \qquad (20)$$

Using Eq. (7), the following expression can be obtained for t≥t_o,

$$\eta \|x(t)\|^2 \le \rho \lambda^{2(t-t_o)} \|x_o\|^2 \qquad (21)$$

If one divides both sides of Eq.(21) by η and takes the positive square root, the uniform exponential stability condition is obtained. □

Theorem: The fuzzy system given in Eq.(6) is uniformly stable if there exists a matrix sequence P(t) that for all t is symmetric and such that

$$\eta I \le P(t) \le \rho I$$
$$A_i^T(t)P(t+1)A_i(t) - P(t) \le 0 \quad, i = 1, ..., r$$

where η and ρ are finite positive constants.

Theorem: Suppose the fuzzy system given in Eq.(6) is uniformly exponentially stable. Then there exists a positive constant δ such that if $\|\Delta A_i(t)\| \le \delta$ for all t and i=1,…, r, then

$$x(t+1) = \sum_{i=1}^{r} h_i(x(t))[A_i(t) + \Delta A_i(t)]x(t)$$

is uniformly exponentially stable.

Theorem: Suppose there exists a matrix sequence P(t) which for all t is symmetric and such that

$$\|P(t)\| \le \rho$$
$$A_i^T(t)P(t+1)A_i(t) - P(t) \le -\eta I \qquad i = 1, ..., r$$

where ρ and η are finite positive constants. Suppose that there exists an integer t_u such that P(t_u) is not positive semidefinite. Then the fuzzy system given in Eq.(6) is not uniformly stable.

4 CONCLUSIONS

The exponential stability is used to estimate how fast the system trajectory approaches and converges to the equilibrium point as time goes to infinity, and it is stricter than asymptotic stability. Therefore, exponential stability guarantees both Lyapunov stability and asymptotic stability but not vice versa.

In this study, some theorems for the stability and instability of the Takagi-Sugeno fuzzy systems are introduced. The consequent part of each T-S rule studied here are time varying. The uniform stability and uniform exponential stability theorems are given for these types of T-S models by examining the existence of a common matrix sequence. Moreover, a perturbation result is presented.

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