LINEAR QUADRATIC GAUSSIAN REGULATORS FOR MULTI-RATE SAMPLED-DATA STOCHASTIC SYSTEMS

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Abstract: In this paper, linear quadratic Gaussian regulators are presented and formalized for multi-rate sampled-data stochastic systems using two well-known approaches: lifting technique and time-variant periodic modeling. It has been demonstrated that both regulators are equivalent at the global frame-period with different computational costs and execution periods. An interesting analysis has been done to demonstrate the convergence of a periodic Kalman filter, used in the periodic regulator, into its equivalent continuous one (Bucy Kalman filter), when the periodicity ratio converges to infinity. In addition to this, in both regulators, multi-rate holds have been used, acting as interfaces between signals at different sampling rates, which may improve the system performance. A numerical example of LQG multi-rate control of a MIMO plant shows the application of both regulators, where in addition to showing the improvement with respect to the single-rate case.

1 INTRODUCTION

In complex real-time control systems, it becomes more and more realistic to assume different sampling periods for different system variables. This is the case of many distributed control systems, where there are more than one processor and many communication channels involved. Moreover, sensor and actuators with different dynamics should be treated at different rates. For all these situations, multi-rate techniques may be used to improve system performances.

In multi-rate sampled-data systems, inputs and outputs are updated at different sampling rates. In most of the cases, multi-rate sampling is due to technological limitations in sensors and actuators. Other limitations may come from computational requirements in real-time applications such as multi-rate sensor fusion, data-missing, mapping, etc.

A general multi-rate sampling could be asynchronous and totally random, although it is generally accepted that a periodicity between sampling-rates of inputs and outputs exists. On the one hand, the m inputs are updated at T_{u_v}, with $v = 1, 2, \ldots, m$; on the other hand, the p outputs are sampled at T_{y_w}, with $w = 1, 2, \ldots, p$. The frame-period is formally defined as $\bar{T} = \mathrm{lcm}(T_{u_v}, T_{y_w})$ which involves all input and output periods, and the base-period is $T = \gcd(T_{u_v}, T_{y_w})$. The ration between periods is $N = \bar{T}/T$, where N is the periodicity of the system. Therefore, inputs and outputs are updated/sampled at multiple time instants of the base-period, with $T_{u_v} = N_{u_v} \cdot T$ and $T_{y_w} = N_{y_w} \cdot T$.

The multi-rate problem has been extensively treated in the last four decades and it is possible to find many contributions dealing with modeling and analysis (Albertos, 1990; Araki and Yamamoto, 1986; Godbout et al., 1990; Khargonekar et al., 1985; Tornero, 1985; Tornero and Armesto, 2003; Tornero et al., 1999), as well as control design of multi-rate systems (Chen and Francis, 1995; Colaneri and de Nicolao, 1995; Qui and Chen, 1999; Tangirala et al., 1999). One of the approaches to treat the modeling phase is to assume an enlarged MIMO system (Khargonekar et al., 1985; Albertos, 1990; Araki and Yamamoto, 1986; Godbout et al., 1990). In these approaches, a discrete time-invariant state equation expressed at frame period $\bar{T}$ is used, with enlarged input and output vectors. Many of these modeling techniques are based on the initial idea of VSD (Kranc, 1957). In (Tornero, 1985), Tornero proposed an interesting time-variant modeling technique for multi-rate systems expressed at base-period T, based on two auxiliary state vectors associated to inputs and outputs. In this approach, usually periodic, input and

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output sampling hold mechanisms are represented by two periodic and diagonal matrices. In (Tornero and Armesto, 2003), it is demonstrated that the approach is equivalent to many other well-known multi-rate modeling techniques.

In this paper, we focus on multi-rate linear quadratic Gaussian regulators, which has been previously treated in (Colaneri and de Nicolao, 1995; Tornero et al., 1999). In (Lee and Tomizuka, 2003), a Kalman filter is developed using the lifting technique. Shah et. al. (Shah et al., 1995) implemented a multi-rate formulation of the iterated EKF on a bioreactor. Kalman filters have also been used in multi-rate digital signal processing and filter banks (analysis/synthesis) (Hong, 1994). In (Zhang et al., 2004) a Kalman filter is obtained based on Itô-Volterra equations for continuous, multi-rate and randomly sampled measurements.

Techniques applied for multi-rate Kalman filter have been also extended to non-linear multi-rate filtering such as the Extended and Unscented Kalman filters (Julier et al., 2000; Julier and Uhlmann, 2002). In this sense, different multi-rate sensor fusion applications in mobile robotics have been already presented (Armesto et al., 2004; Armesto and Tornero, 2004).

In the paper, two different but equivalent multi-rate LQG regulators are presented, based on periodic (time-variant) and on lifting techniques (time-invariant), which can be summarized in figure 1. One of the most important contributions comes from the fact that multi-rate holds are integrated into LQG regulators. Another contribution is the demonstration of convergence of discrete-time Kalman filter to the continuous one (Bucy-Kalman), when the periodicity ratio goes to infinity. In addition, multi-rate holds are defined according to general primitive functions, which can generate conventional ZOH and FOH as well as others holds based on Bezier, exponential or sinusoidal functions (Armesto and Tornero, 2003).

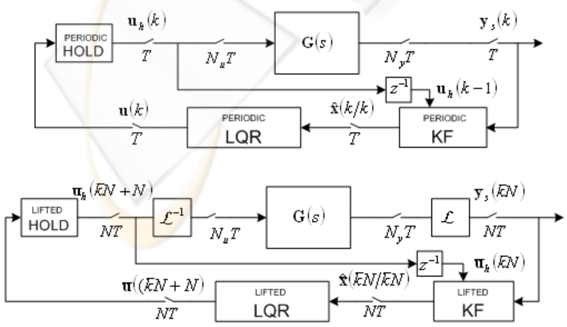

Figure 1: Multi-rate LQG control structures.

2 MULTI-RATE TIME-VARIANT PERIODIC LINEAR QUADRATIC GAUSSIAN REGULATOR

2.1 Sampled-data System Model

Suppose a continuous-time stochastic system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c \cdot \mathbf{x}(t) + \mathbf{B}_c \cdot \mathbf{u}(t) + \mathbf{G}_c \cdot \mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{v}(t)$$

where $\mathbf{A}_c \in \Re^{n \times n}$ is the state matrix, $\mathbf{B}_c \in \Re^{n \times m}$ input matrix and $\mathbf{C} \in \Re^{p \times n}$ output matrix, being n, m and p the dimensions of the state $\mathbf{x}(t)$, input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$, respectively; $\mathbf{G}_c \in \Re^{n \times g}$ is the system noise transmission matrix, coupling the system noises with the state; $\mathbf{w}(t) \in \Re^{g \times 1}$ and $\mathbf{v}(t) \in \Re^{p \times 1}$ are independent Gaussian Wiener processes with covariances $\mathbf{Q}_c \in \Re^{g \times g}$ and $\mathbf{R}_c \in \Re^{p \times p}$, respectively.

The sampled-data system with discrete inputs at base period is (Loan, 1978):

$$\mathbf{x}((k+1)T) = \mathbf{A}(T) \cdot \mathbf{x}(kT) + \mathbf{B}(T) \cdot \mathbf{u}(kT) + \mathbf{w}(kT) \quad (1)$$

$$\mathbf{y}(kT) = \mathbf{C} \cdot \mathbf{x}(kT) + \mathbf{v}(kT) \quad (2)$$

with,

$$\mathbf{A} = \mathbf{A}(T) = e^{\mathbf{A}_c T} \quad \mathbf{B} = \mathbf{B}(T) = \int_0^T e^{\mathbf{A}_c (T-\tau)} \mathbf{B}_c d\tau$$

$$\mathbf{w}(kT) = \int_0^T e^{\mathbf{A}_c (T-\tau)} \mathbf{G}_c \mathbf{w}(\tau - kT) d\tau$$

$$\mathbf{v}(kT) = \mathbf{C} \int_0^T \int_0^t e^{\mathbf{A}_c (T-\tau)} \mathbf{G}_c d\mathbf{w}(\tau - kT) dt + \int_0^T d\mathbf{v}(\tau - kT)$$

In the remainder of the paper, we use notation $\mathbf{x}(k) = \mathbf{x}(kT)$, $\mathbf{u}(k) = \mathbf{u}(kT)$, $\mathbf{y}(k) = \mathbf{y}(kT)$, $\mathbf{w}(k) = \mathbf{w}(kT)$ and $\mathbf{v}(k) = \mathbf{v}(kT)$, with discrete covariances $\mathbf{Q} = \mathbf{Q}(T)$ and $\mathbf{R} = \mathbf{R}(T)$ computed as (Colaneri and de Nicolao, 1995),

$$\mathbf{Q} = \int_0^T e^{\mathbf{A}_c (T-\tau)} \cdot \mathbf{G}_c \cdot \mathbf{Q}_c \cdot \mathbf{G}_c^T \cdot e^{\mathbf{A}_c^T (T-\tau)} d\tau$$

$$\mathbf{R} = \mathbf{C} \int_0^T \left\{ \left[\int_\tau^T e^{\mathbf{A}_c (s-\tau)} \mathbf{G}_c ds \right] \mathbf{Q}_c \left[\int_\tau^T \mathbf{G}_c^T e^{\mathbf{A}_c^T (s-\tau)} ds \right] \right\} d\tau \mathbf{C}^T + \mathbf{R}_c T$$

2.2 Multi-rate Time-Variant Periodic High Order Holds

Multi-rate high-order holds (MR-HOH) are used as multi-rate interfaces between signals at different sampling frequencies (Tornero and Tomizuka, 2002). Low-frequency signals are extrapolated to high frequency, usually at base-period. In (Armesto and Tornero, 2003), it was proposed a methodology for designing MR-HOHs based on primitive functions such as polynomial extrapolation, approximation functions (Bezier) and even non-polynomial

Table 1: Multi-rate High Order Holds Based on Polynomial Extrapolation Functions.

Name	Order	Primitive function $\mathbf{u}_h(t)$
ZOH	0	$\mathbf{u}(t_j)$
FOH	1	$\left(1+\frac{t-t_j}{t_j-t_{j-1}}\right)\cdot\mathbf{u}(t_j)-\frac{t-t_j}{t_j-t_{j-1}}\cdot\mathbf{u}(t_{j-1})$
NOH	n	$\sum_{l=0}^{n}\left[\prod_{\substack{q=0\\q\neq l}}^{n}\left[\frac{t-t_{j-q}}{t_{j-l}-t_{j-q}}\right]\cdot\mathbf{u}(t_{j-l})\right]$

Table 2: Multi-rate High Order Holds Based on Polynomial Extrapolation Discrete Functions.

Name	Order	Primitive function $\mathbf{u}_h(t)$
ZOH	0	$\mathbf{u}(jN_uT)$
FOH	1	$\left(1+\frac{i}{N_u}\right)\cdot\mathbf{u}(jN_uT)-\frac{i}{N_u}\cdot\mathbf{u}((j-1)N_uT)$
NOH	n	$\sum_{l=0}^{n}\left[\prod_{\substack{q=0\\q\neq l}}^{n}\left[\frac{i+qN_u}{(q-l)N_u}\right]\cdot\mathbf{u}((j-l)N_uT)\right]$

functions (exponential, sinusoidal, etc). In (Kabamba, 1987), other Generalized Sampled-data Hold Functions (GSHF) can be found.

Multi-rate sampled-data systems are obtained combining different MR-HOH and the continuous-time, providing different system performance (Braslavsky et al., 1998).

In Table 1 it can be found the primitive functions for polynomial extrapolation functions used to generate conventional ZOH, FOH and even the general case NOH (N-th Order Hold).

Primitive functions use signal samples at low frequency as control point for generating an extrapolated continuous-time signal,

$$\mathbf{u}_h(t)=\sum_{l=0}^{n}\mathbf{f}_{n,l}(t,t_j,t_{j-1},\ldots,t_{j-l})\cdot\mathbf{u}(t_{j-l})=\sum_{l=0}^{n}\mathbf{f}_{n,l}(\cdot)\mathbf{u}(t_{j-l})$$

where n is the order of the hold device and t_j is the time of the last updated input, with $t_j \leq t < t_{j+1}$.

The discrete function expressed at base-period with regular sampling is,

$$\mathbf{u}_h(k)=\sum_{l=0}^{n}\mathbf{f}^*_{n,l}(i,N_u)\cdot\mathbf{u}((j-l)N_uT)$$

where $t_j = j\cdot N_u\cdot T$, $t_j-t_{j-1}=N_u$ and $i=\mathrm{Mod}(k,N_u)=k-jN_u$. Table 2 shows the signal sampled at high frequency for ZOH, FOH and NOH.

In order to describe the multi-rate input sampling at base-period, a periodic diagonal matrix is defined (Tornero, 1985),

$$\mathbf{\Delta}_u(k)=\mathrm{diag}\{\delta^{u_v}(k), v=1,2,\ldots,m\}$$

$$\delta^{u_v}(k)=\begin{cases}1 & \text{if } \mathrm{Mod}(k,N_{u_v})=0\\ 0 & \text{otherwise}\end{cases}$$

According to this, the periodic representation for multi-rate high-order holds (Armesto and Tornero, 2003) is ,

$$\upsilon_u(k+1)=\mathbf{A}_h(k)\cdot\upsilon_u(k)+\mathbf{B}_h(k)\cdot\mathbf{u}(k) \quad (3)$$

$$\mathbf{u}_h(k)=\mathbf{C}_h(k)\cdot\upsilon_u(k)+\mathbf{D}_h(k)\cdot\mathbf{u}(k) \quad (4)$$

where $\upsilon_u(k)$ is an auxiliary state vector related to the input hold mechanism. The periodic multi-rate hold matrices are defined as,

$$\mathbf{A}_h(k)=\begin{bmatrix}\mathbf{I}-\mathbf{\Delta}_u(k) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{\Delta}_u(k) & \mathbf{I}-\mathbf{\Delta}_u(k) & \cdots & \mathbf{0}\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{\Delta}_u(k) & \mathbf{I}-\mathbf{\Delta}_u(k)\end{bmatrix} \quad \mathbf{B}_h(k)=\begin{bmatrix}\mathbf{\Delta}_u(k)\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}$$

$$\mathbf{C}_h(k)=\begin{bmatrix}\mathbf{f}^*_{n,0}(\cdot) & \mathbf{f}^*_{n,1}(\cdot) & \cdots & \mathbf{f}^*_{n,n}(\cdot)\end{bmatrix}\bullet[\mathbf{I}-\mathbf{\Delta}_u(k)]\mathbf{D}_h(k)=\mathbf{\Delta}_u(k)$$

Despite of the mathematical complexity, the periodic state-space representation for a MR-ZOH is simply,

$$\upsilon_u(k+1)=(\mathbf{I}-\mathbf{\Delta}_u(k))\cdot\upsilon_u(k)+\mathbf{\Delta}_u(k)\cdot\mathbf{u}(k)$$

$$\mathbf{u}_h(k)=(\mathbf{I}-\mathbf{\Delta}_u(k))\cdot\upsilon_u(k)+\mathbf{\Delta}_u(k)\cdot\mathbf{u}(k)$$

which means that the input $\mathbf{u}(k)$ is incorporated into the state on every update (at low frequency) and the output $\mathbf{u}_h(k)$ maintains this value until the next update. For a MR-FOH the formulation is,

$$\begin{bmatrix}\upsilon_u(k+1)\\ \upsilon_u(k-N_u+1)\end{bmatrix}=\begin{bmatrix}\mathbf{I}-\mathbf{\Delta}_u(k) & \mathbf{0}\\ \mathbf{\Delta}_u(k) & \mathbf{I}-\mathbf{\Delta}_u(k)\end{bmatrix}\cdot\begin{bmatrix}\upsilon_u(k)\\ \upsilon_u(k-N_u)\end{bmatrix}+\begin{bmatrix}\mathbf{\Delta}_u(k)\\ \mathbf{0}\end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{u}_h(k)=\begin{bmatrix}1+\frac{i}{N_u} & -\frac{i}{N_u}\end{bmatrix}\bullet[\mathbf{I}-\mathbf{\Delta}_u(k)]\begin{bmatrix}\upsilon_u(k)\\ \upsilon_u(k-N_u)\end{bmatrix}+\mathbf{\Delta}_u(k)\mathbf{u}(k)$$

where $\bullet$ represents the inner product. In this case, the present updated input and the last updated input are stored in the state and used to generate a ramp based on these values at the output. In section 3.1 it is shown a very simple numerical example of the transfer function of several holds.

2.3 Multi-rate Time-Variant Periodic Stochastic System

A general multi-rate periodic stochastic system can be seen as the concatenation of a multi-rate high order hold (3) and (4) and the discrete-time stochastic model (1) and (2) at base-period. Note that, we have expressed the multi-rate stochastic system in terms of k instead of $k+1$ because the LQG requires the estimation of $\mathbf{x}(k)$ in *real-time*, that is at the present sampling period k,

$$\mathbf{x}(k)=\mathbf{A}\cdot\mathbf{x}(k-1)+\mathbf{B}\cdot\mathbf{u}_h(k-1)+\mathbf{w}(k-1) \quad (5)$$

$$\mathbf{y}_s(k)=\mathbf{C}_s(k)\cdot\mathbf{x}(k)+\mathbf{v}_s(k) \quad (6)$$

where $\mathbf{y}_s(k)$ only contains measured outputs, $\mathbf{C}_s(k)$ is the row-reduced matrix of $\mathbf{C}$ and $\mathbf{v}_s(k)$ is the reduced noise vector, respectively. In the remainder of the paper, matrices and vectors affected by the multi-rate sampling will be also denoted with sub-index s. The size-varying output vector is defined as,

$$y_w(k)\subset\mathbf{y}_s(k)\in\Re^{p(k)\times 1} \text{ if } \delta^{y_w}(k)=1$$

$$\delta^{y_w}(k)=\begin{cases}1 & \text{if } \mathrm{Mod}(k,N_y)=0\\ 0 & \text{otherwise}\end{cases}$$

being $p(k)$ the number of sampled outputs at a given time instant k. In fact, the complete state-space representation of the multi-rate system including the hold is,

$$\mathbf{x}_{\mathrm{MR}}(k)=\mathbf{A}_{\mathrm{MR}}(k-1)\mathbf{x}_{\mathrm{MR}}(k-1)+\mathbf{B}_{\mathrm{MR}}(k-1)\mathbf{u}(k-1)+\begin{bmatrix}\mathbf{w}(k-1)\\ \mathbf{0}\end{bmatrix}$$

$$\mathbf{y}_s(k)=\mathbf{C}_{\mathrm{MR}}(k)\cdot\mathbf{x}_{\mathrm{MR}}(k)+\mathbf{v}_s(k)$$

where $\mathbf{x}_{\mathrm{MR}}(k) = [\mathbf{x}(k)\ \upsilon_u(k)]^T$ and,

$$\mathbf{A}_{\mathrm{MR}}(k) = \begin{bmatrix} \mathbf{A} & \mathbf{BC}_h(k) \\ \mathbf{0} & \mathbf{A}_h(k) \end{bmatrix}\ \mathbf{B}_{\mathrm{MR}}(k) = \begin{bmatrix} \mathbf{BD}_h(k) \\ \mathbf{B}_h(k) \end{bmatrix}\ \mathbf{C}_{\mathrm{MR}}(k) = [\mathbf{C}_s(k)\ \mathbf{0}]$$

2.4 Multi-rate Time-Variant Periodic Kalman Filter

According to the periodic multi-rate stochastic system (5) and (6), we define the periodic estimation model,

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}\cdot\hat{\mathbf{x}}(k-1|k-1) + \mathbf{B}\cdot\mathbf{u}_h(k-1)$$
$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}_s(k)\cdot(\mathbf{y}_s(k) - \mathbf{C}_s(k)\cdot\hat{\mathbf{x}}(k|k-1))$$

Note that the state of the hold is not included, since it is completely known. Therefore, input of the estimation model is the output of the hold. In this sense, equations related with the Kalman Gain $\mathbf{K}_s(k)$ and output covariance matrix $\mathbf{S}_s(k)$ are affected by the multi-rate output sampling, while the equation related with the state covariance prediction $\mathbf{P}(k|k-1)$ is not affected. Therefore, the multi-rate periodic Kalman filter (MR-PKF) equations are,

$$\mathbf{P}(k|k-1) = \mathbf{A}\cdot\mathbf{P}(k-1|k-1)\cdot\mathbf{A}^T + \mathbf{Q}$$
$$\mathbf{K}_s(k) = \mathbf{P}(k|k-1)\cdot\mathbf{C}_s^T(k)\cdot\mathbf{S}_s^{-1}(k)$$
$$\mathbf{S}_s(k) = \mathbf{C}_s(k)\cdot\mathbf{P}(k|k-1)\cdot\mathbf{C}_s^T(k) + \mathbf{R}_s(k)$$
$$\mathbf{P}(k|k) = \mathbf{P}(k|k-1) - \mathbf{K}_s(k)\cdot\mathbf{C}_s(k)\cdot\mathbf{P}(k|k-1)$$

Special attention requires the case $p(k)=0$, where outputs are not sampled. The output vector is void and also each vector/matrix denoted with sub-index s. It is not possible to correct the state and its covariance, and they both are simply predicted.

2.5 Multi-rate Time-Variant Periodic Linear Quadratic Regulator

Suppose a single-rate index as follows,

$$J_{SR} = \frac{1}{2}\sum_{k=0}^{\infty} \left[\mathbf{x}^T(k)\quad \mathbf{u}^T(k)\right]\cdot\begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{M}} \\ \tilde{\mathbf{M}}^T & \tilde{\mathbf{R}} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix} \quad (7)$$

with $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{R}}$ weighting matrices for states and inputs previously computed from a discrete-time continuous-time index,

$$J_c = \frac{1}{2}\int_0^{\infty} \left[\mathbf{x}^T(t)\quad \mathbf{u}^T(t)\right]\cdot\begin{bmatrix} \mathbf{Q}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_c \end{bmatrix}\cdot\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \quad (8)$$

with,

$$\tilde{\mathbf{Q}} = \int_0^T e^{\mathbf{A}_c^T t}\tilde{\mathbf{Q}}_c e^{\mathbf{A}_c t} dt$$
$$\tilde{\mathbf{M}} = \int_0^T e^{\mathbf{A}_c^T t}\tilde{\mathbf{Q}}_c\left[\int_0^t e^{\mathbf{A}_c \tau}\mathbf{B}_c d\tau\right] dt$$
$$\tilde{\mathbf{R}} = \int_0^T \left[\int_0^t \mathbf{B}_c^T e^{\mathbf{A}_c^T \tau} d\tau\right]\tilde{\mathbf{Q}}_c\left[\int_0^t e^{\mathbf{A}_c \tau}\mathbf{B}_c d\tau\right] dt + \tilde{\mathbf{R}}_c\cdot T$$

The multi-rate periodic equivalent index to the single-rate one considering the hold is,

$$J_{\mathrm{MR}} = \frac{1}{2}\sum_{k=0}^{\infty} [\mathbf{x}^T(k)\ \mathbf{u}_h^T(k)]\begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{M}} \\ \tilde{\mathbf{M}}^T & \tilde{\mathbf{R}} \end{bmatrix}\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}_h(k) \end{bmatrix}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty} [\mathbf{x}_{\mathrm{MR}}^T(k)\ \mathbf{u}(k)]\begin{bmatrix} \mathbf{Q}_{\mathrm{MR}}(k) & \mathbf{M}_{\mathrm{MR}}(k) \\ \mathbf{M}_{\mathrm{MR}}^T(k) & \mathbf{R}_{\mathrm{MR}}(k) \end{bmatrix}\begin{bmatrix} \mathbf{x}_{\mathrm{MR}}(k) \\ \mathbf{u}(k) \end{bmatrix}$$

where,

$$\mathbf{Q}_{\mathrm{MR}}(k) = \begin{bmatrix} \tilde{\mathbf{Q}} & \tilde{\mathbf{M}}\mathbf{C}_s(k) \\ \mathbf{C}_s^T(k)\tilde{\mathbf{M}} & \mathbf{C}_s^T(k)\tilde{\mathbf{R}}\mathbf{C}_s(k) \end{bmatrix}\ \mathbf{M}_{\mathrm{MR}}(k) = \begin{bmatrix} \tilde{\mathbf{M}}\mathbf{D}_h(k) \\ \mathbf{C}_s^T(k)\tilde{\mathbf{R}}\mathbf{C}_s(k) \end{bmatrix}$$
$$\mathbf{R}_{\mathrm{MR}}(k) = \mathbf{D}_h^T(k)\tilde{\mathbf{R}}\mathbf{D}_h(k)$$

The optimal multi-rate control input is obtained from the solution of the N-Periodic Riccati equation (Tornero et al., 1999),

$$\tilde{\mathbf{A}}_{\mathrm{MR}}(k) = \mathbf{A}_{\mathrm{MR}}(k) - \mathbf{B}_{\mathrm{MR}}(k)\cdot\mathbf{R}_{\mathrm{MR}}^{-1}(k)\cdot\mathbf{M}_{\mathrm{MR}}^T(k)$$
$$\mathbf{S}_{\mathrm{MR}}(k) = \tilde{\mathbf{A}}_{\mathrm{MR}^T}(k)\cdot\mathbf{S}_{\mathrm{MR}}(k+1)\cdot[\tilde{\mathbf{A}}_{\mathrm{MR}}(k) + \mathbf{K}_{\mathrm{MR}}(k))] +$$
$$+ \mathbf{Q}_{\mathrm{MR}}(k) - \mathbf{M}_{\mathrm{MR}}(k)\cdot\mathbf{R}_{\mathrm{MR}}^{-1}(k)\cdot\mathbf{M}_{\mathrm{MR}}^T(k)$$
$$\mathbf{K}_{\mathrm{MR}}(k) = [\mathbf{R}_{\mathrm{MR}}(k) + \mathbf{B}_{\mathrm{MR}}^T(k)\cdot\mathbf{S}_{\mathrm{MR}}(k+1)\cdot\mathbf{B}_{\mathrm{MR}}(k)]^{-1}\cdot$$
$$\cdot\mathbf{B}_{\mathrm{MR}}(k)\cdot\mathbf{S}_{\mathrm{MR}}(k+1)\cdot\tilde{\mathbf{A}}_{\mathrm{MR}}(k)$$
$$\mathbf{u}(k) = -(\mathbf{R}_{\mathrm{MR}}^{-1}(k)\cdot\mathbf{M}_{\mathrm{MR}}^T(k) + \mathbf{K}_{\mathrm{MR}}(k))\cdot\mathbf{x}(k)$$

Remark 1 $\mathbf{R}_{\mathrm{MR}}(k)$ *is singular if non one input is updated. This apparent problem can be solved since elements related with non updated inputs do not affect to the index (Colaneri and de Nicolao, 1995).*

3 MULTI-RATE LIFTED LINEAR QUADRATIC GAUSSIAN REGULATOR

3.1 Multi-rate Lifted High Order Holds

The lifting technique for the dual-rate HOH gives the lifted transfer function expressed at frame-period,

$$\bar{\mathbf{G}}_h(\bar{z}) = [\mathbf{I}\ \sum_{l=0}^{n}\mathbf{f}_{n,l}^*(1)\cdot\bar{z}^{-l}\ \ldots\ \sum_{l=0}^{n}\mathbf{f}_{n,l}^*(N-1)\cdot\bar{z}^{-l}]^T$$

where inputs are updated once per frame-period. This transfer function is derived from the state-space representation is as follows,

$$\bar{\upsilon}_u(\bar{k}N+N) = \bar{\mathbf{A}}_h\cdot\bar{\upsilon}_u(\bar{k}N) + \bar{\mathbf{B}}_h\cdot\bar{\mathbf{u}}(\bar{k}N)$$
$$\bar{\mathbf{u}}_h(\bar{k}N) = \bar{\mathbf{C}}_h\cdot\bar{\upsilon}_u(\bar{k}N) + \bar{\mathbf{D}}_h\cdot\bar{\mathbf{u}}(\bar{k}N)$$

where,

$$\bar{\mathbf{G}}_h(\bar{z}) = \bar{\mathbf{C}}_h\ (\bar{z}\mathbf{I} - \bar{\mathbf{A}}_h)^{-1}\ \bar{\mathbf{B}}_h + \bar{\mathbf{D}}_h$$

$$\bar{\mathbf{A}}_h = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}\quad \bar{\mathbf{B}}_h = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

$$\bar{\mathbf{C}}_h = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_{n,1}^*(1) & \cdots & \mathbf{f}_{n,n}^*(1) & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{f}_{n,1}^*(N-1) & \cdots & \mathbf{f}_{n,n}^*(N-1) & \mathbf{0} \end{bmatrix}\quad \bar{\mathbf{D}}_h = \begin{bmatrix} \mathbf{I} \\ \mathbf{f}_{n,0}^*(1) \\ \vdots \\ \mathbf{f}_{n,0}^*(N-1) \end{bmatrix}$$

Table 3: Numerical Examples of Lifted Transfer Function of MR-HOHs.

N	DR-ZOH	DR-FOH	DR-SOH
1	$\mathbf{G}_h = 1$	$\mathbf{G}_h = 1$	$\mathbf{G}_h = 1$
2	$\mathbf{G}_h = \begin{bmatrix}1\\1\end{bmatrix}$	$\mathbf{G}_h = \begin{bmatrix}1\\ \frac{3\bar{z}-1}{2\bar{z}}\end{bmatrix}$	$\mathbf{G}_h = \begin{bmatrix}1\\ \frac{15\bar{z}^2-10\bar{z}+3}{8\bar{z}^2}\end{bmatrix}$
3	$\mathbf{G}_h = \begin{bmatrix}1\\1\\1\end{bmatrix}$	$\mathbf{G}_h = \begin{bmatrix}1\\ \frac{4\bar{z}-1}{3\bar{z}}\\ \frac{5\bar{z}-2}{3\bar{z}}\end{bmatrix}$	$\mathbf{G}_h = \begin{bmatrix}1\\ \frac{14\bar{z}^2-7\bar{z}+2}{9\bar{z}^2}\\ \frac{20\bar{z}^2-16\bar{z}+5}{9\bar{z}^2}\end{bmatrix}$

Lifted MR-HOH for a periodic sampling can be obtained using successive substitution of equation (3) and (4). In that case, the lifted input vector is $\bar{\mathbf{u}}(\bar{k}N) = [\mathbf{u}^T(\bar{k}N-N)\ \ldots\ \mathbf{u}^T(\bar{k}N-2N_u)\ \mathbf{u}^T(\bar{k}N-N_u)]^T$. In order to show that despite of the mathematical complexity the implementation of MR-HOH is very simple, since at the end they are composed with simple transfer functions. In this sense, table 3 shows the lifted transfer functions of ZOH, FOH and SOH.

3.2 Multi-rate Lifted Stochastic System

A general multi-rate lifted stochastic system is obtained by combining multi-rate holds with the stochastic system,

$$\bar{\mathbf{x}}(\bar{k}N) = \bar{\mathbf{A}}\cdot\bar{\mathbf{x}}((\bar{k}-1)N) + \bar{\mathbf{B}}\cdot\bar{\mathbf{u}}_h(\bar{k}N) + \bar{\mathbf{G}}\cdot\bar{\mathbf{w}}(\bar{k}N)$$
$$\bar{\mathbf{y}}_s(\bar{k}N) = \bar{\mathbf{C}}_s\cdot\bar{\mathbf{x}}((\bar{k}-1)N) + \bar{\mathbf{D}}_s\cdot\bar{\mathbf{u}}_h(\bar{k}N) + \tilde{\mathbf{G}}_s\cdot\bar{\mathbf{w}}(\bar{k}N) + \bar{\mathbf{v}}_s(\bar{k}N)$$

with lifted signals,

$$\bar{\mathbf{w}}(\bar{k}N) = [\mathbf{w}^T(\bar{k}N-N)\ \ldots\ \mathbf{w}^T(\bar{k}N-2)\ \mathbf{w}^T(\bar{k}N-1)]^T$$
$$\bar{\mathbf{y}}_s(\bar{k}N) = [\mathbf{y}^T(\bar{k}N-N+N_y)\ \ldots\ \mathbf{y}^T(\bar{k}N-N_y)\ \mathbf{y}^T(\bar{k}N)]^T$$
$$\bar{\mathbf{v}}_s(\bar{k}N) = [\mathbf{v}^T(\bar{k}N-N+N_y)\ \ldots\ \mathbf{v}^T(\bar{k}N-N_y)\ \mathbf{v}^T(\bar{k}N)]^T$$

and,

$$\bar{\mathbf{A}} = \mathbf{A}^N$$
$$\bar{\mathbf{B}} = [\mathbf{A}^{N-1}\cdot\mathbf{B}\quad \mathbf{A}^{N-2}\cdot\mathbf{B}\quad \ldots\quad \mathbf{B}]$$
$$\bar{\mathbf{G}} = [\mathbf{A}^{N-1}\quad \mathbf{A}^{N-2}\quad \ldots\quad \mathbf{I}]$$
$$\bar{\mathbf{C}}_s = \begin{bmatrix}\mathbf{C}\cdot\mathbf{A}^{N_y}\\ \mathbf{C}\cdot\mathbf{A}^{2N_y}\\ \vdots\\ \mathbf{C}\cdot\mathbf{A}^{N}\end{bmatrix}$$
$$\bar{\mathbf{D}}_s = \begin{bmatrix}\mathbf{C}\cdot\mathbf{B} & \mathbf{0} & \ldots & \mathbf{0}\\ \mathbf{C}\cdot\mathbf{A}^{N_y}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{B} & \ldots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{C}\cdot\mathbf{A}^{N-N_y}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{A}^{N-2N_y}\cdot\mathbf{B} & \ldots & \mathbf{C}\cdot\mathbf{B}\end{bmatrix}$$
$$\tilde{\mathbf{G}}_s = \begin{bmatrix}\mathbf{C} & \mathbf{0} & \ldots & \mathbf{0}\\ \mathbf{C}\cdot\mathbf{A}^{N_y} & \mathbf{C} & \ldots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{C}\cdot\mathbf{A}^{N-N_y} & \mathbf{C}\cdot\mathbf{A}^{N-2N_y} & \ldots & \mathbf{C}\end{bmatrix}$$

In fact, the complete state-space representation, including the hold, is:

$$\bar{\mathbf{x}}_{\mathrm{MR}}(\bar{k}N) = \bar{\mathbf{A}}_{\mathrm{MR}}\bar{\mathbf{x}}_{\mathrm{MR}}((\bar{k}-1)N) + \bar{\mathbf{B}}_{\mathrm{MR}}\bar{\mathbf{u}}(\bar{k}N) + \begin{bmatrix}\bar{\mathbf{G}}\bar{\mathbf{w}}(\bar{k}N)\\ \mathbf{0}\end{bmatrix}$$
$$\bar{\mathbf{y}}_s(\bar{k}N) = \bar{\mathbf{C}}_{\mathrm{MR}}\bar{\mathbf{x}}_{\mathrm{MR}}((\bar{k}-1)N) + \bar{\mathbf{D}}_{\mathrm{MR}}\bar{\mathbf{u}}(\bar{k}N) + \tilde{\mathbf{G}}_s\bar{\mathbf{w}}(\bar{k}N) + \bar{\mathbf{v}}_s(\bar{k}N)$$

where $\bar{\mathbf{x}}_{\mathrm{MR}}(\bar{k}N) = [\bar{\mathbf{x}}(\bar{k}N)\ \bar{\upsilon}_u(\bar{k}N)]^T$ and,

$$\bar{\mathbf{A}}_{\mathrm{MR}} = \begin{bmatrix}\bar{\mathbf{A}} & \bar{\mathbf{B}}\cdot\bar{\mathbf{C}}_h\\ \mathbf{0} & \bar{\mathbf{A}}_h\end{bmatrix} \qquad \bar{\mathbf{B}}_{\mathrm{MR}} = \begin{bmatrix}\bar{\mathbf{B}}\cdot\bar{\mathbf{D}}_h\\ \bar{\mathbf{B}}_h\end{bmatrix}$$
$$\bar{\mathbf{C}}_{\mathrm{MR}} = [\bar{\mathbf{C}}_s\quad \bar{\mathbf{D}}_s\cdot\bar{\mathbf{C}}_h] \qquad \bar{\mathbf{D}}_{\mathrm{MR}} = \bar{\mathbf{D}}_s\cdot\bar{\mathbf{D}}_h$$

3.3 Multi-rate Lifted Kalman Filter (MR-LKF)

The lifted estimation model is,

$$\hat{\bar{\mathbf{x}}}(\bar{k}N|(\bar{k}-1)N) = \bar{\mathbf{A}}\cdot\hat{\bar{\mathbf{x}}}((\bar{k}-1)N|(\bar{k}-1)N) + \bar{\mathbf{B}}\cdot\bar{\mathbf{u}}_h(\bar{k}N)$$
$$\hat{\bar{\mathbf{x}}}(\bar{k}N|\bar{k}N) = \hat{\bar{\mathbf{x}}}(\bar{k}N|(\bar{k}-1)N) +$$
$$+\bar{\mathbf{K}}_s(\bar{k}N)\cdot(\bar{\mathbf{y}}_s(\bar{k}N) - \bar{\mathbf{C}}_s\cdot\hat{\bar{\mathbf{x}}}((\bar{k}-1)N|(\bar{k}-1)N) - \bar{\mathbf{D}}_s\cdot\bar{\mathbf{u}}(\bar{k}N))$$

where $\bar{\mathbf{K}}_s(\bar{k}N)$ is the Kalman correction matrix. Once again, the state of the hold is not estimated since it is assumed to be completely known.

Kalman filter equations for the lifted model are,

$$\bar{\mathbf{P}}(\bar{k}N|(\bar{k}-1)N) = \bar{\mathbf{A}}\cdot\bar{\mathbf{P}}((\bar{k}-1)N|(\bar{k}-1)N)\cdot\bar{\mathbf{A}}^T + \bar{\mathbf{G}}\cdot\bar{\mathbf{Q}}\cdot\bar{\mathbf{G}}^T$$
$$\bar{\mathbf{P}}_{xy,s}(\bar{k}N|(\bar{k}-1)N) = \bar{\mathbf{C}}_s\bar{\mathbf{P}}((\bar{k}-1)N|(\bar{k}-1)N)\bar{\mathbf{A}}^T + \tilde{\mathbf{G}}_s\bar{\mathbf{Q}}\bar{\mathbf{G}}^T$$
$$\bar{\mathbf{P}}_{yy,s}(\bar{k}N) = \bar{\mathbf{C}}_s\bar{\mathbf{P}}((\bar{k}-1|)N|(\bar{k}-1|)N)\bar{\mathbf{C}}_s^T + \tilde{\mathbf{G}}_s\bar{\mathbf{Q}}\tilde{\mathbf{G}}_s^T + \bar{\mathbf{R}}_s$$
$$\bar{\mathbf{K}}_s(\bar{k}N) = \bar{\mathbf{P}}_{xy,s}^T(\bar{k}N|(\bar{k}-1)N)\cdot\bar{\mathbf{P}}_{yy,s}^{-1}(\bar{k}N) \tag{9}$$
$$\bar{\mathbf{P}}(\bar{k}N|\bar{k}N) = \bar{\mathbf{P}}(\bar{k}N|(\bar{k}-1)N) - \bar{\mathbf{K}}_s(\bar{k}N)\cdot\bar{\mathbf{P}}_{xy,s}(\bar{k}N|(\bar{k}-1)$$

where the enlarged covariance matrices are,

$$\bar{\mathbf{Q}} = \mathrm{E}[\bar{\mathbf{w}}(\bar{k}N)\cdot\bar{\mathbf{w}}^T(\bar{k}N)] \quad \bar{\mathbf{R}}_s = \mathrm{E}[\bar{\mathbf{v}}_s(\bar{k}N)\cdot\bar{\mathbf{v}}_s^T(\bar{k}N)]$$

3.4 Multi-rate Lifted Linear Quadratic Regulator

The multi-rate lifted equivalent index to the single-rate one (8) considering the hold is,

$$\bar{J}_{\mathrm{MR}} = \frac{1}{2}\sum_{\bar{k}=0}^{\infty}[\bar{\mathbf{x}}_{\mathrm{MR}}^{\mathbf{T}}(\bar{k}N)\ \bar{\mathbf{u}}^{\mathbf{T}}(\bar{k}N)]\cdot\begin{bmatrix}\bar{\mathbf{Q}}_{\mathrm{MR}} & \bar{\mathbf{M}}_{\mathrm{MR}}\\ \bar{\mathbf{M}}_{\mathrm{MR}}^T & \bar{\mathbf{R}}_{\mathrm{MR}}\end{bmatrix}\cdot\begin{bmatrix}\bar{\mathbf{x}}_{\mathrm{MR}}(\bar{k}N)\\ \bar{\mathbf{u}}(\bar{k}N)\end{bmatrix}$$

where,

$$\begin{bmatrix}\bar{\mathbf{Q}}_{\mathrm{MR}} & \bar{\mathbf{M}}_{\mathrm{MR}}\\ \bar{\mathbf{M}}_{\mathrm{MR}}^T & \bar{\mathbf{R}}_{\mathrm{MR}}\end{bmatrix} = \begin{bmatrix}\bar{\tilde{\mathbf{A}}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \bar{\tilde{\mathbf{B}}}\bar{\mathbf{C}}_h & \bar{\tilde{\mathbf{B}}}\bar{\mathbf{D}}_h\end{bmatrix}^T\begin{bmatrix}\bar{\tilde{\mathbf{Q}}} & \bar{\tilde{\mathbf{M}}}\\ \bar{\tilde{\mathbf{M}}}^T & \bar{\tilde{\mathbf{R}}}\end{bmatrix}\begin{bmatrix}\bar{\tilde{\mathbf{A}}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \bar{\tilde{\mathbf{B}}}\bar{\mathbf{C}}_h & \bar{\tilde{\mathbf{B}}}\bar{\mathbf{D}}_h\end{bmatrix}$$

with,

$$\bar{\tilde{\mathbf{Q}}} = \mathrm{diag}\{\tilde{\mathbf{Q}},\ldots,\tilde{\mathbf{Q}}\}\ \bar{\tilde{\mathbf{M}}} = \mathrm{diag}\{\tilde{\mathbf{M}},\ldots,\tilde{\mathbf{M}}\}\ \bar{\tilde{\mathbf{R}}} = \mathrm{diag}\{\tilde{\mathbf{R}},\ldots,\tilde{\mathbf{R}}\}$$

$$\bar{\tilde{\mathbf{A}}} = \begin{bmatrix}\mathbf{I}\\ \mathbf{A}\\ \vdots\\ \mathbf{A}^N\end{bmatrix} \quad \bar{\tilde{\mathbf{B}}} = \begin{bmatrix}\mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\ \mathbf{B} & \ldots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \ddots & \mathbf{0}\\ \mathbf{A}^{N-1}\mathbf{B} & \ldots & \mathbf{B} & \mathbf{0}\end{bmatrix}$$

The optimal control input is obtained from the solution of the Riccati equation in a similar way,

$$\bar{\mathbf{S}}_{\mathrm{MR}}(\bar{k}N) = \bar{\mathbf{Q}}_{\mathrm{MR}} - \bar{\mathbf{M}}_{\mathrm{MR}}\bar{\mathbf{R}}_{\mathrm{MR}}^{-1}\bar{\mathbf{M}}_{\mathrm{MR}}^T + [\bar{\mathbf{A}}_{\mathrm{MR}}^T - \bar{\mathbf{M}}_{\mathrm{MR}}^T\bar{\mathbf{R}}_{\mathrm{MR}}^{-1}\bar{\mathbf{B}}_{\mathrm{MR}}^T]$$
$$\cdot\bar{\mathbf{S}}_{\mathrm{MR}}(\bar{k}N+N)\cdot[\bar{\mathbf{A}}_{\mathrm{MR}} - \bar{\mathbf{B}}_{\mathrm{MR}}^T(\bar{\mathbf{R}}_{\mathrm{MR}}^{-1}\bar{\mathbf{M}}_{\mathrm{MR}}^T + \bar{\mathbf{K}}_{\mathrm{MR}}(\bar{k}N))]$$
$$\bar{\mathbf{K}}_{\mathrm{MR}}(\bar{k}N) = [\bar{\mathbf{R}}_{\mathrm{MR}} + \bar{\mathbf{B}}_{\mathrm{MR}}^T\bar{\mathbf{S}}_{\mathrm{MR}}(\bar{k}N+N)\bar{\mathbf{B}}_{\mathrm{MR}}]^{-1}\bar{\mathbf{B}}^T\bar{\mathbf{S}}_{\mathrm{MR}}(\bar{k}N+N)\cdot$$
$$\cdot(\bar{\mathbf{A}}_{\mathrm{MR}} - \bar{\mathbf{B}}_{\mathrm{MR}}\bar{\mathbf{R}}_{\mathrm{MR}}^{-1}\bar{\mathbf{M}}_{\mathrm{MR}}^T)$$
$$\bar{\mathbf{u}}(\bar{k}N+N) = -(\bar{\mathbf{R}}_{\mathrm{MR}}^{-1}\bar{\mathbf{M}}_{\mathrm{MR}}^T + \bar{\mathbf{K}}_{\mathrm{MR}}(\bar{k}N))\cdot\bar{\mathbf{x}}(\bar{k}N)$$

4 MULTI-RATE BUCY-KALMAN FILTER

The limit case, when $N \to \infty$ with constant frame-period, both discrete-time Kalman filters converge to their equivalent continuous-ones. Continuous lifting technique must be applied to converge MR-LKF into its equivalent one (Bamieh et al., 1991). In this paper the convergence of MR-PKF to the Bucy-Kalman filter (continuous) is shown.

The prediction and correction covariance equations for a sampled-data system is,

$$\mathbf{P}(t+T|t) = e^{\mathbf{A}_c T} \cdot \mathbf{P}(t|t) \cdot e^{\mathbf{A}_c^T T} + \mathbf{Q}(T)$$
$$\mathbf{S}_s(t) = \mathbf{C}_s(t)\mathbf{P}(t+T|t)\mathbf{C}_s^T(t) + \mathbf{R}_s(T)$$
$$\mathbf{K}_s(t+T) = \mathbf{P}(t+T|t) \cdot \mathbf{C}_s^T(t) \cdot \mathbf{S}_s^{-1}(t)$$
$$\mathbf{P}(t+T|t+T) = \mathbf{P}(t+T|t) - \mathbf{K}_s(t+T) \cdot \mathbf{C}_s(t) \cdot \mathbf{P}(t+T|t)$$

being $\mathbf{C}_s(t)$ the output matrix considering the multi-rate sampling. The derivate of the covariance is,

$$\dot{\mathbf{P}}(t|t) = \lim_{T\to 0} \frac{\mathbf{P}(t+T|t+T) - \mathbf{P}(t|t)}{T}$$
$$\dot{\mathbf{P}}(t|t) = \lim_{T\to 0} \frac{\partial \mathbf{P}(t+T|t)}{\partial T} - \lim_{T\to 0} \frac{\partial \mathbf{K}_s(t+T)}{\partial T} \cdot \mathbf{C}_s(t) \cdot \mathbf{P}(t+T|t) -$$
$$- \lim_{T\to 0} \mathbf{K}_s(t+T) \cdot \mathbf{C}_s(t) \cdot \frac{\partial \mathbf{P}(t+T|t)}{\partial T}$$

Operating and re-ordering some terms,

$$\lim_{T\to 0} \frac{\partial \mathbf{P}(t+T|t)}{\partial T} = \tilde{\dot{\mathbf{P}}}(t|t) = \mathbf{A}_c \cdot \mathbf{P}(t|t) + \mathbf{P}(t|t) \cdot \mathbf{A}_c^T + \mathbf{G}_c \cdot \mathbf{Q}_c \cdot \mathbf{G}_c^T$$
$$\lim_{T\to 0} \frac{\partial \mathbf{K}_s(t+T)}{\partial T} \mathbf{C}_s(t)\mathbf{P}(t+T|t) = \tilde{\dot{\mathbf{P}}}(t|t)\mathbf{C}_s(t)\mathbf{K}_s^T(t) -$$
$$- \mathbf{K}_s(t)\dot{\mathbf{S}}_s(t)\mathbf{K}_s^T(t)$$
$$\lim_{T\to 0} \mathbf{K}_s(t+T)\mathbf{C}_s(t) \frac{\partial \mathbf{P}(t+T|t)}{\partial T} = \mathbf{K}_s(t)\mathbf{C}_s(t)\tilde{\dot{\mathbf{P}}}(t|t)$$

where $\dot{\mathbf{S}}_s^{-1}(t) = -\mathbf{S}_s^{-1}(t)\dot{\mathbf{S}}_s(t)\mathbf{S}_s^{-1}(t)$. Operating, we get to multi-rate Bucy-Kalman filter equations:

$$\tilde{\dot{\mathbf{P}}}(t|t) = \mathbf{A}_c \cdot \mathbf{P}(t|t) + \mathbf{P}(t|t) \cdot \mathbf{A}_c^T + \mathbf{G}_c \cdot \mathbf{Q}_c \cdot \mathbf{G}_c^T$$
$$\mathbf{K}_s(t) = \tilde{\dot{\mathbf{P}}}(t|t)\mathbf{C}_s^T(t)\left[\mathbf{C}_s(t)\tilde{\dot{\mathbf{P}}}(t|t)\mathbf{C}_s^T(t) + \mathbf{R}_s(t)\right]^{-1}$$
$$\dot{\mathbf{P}}(t|t) = \tilde{\dot{\mathbf{P}}}(t|t) - \mathbf{K}_s(t) \cdot \mathbf{C}_s(t) \cdot \mathbf{P}(t|t)$$

Finally, the state estimation is as follows:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_c \cdot \hat{\mathbf{x}}(t) + \mathbf{B}_c \cdot \mathbf{u}_h(t) + \mathbf{K}_s(t)(\mathbf{y}_s(t) - \mathbf{C}_s(t) \cdot \hat{\mathbf{x}}(t))$$

5 NUMERICAL EXAMPLE

Suppose the weak-coupled plant proposed by (Araki and Yamamoto, 1986; Godbout et al., 1990), with inputs u_1 and u_2 updated at $T_{u_1} = 0.1 sec.$ and $T_{u_2} = 0.15$ sec. and outputs y_1 and y_2 sampled at $T_{y_1} = 0.15$ sec. and $T_{y_2} = 0.1$ sec., respectively. The base-period is $T = 0.05$ sec. and the frame-period $\bar{T} = 0.3$ sec., thus $N = 6$.

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c \cdot \mathbf{x}(t) + \mathbf{B}_c \cdot \mathbf{u}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{v}(t)$$

with,

$$\mathbf{x}(t) = [x_1\ x_2\ x_3]^T,\ \mathbf{u}(t) = [u_1\ u_2]^T,\ \mathbf{y}(t) = [y_1\ y_2]^T$$
$$\mathbf{A}_c = \mathrm{diag}\{-2.5, -2, -1\}$$
$$\mathbf{B}_c = \begin{pmatrix} 2.5 & 0 \\ 10 & -1.2 \\ 5/6 & 1 \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} -4 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

where noise covariances are $\mathbf{Q}_c = \mathrm{diag}\{0.05, 0.3, 0.1\}$ and $\mathbf{R}_c = \mathrm{diag}\{0.1, 0.1\}$.

The lifted input and output vectors are:

$$\bar{\mathbf{u}}(\bar{k}N) = [u_1(\bar{k}N-N)\ u_2(\bar{k}N-N)\ u_1(\bar{k}N-4)\ u_2(\bar{k}N-3)\ u_1(\bar{k}N-2)]^T$$
$$\bar{\mathbf{y}}_s(\bar{k}N) = [y_2(\bar{k}N-4)\ y_1(\bar{k}N-3)\ y_2(\bar{k}N-2)\ y_1(\bar{k}N)\ y_2(\bar{k}N)]^T$$

while the periodic time-variant vectors and matrices are,

$$\mathbf{\Delta}_u(0) = \mathbf{I}\ \mathbf{\Delta}_u(1) = \mathbf{0}\ \mathbf{\Delta}_u(2) = \mathbf{\Delta}_u(4) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ \mathbf{\Delta}_u(3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{y}_s(0) = [y_1\ y_2]^T,\ \mathbf{y}_s(1) = \mathbf{y}_s(5) = \emptyset,\ \mathbf{y}_s(2) = \mathbf{y}_s(4) = y_2,\ \mathbf{y}_s(3) = y_1$$

Figure 2 shows the state covariance computation using a MR-LKF and MR-PKF for unit step with ZOH. The MR-LKF is combined with the fast sampling model of the plant (inter-sampling prediction). It can be observed that the covariance (and also the estimation error, which is not represented), are the same at every frame-period, therefore both filters are equivalent at those time instants. However, the advantage of the multi-rate periodic time-variant approach is that estimations are performed at a fast sampling rate, while the lifting approach is in *open-loop* until all inputs and output of the frame-period have been processed.

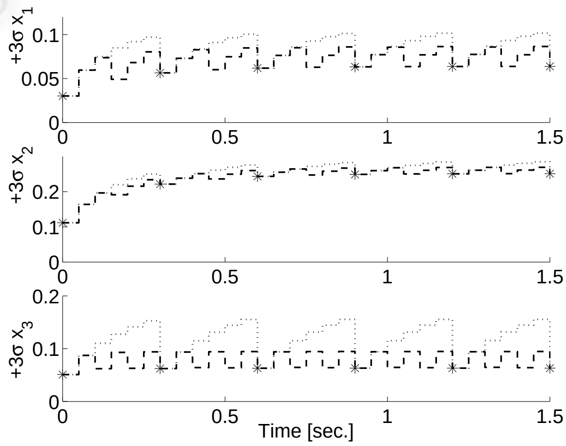

Figure 2: State covariance for MR-LFK (dotted line) and MR-PKF (dashed line).

Now, can compare the equivalence of the periodic and lifted LQG control using a ZOH. In figure 3, it is depicted the state evolution with initial values $\mathbf{x}(0) = [0.5\ -0.4\ 0.3]^T$, and weighting matrices $\tilde{\mathbf{Q}}_c = 10 \cdot \mathbf{I}_{3\times 3}$, $\tilde{\mathbf{R}}_c = 0.1 \cdot \mathbf{I}_{2\times 2}$. It can be appreciated that in both cases, they give the same estimation each frame-period, however in the periodic approach the state estimation is much more closer to the real state. This is because the lifted LQG is in open-loop among frame-periods. Figure 3 shows the state evolution for the proposed example, where the same conclusions can be obtained.

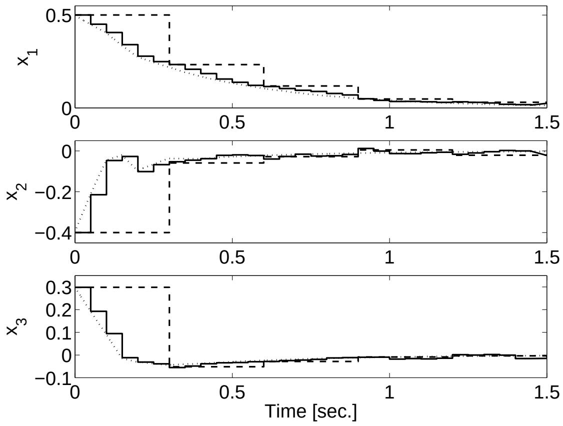

Figure 3: State estimation in LQG control, with the periodic regulator (solid line) with the lifted regulator (dashed line) and the real state evolution of multi-rate LQR control (dotted-line).

6 CONCLUSIONS

In this paper, we have presented and formalized two Linear Quadratic Gaussian regulators for multi-rate sampled-data systems, using time invariant (lifting technique) and time variant (periodic) modeling. In the numerical example, it has been shown that both approaches are equivalent at frame period, since they are representing the same sample-data system.

The lifted LQG regulator is expressed with constant coefficient matrices and executed at the frame-period. This implies a partial open loop control among frame-periods. On the contrary, the periodic LQG is executed at base-period and consequently, it is a time-variant system. In addition, this regulator is not restricted to periodic sampling and can also be used for dealing with data-missing problems without recomputing the system model.

In both regulators, multi-rate holds are integrated as signal interfaces for different frequencies. General multi-rate holds are based on primitive functions. In particular, from polynomial functions it is possible to generate multi-rate ZOH, FOH, etc. Other less-conventional holds based on non-polynomial functions may improve signal reconstruction and even system behavior. The wide variety of holds proposed directly in this paper or indirectly in other authors' references, open a research field to be exploited.

An interesting analysis has been done to demonstrate the convergence of the Periodic Kalman filter to its equivalent continuous one (Bucy Kalman filter), when the periodicity ratio converges to infinity. A new approach related with Bucy Kalman filter is the fact that continuous signals with discontinuities can be easily incorporated for estimation.

The multi-rate periodic modeling can be also used in non-linear filters such as Extended and Unscented Kalman Filters and Particle Filters, widely used in mobile robotics.

Many technological limitations associated to sensors and actuators can be overtaken in a systematic way, using the multi-rate sampling approaches used in this paper.

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