BOUNDS FOR THE SOLUTION OF DISCRETE COUPLED LYAPUNOV EQUATION

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Abstract: Upper and lower matrix bounds for the solution of the discrete time coupled algebraic Lyapunov equation for linear discrete-time system with Markovian jumps in parameters are developed. The bounds of the maximal, minmal eigenvalues, the summation of eigenvalues, trace and determinant are also given.

1 INTRODUCTION

It is well known that algebraic Lyapunov and Riccati equations are widely applied to various engineering areas including different problems in signal processing and, especially, control theory. In the area of control system analysis and design, these equations play crucial role in system stability and boundedness analysis, optimal and robust controllers and filters design, the transient behavior estimates, etc. During the past two decades many bounds for the solution of various types of algebraic Lyapunov and Riccati equation have been reported. The surveys of such results can be found in (Mori and Derese, 1984), (Komaroff, 1996), (Kwon et al., 1996), (Czornik and Nawrat, 2000). The reasons that the problem to estimate upper and lower bounds of these equations has become an attractive topic are that the bounds are also applied to solve many control problems such as stability analysis (Lee et al., 1995), (Patel and Toda, 1980), time-delay system controller design (Mori et al., 1983), estimation of the minimal cost and the suboptimal controller design (Langholz, 1979), convergence of numerical algorithms (Allwright,1980), robust stabilization problem (Boukas et al., 1997). Eigenvalue bounds can be also used to determine whether or not the system under consideration possesses the singularly perturbed structure (Gajic and Qureshi, 1995). An excellent motivation to study the bounds for Lyapunov equation is given in (Gajic and Qureshi, 1995) (Section 2.2). The authors advocated the results in this area by saying that sometimes we are just interested in the general behavior of the underlying system and then the behavior can be determined by examining certain bounds on the parameters of the solution instead of the full solution.

Considering the linear dynamical systems with Markovian jumps in parameter values, which have recently attracted a great deal of interest, instead of one equation a set of coupled algebraic equations arises. They are called coupled algebraic Riccati and coupled Lyapunov equation. All the reasons mentioned above could be repeated to show how the bounds for coupled algebraic Lyapunov equations can be used. Bounds for the coupled Riccati equation have been already obtained in (Czornik and Swierniak,2001a) and (Czornik and Swierniak, 2001b). To our knowledge this paper is the first where the bounds for coupled algebraic Lyapunov equations are established.

The eigenvalues $\lambda_i(X)$, where $i = 1, ..., n$, of a symmetric matrix $X \in R^{n\times n}$ are assumed to be arranged such that

$$\lambda_1(X) \geq \lambda_2(X) \geq ... \geq \lambda_n(X).$$

When we consider the discrete time jump linear system the following discrete coupled algebraic Lyapunov equation (DCALE) arises (Chizeck et al., 1986):

$$P_i = Q_i + A_i' F_i A_i \tag{1}$$

where

$$F_i = \sum_{j\in S} p_{ij} P_j \tag{2}$$

and $A_i, Q_i, P_i \in R^{n\times n}$, $p_{ij} \in [0,1]$, $\sum_{j\in S} p_{ij} = 1$, $i \in S$, S is a finite set. The numbers p_{ij} are the transitions probabilities of a Markov chain.

We need the following lemma.

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Lemma 1 *(Marshall and Olkin, 1979)Let* $X, Y \in R^{n\times n}$ *with* $X = X'$, $Y = Y'$, $X, Y \geq 0$. *Then the following inequalities hold*

$$\lambda_{i+j-1}(XY) \leq \lambda_i(X)\lambda_j(Y), \text{ if } i+j \leq n+1 \quad (3)$$

$$\lambda_{i+j-n}(XY) \geq \lambda_i(X)\lambda_j(Y), \text{ if } i+j \geq n+1 \quad (4)$$

$$\sum_{k=1}^{l} \lambda_k (X+Y) \leq \sum_{k=1}^{l} \lambda_k (X) + \sum_{k=1}^{l} \lambda_k (Y) \quad (5)$$

$$\sum_{k=1}^{l} \lambda_{n-k+1} (X+Y) \geq$$

$$\sum_{k=1}^{l} \lambda_{n-k+1} (X) + \sum_{k=1}^{l} \lambda_{n-k+1} (Y). \quad (6)$$

2 MAIN RESULTS

The next theorem contains the main result of the paper.

Theorem 2 *For the eigenvalues* $\lambda_k(P_i)$, $k = 1, ..., n, i \in S$ *of positive definite solution* $P_i, i \in S$ *of DCALE (1), the following inequalities hold*

$$\sum_{k=1}^{l} \lambda_k (P_i) \leq \sum_{k=1}^{l} \lambda_k (Q_i) + \left(\max_{j\in S} p_{ij}\right) \lambda_1 (A_i A_i') \cdot$$

$$\cdot \frac{\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (Q_i)}{1 - \max_{j\in S} \lambda_1 \left(A_j A_j'\right) \max_{j\in S} \sum_{i\in S} p_{ij}} =$$

$$= \alpha(l, i), \quad (7)$$

for $l = 1, ..., n$, *if* $\max_{j\in S} \lambda_1 \left(A_j A_j'\right) \max_{j\in S} \sum_{i\in S} p_{ij} < 1$, *and*

$$\sum_{k=1}^{l} \lambda_{n-k+1} (P_i) \geq$$

$$\sum_{k=1}^{l} \lambda_{n-k+1} (Q_i) + \left(\min_{j\in S} p_{ij}\right) \min_{j\in S} \lambda_n \left(A_j A_j'\right)$$

$$\frac{\sum_{i\in S}\sum_{k=1}^{l} \lambda_{n-k+1} (Q_i)}{1 - \min_{j\in S} \lambda_n \left(A_j A_j'\right) \min_{j\in S} \sum_{i\in S} p_{ij}} =$$

$$= \beta(l, i), \quad (8)$$

for $l = 1, ..., n$, *if* $\min_{j\in S} \lambda_n \left(A_j A_j'\right) \min_{j\in S} \sum_{i\in S} p_{ij} < 1$.

Proof. From (1) it follows, by using (5) and (3), that

$$\sum_{k=1}^{l} \lambda_k (P_i) \leq \sum_{k=1}^{l} \lambda_k (Q_i) + \sum_{k=1}^{l} \lambda_k (A_i' F_i A_i) =$$

$$= \sum_{k=1}^{l} \lambda_k (Q_i) + \sum_{k=1}^{l} \lambda_k (F_i A_i A_i')$$

$$\leq \sum_{k=1}^{l} \lambda_k (Q_i) + \lambda_1 (A_i A_i') \sum_{k=1}^{l} \lambda_k (F). \quad (9)$$

Applying (5) to (2) leads to

$$\sum_{k=1}^{l} \lambda_k (F) \leq \sum_{j\in S} \left(p_{ij} \sum_{k=1}^{l} \lambda_k (P_j) \right). \quad (10)$$

Combining (9) with (10) yields to

$$\sum_{k=1}^{l} \lambda_k (P_i) \leq \sum_{k=1}^{l} \lambda_k (Q_i) +$$

$$+ \max_{j\in S} \lambda_1 \left(A_j A_j'\right) \sum_{j\in S} \left(p_{ij} \sum_{k=1}^{l} \lambda_k (P_j) \right). \quad (11)$$

Summing the above inequality over $i \in S$ we have

$$\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (P_i) \leq \sum_{i\in S}\sum_{k=1}^{l} \lambda_k (Q_i)$$

$$+ \max_{j\in S} \lambda_1 \left(A_j A_j'\right) \sum_{i,j\in S} \left(p_{ij} \sum_{k=1}^{l} \lambda_k (P_j) \right) =$$

$$\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (Q_i) + \max_{j\in S} \lambda_1 \left(A_j A_j'\right) \cdot$$

$$\cdot \sum_{j\in S} \left(\left(\sum_{i\in S} p_{ij} \right) \sum_{k=1}^{l} \lambda_k (P_j) \right) \leq$$

$$\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (Q_i) + \max_{j\in S} \lambda_1 \left(A_j A_j'\right) \cdot$$

$$\cdot \left(\max_{j\in S} \sum_{i\in S} p_{ij} \right) \sum_{i\in S}\sum_{k=1}^{l} \lambda_k (P_i).$$

Solving this inequality respect to $\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (P_i)$ and taking into account that

$$\max_{j\in S} \lambda_1 \left(A_j A_j'\right) \max_{j\in S} \sum_{i\in S} p_{ij} < 1$$

we obtain

$$\sum_{i\in S}\sum_{k=1}^{l} \lambda_k (P_i) \leq$$

$$\frac{\sum_{i\in S}\sum_{k=1}^{l}\lambda_k\left(Q_i\right)}{1-\max\limits_{j\in S}\lambda_1\left(A_jA_j'\right)\max\limits_{j\in S}\sum_{i\in S}p_{ij}}. \tag{12}$$

(9) implies also that

$$\sum_{k=1}^{l}\lambda_k\left(P_i\right)\leq\sum_{k=1}^{l}\lambda_k\left(Q_i\right)+$$
$$\lambda_1\left(A_iA_i'\right)\left(\sum_{j\in S}\left(p_{ij}\sum_{k=1}^{l}\lambda_k\left(P_j\right)\right)\right)\leq$$
$$\sum_{k=1}^{l}\lambda_k\left(Q_i\right)+\left(\max_{j\in S}p_{ij}\right)\cdot$$
$$\cdot\lambda_1\left(A_iA_i'\right)\sum_{j\in S}\sum_{k=1}^{l}\lambda_k\left(P_j\right). \tag{13}$$

Applying (12) on the right hand side of (13) we have (7).

To proof (8) let's observe that the use of (6) and (4) to (1) to gives

$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)\geq\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+$$
$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(A_i'F_iA_i\right)=$$
$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+\sum_{k=1}^{l}\lambda_{n-k+1}\left(F_iA_iA_i'\right)\geq$$
$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+\lambda_n\left(A_iA_i'\right)\cdot$$
$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(F_i\right)\geq\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+$$
$$\min_{j\in S}\lambda_n\left(A_jA_j'\right)\sum_{k=1}^{l}\lambda_{n-k+1}\left(F_i\right). \tag{14}$$

Summing (14) over $i\in S$ we have

$$\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)\geq\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+$$
$$\min_{j\in S}\lambda_n\left(A_jA_j'\right)\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(F_i\right). \tag{15}$$

Applying (6) to (2) leads to

$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(F_i\right)\geq\sum_{j\in S}p_{ij}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_j\right). \tag{16}$$

Combining (15) with (16) yields to

$$\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)\geq\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+$$
$$\min_{j\in S}\lambda_n\left(A_jA_j'\right)\cdot$$
$$\cdot\sum_{i\in S}\left(\sum_{j\in S}p_{ij}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_j\right)\right)=$$
$$\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+\min_{j\in S}\lambda_n\left(A_jA_j'\right)\cdot$$
$$\cdot\sum_{j\in S}\left(\left(\sum_{i\in S}p_{ij}\right)\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_j\right)\right)\geq$$
$$\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+\min_{j\in S}\lambda_n\left(A_jA_j'\right)\cdot$$
$$\cdot\min_{j\in S}\sum_{i\in S}p_{ij}\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right).$$

Solving this inequality with respect to $\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)$ and taking into account that

$$\min_{j\in S}\lambda_n\left(A_jA_j'\right)\min_{j\in S}\sum_{i\in S}p_{ij}<1$$

we obtain

$$\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)\geq$$
$$\frac{\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)}{1-\min\limits_{j\in S}\lambda_n\left(A_jA_j'\right)\min\limits_{j\in S}\sum_{i\in S}p_{ij}}. \tag{17}$$

Combining (14) and (16) we conclude that

$$\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right)\geq\sum_{k=1}^{l}\lambda_{n-k+1}\left(Q_i\right)+\left(\min_{j\in S}p_{ij}\right)\cdot$$
$$\cdot\min_{j\in S}\lambda_n\left(A_jA_j'\right)\sum_{i\in S}\sum_{k=1}^{l}\lambda_{n-k+1}\left(P_i\right).$$

Applying (17) to the right hand side of the above inequality we obtain (8). ■

Using the Theorem 2 we can establish the following general matrix bound for the solution of DCALE (1).

Theorem 3 *For the positive definite solution $P_i, i\in S$ of DCALE (1) we have*

$$P_i\leq\left(\sum_{j\in S}p_{ij}\alpha\left(1,j\right)\right)A_i'A_i+Q_i, \tag{18}$$

if $\max\limits_{j\in S} \lambda_1 \left(A_j A_j'\right) \max\limits_{j\in S} \sum_{i\in S} p_{ij} < 1$ *and*

$$P_i \geq \left(\sum_{j\in S} p_{ij}\beta\left(1,j\right)\right) A_i' A_i + Q_i \qquad (19)$$

if $\min\limits_{j\in S} \lambda_n \left(A_j A_j'\right) \min\limits_{j\in S} \sum_{i\in S} p_{ij} < 1$, *where* $\alpha\left(1,j\right)$ *and* $\beta\left(1,j\right)$ *are given in Theorem 2.*

Proof. In (Rugh, 1993) it has been shown that for any symmetric matrix $T \in R^{n\times n}$ and $x \in R^n$

$$\lambda_n(T)x'x \leq x'Tx \leq \lambda_1(T)x'x.$$

Using this inequality to (1) we have

$$\left(\sum_{j\in S} p_{ij}\lambda_n(P_j)\right) A_i'A_i + Q_i \leq P_i \leq$$

$$\leq \left(\sum_{j\in S} p_{ij}\lambda_1(P_j)\right) A_i'A_i + Q_i.$$

Combining this inequality with (7) and (8) for $l = 1$ we get the conclusions of the theorem. ■

From Theorem 3 on the obvious way the bounds for $\det\left(P_i\right)$, $tr\left(P_i\right)$, $\lambda_i(P)$ can be obtained and they are collected in the next Remark.

Remark 1 *For the positive definite solution* $P_i, i \in S$ *of DCALE (1) we have*

$$tr\left(P_i\right) \leq \left(\sum_{j\in S} p_{ij}\alpha\left(1,j\right)\right) tr\left(A_i'A_i\right) + trQ_i,$$

$$\det\left(P_i\right) \leq \det\left[\left(\sum_{j\in S} p_{ij}\alpha\left(1,j\right)\right)\left(A_i'A_i\right) + Q_i\right],$$

$$\lambda_k\left(P_i\right) \leq \lambda_k\left[\left(\sum_{j\in S} p_{ij}\alpha\left(1,j\right)\right)\left(A_i'A_i\right) + Q_i\right],$$

if $\max\limits_{j\in S} \lambda_1 \left(A_j A_j'\right) \max\limits_{j\in S} \sum_{i\in S} p_{ij} < 1$ *and*

$$tr\left(P_i\right) \geq \left(\sum_{j\in S} p_{ij}\beta\left(1,j\right)\right) tr\left(A_i'A_i\right) + tr\left(Q_i\right)$$

$$\det\left(P_i\right) \geq \det\left[\left(\sum_{j\in S} p_{ij}\beta\left(1,j\right)\right)\left(A_i'A_i\right) + Q_i\right]$$

$$\lambda_k\left(P_i\right) \geq \lambda_k\left[\left(\sum_{j\in S} p_{ij}\beta\left(1,j\right)\right) tr\left(A_i'A_i\right) + Q_i\right],$$

if $\min\limits_{j\in S} \lambda_n \left(A_j A_j'\right) \min\limits_{j\in S} \sum_{i\in S} p_{ij} < 1$. *Where* $\alpha\left(1,j\right)$ *and* $\beta\left(1,j\right)$ *are given in Theorem 2.*

Now we have bounds of $\lambda_1(P_i)$ and $tr\left(P_i\right)$ in Theorem 2 and in Corollary 1 similar for the lower bounds of $\lambda_n(P_i)$ and $tr\left(P_i\right)$, but in general is difficult to say which one are better, however the example presented in the next section suggests that the bounds from Theorem 3 can be better.

3 NUMERICAL EXAMPLE

Consider the following fourth-order jump linear system with three switching modes (Gajic and Qureshi, 1995): $S = \{1, 2, 3\}$

$$[p_{ij}]_{i,j\in S} = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0.0667 & 0.0665 & 0.0844 & -0.2257 \\ 0.1383 & -0.1309 & 0.0797 & 0.1162 \\ 0.0658 & 0.0298 & 0.0645 & -0.1018 \\ -0.2283 & 0.2438 & -0.1990 & 0.2997 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.1885 & -0.3930 & -0.0894 & -0.1919 \\ -0.4230 & 0.3598 & -0.1224 & -0.1548 \\ 0.0350 & -0.1950 & -0.1967 & -0.1017 \\ -0.2648 & -0.2440 & -0.0542 & 0.0484 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.2746 & 0.0634 & 0.3414 & -0.0692 \\ 0.0796 & 0.4167 & 0.0283 & -0.1207 \\ -0.1607 & 0.0344 & -0.2227 & 0.1617 \\ 0.1175 & -0.2969 & 0.4149 & 0.3314 \end{bmatrix}$$

$$Q_1 = Q_2 = Q_3 = I_4.$$

For the solution P_1, P_2, P_3 we have

$$\lambda_1(P_1) = 1.3533, \lambda_2(P_1) = 1.1182, \lambda_3(P_1) = 1.0124,$$
$$\lambda_4(P_1) = 1.0000,$$
$$\lambda_1(P_2) = 1.7003, \lambda_2(P_2) = 1.2309, \lambda_3(P_2) = 1.0979,$$
$$\lambda_4(P_2) = 1.0104,$$
$$\lambda_1(P_3) = 1.6385, \lambda_2(P_3) = 1.3763, \lambda_3(P_3) = 1.0665,$$
$$\lambda_4(P_3) = 1.0019,$$

(7) and (8) give the following bounds

$$\lambda_1(P_1) \leq 3.5806, \lambda_4(P_1) \geq 1$$
$$\lambda_1(P_2) \leq 4.7421, \lambda_4(P_2) \geq 1$$
$$\lambda_1(P_3) \leq 6.2760, \lambda_4(P_3) \geq 1$$

which are not satisfying. However (18) and gives

$$\begin{aligned} 1.0000 &\leq \lambda_4(P_1) \leq 1.0000, 1.0101 \leq \lambda_3(P_1) \leq 1.0562, \\ 1.0809 &\leq \lambda_2(P_1) \leq 1.4486, 1.2928 \leq \lambda_1(P_1) \leq 2.6237, \\ 1.0098 &\leq \lambda_4(P_2) \leq 1.0445, 1.0807 \leq \lambda_3(P_2) \leq 1.3667, \\ 1.2054 &\leq \lambda_2(P_2) \leq 1.9334, 1.5095 \leq \lambda_1(P_2) \leq 3.3156, \\ 1.0014 &\leq \lambda_4(P_3) \leq 1.0082, 1.0550 \leq \lambda_3(P_3) \leq 1.3196, \\ 1.3130 &\leq \lambda_2(P_3) \leq 2.8202, 1.5134 \leq \lambda_1(P_3) \leq 3.9861. \end{aligned}$$

4 CONCLUSION

Upper and lower matrix bounds for the solution of DCALE have been developed. By these bounds, the corresponding eigenvalue bounds (i.e. for each eigenvalues including the extreme ones, the trace and the determinant) have been defined in turn.

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