A PARAMETERIZED POLYHEDRA APPROACH FOR THE EXPLICIT ROBUST MODEL PREDICTIVE CONTROL

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Keywords: Predictive control, constraints, parameterized polyhedra, multiparametric optimization.

Abstract: The paper considers the discrete-time linear time-invariant systems affected by input disturbances. The goal is to construct the robust model predictive control (RMPC) law taking into account the constraints existence from the design stage. The explicit formulation of the controller is found by exploiting the fact that the optimum of a min-max multi-parametric program is placed on the parameterized vertices of a parameterized polyhedron. As these vertices have specific validity domains, the control law has the form of a piecewise linear function of the current state. Its evaluation replaces the time-consuming on-line optimization problems.

1 INTRODUCTION

Model Predictive Control (MPC) enjoys a remarkable reputation among the control design techniques for process industries. In the beginnings, practitioners used MPC in the unconstrained closed forms due to its simplicity and versatility and dealt with the constraints violation a posteriori. In the '90s, theoreticians proved that constraints could be included at the design stage with excellent results towards the feasibility, stability or robustness. The inconvenience, which represented also an impasse in applying the constrained predictive control to high sampling rate systems, was the relative high complexity of the optimization problem to be solved at each sampling period. Lately, the constrained MPC paradigm was reformulated in terms of LMI (Kothare et al., 1996) with a reduction of computational time but the class of system to be controlled was still limited.

An improvement from the on-line computational point of view can be achieved if the explicit solution of the MPC optimization problem is formulated. In this way, at each sampling time, a piecewise linear function has to be evaluated. In fact the MPC strategy is based on a multi-parametric optimization problem as both the global optimum and the set of constraints are parameter dependent. In the nominal case corresponding with a quadratic optimization problem and linear constraints, the explicit solution was investigated with success using an algebraic approach in (Bemporad et al., 2002b), geometrical arguments in (Seron et al., 2002), (Olaru and Dumur, 2004) and lately dynamic programming (Goodwin et al., 2004).

In the case of robust MPC, the explicit solution is somehow more difficult to achieve as the optimization problem is based on a min-max cost function. It was successfully tackled in (Bemporad et al., 2001) but the alternative methods do not present similar solutions so far. The current work is trying to compensate this setback through an explicit solution for the robust MPC by geometrical base. The method is based on the concept of parameterized polyhedra (Loechner and Wilde, 1997) and their correspondent parameterized vertices where the optimal solution is founded.

2 ROBUST MPC FORMULATION

Consider the MPC problem formulated for a discrete-time linear time-invariant system affected by an input disturbance:

$$x_{t+1} = Ax_t + Bu_t + Ev_t \qquad (1)$$

and subject to a set of linear constraints:

$$Cx_t + Du_t \leq d \qquad (2)$$

The vectors $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ represent the states and inputs while $v_t \in \mathbb{R}^p$ is the unknown vector of disturbances lying inside a polytope containing the

Olaru S. and Dumur D. (2005).
A PARAMETERIZED POLYHEDRA APPROACH FOR THE EXPLICIT ROBUST MODEL PREDICTIVE CONTROL.
In *Proceedings of the Second International Conference on Informatics in Control, Automation and Robotics - Signal Processing, Systems Modeling and Control*, pages 3-10
DOI: 10.5220/0001178400030010

origin defined by a set of linear constraints:

$$V = \{v \,| Mv \leqslant l; l \geqslant 0\} \tag{3}$$

In the following, the pair (A, B) is supposed to be stabilizable and it is assumed that the full measurement of the current state is available at each time t.

MPC is an optimization based technique. In opposition to the nominal case where quadratic cost functions are used (Maciejowski, 2002), (Rossiter, 2003), in the case of models affected by disturbance, a min-max optimization is preferred, resulting a RMPC formulation:

$$\min_{u_t,\ldots,u_{t+N_u-1}} \left\{ \max_{v_t,\ldots,v_{t+N-1}} \left\{ S_{P_\lambda}(x_{t+N|t}) + \right.\right. \\ \left.\left. + \sum_{k=1}^{N-1} \left\| Q x_{t+k|t} \right\|_\infty + \sum_{k=0}^{N_u-1} \left\| R u_{t+k} \right\|_\infty \right\} \right\} \tag{4}$$

$$\begin{aligned} \text{s.t.:}\quad & C x_{t+k|t} + D u_{t+k} \leqslant d, k = 1, \ldots, N \\ & M v_{t+k} \leqslant l, k = 0, \ldots N-1 \\ & x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k} + E v_{t+k}, \\ & k \geqslant 0, x_{t+N|t} \in P_\lambda \end{aligned} \tag{5}$$

with Q, R weighting matrices,$\|*\|_\infty \triangleq \max_{i=1,\ldots,r}\left(*^i\right)$, where $*^i$ is the i-th element of the vector $* \in \mathbb{R}^r$. The state predictions $x_{t+k|t}$ are obtained based on the current state vector x_t and by applying the input sequence $u_t, \ldots, u_{t+Nu-1}$, to model (1) over a control horizon. Note that, in the general case, the control (N_u) and the prediction (N) horizons might be different if the control vector has a fix formulation for $N_u \leqslant k \leqslant N$. Conversely, the disturbance sequence $v_t, \ldots, v_{t+N-1}$ affects the prediction over the whole prediction horizon.

The stability of the MPC scheme depends on the chosen horizons and on the terminal cost. In order to guarantee the stability, an infinite prediction horizon should be used. Such a choice transforms (4)-(5) in an intractable problem. The solution is then to choose a finite prediction horizon and to consider that after this point the system trajectory is brought inside a positively invariant set, P, that can be computed off-line (Kerrigan, 2000). To this terminal region a function $S_{P_\lambda}(x)$ can be associated, appearing in (4) as a terminal cost penalizing the evolution from N to ∞.

Applying a *receding horizon strategy* the optimization (4)-(5) is solved at each sampling time t using the measured state vector x_t (playing the role of parameter for the optimization). If $\mathbf{k}_\mathbf{u}^*(x_t) = \left\{u_t^*, \ldots, u_{t+N_u-1}^*\right\}$ is the solution to (4)-(5), the input applied to the system (1) is the first value of this sequence $\mathbf{k}_\mathbf{u}^*(x_t)$ such that $u_t = u_t^*$, the other values are abandoned and the procedure is restarted.

A special concern must be given to the choice of the control horizon. Indeed, this parameter is sensitive as it reflects the number of degrees of freedom available to ensure the constraints fulfillment for all possible combinations of disturbances. On the other hand, with less control alternatives the computational load is diminished. In the robust MPC case, the control horizon is generally equal with the prediction horizon $N_u = N$, as the cumulative effect of the worst case disturbances needs an important control counterpart.

$$\min_{u_t} \left\{ \max_{v_t} \left\{ \min_{u_{t+1}} \ldots \min_{u_{t+N_u-1}} \left\{ \max_{v_{t+N_u},\ldots,v_{t+N-1}} \right.\right.\right. \\ \left.\left.\left. S_{P_\lambda}(x_{t+N|t}) + \sum_{k=1}^{N-1} \left\| Q x_{t+k|t} \right\|_\infty + \sum_{k=0}^{N_u-1} \left\| R u_{t+k} \right\|_\infty \right\} \cdots \right\} \tag{6}$$

or equivalently in a "closed loop" formulation:

$$\min_{u_t} \left\{ \|R u_t\|_\infty + \max_{v_t} \left\{ \left\| Q x_{t+1|t} \right\|_\infty + \right.\right. \\ + \min_{u_{t+1}} \left\{ \ldots + \min_{u_{t+N_u-1}} \left\{ \|R u_{t+N_u-1}\|_\infty + \right.\right. \\ \left.\left.\left.\left. + \max_{v_{t+N_u-1},\ldots,v_{t+N}} \left\{ S_{P_\lambda}(x_{t+N|t}) + \sum_{k=N_u}^{N-1} \left\| Q x_{t+k|t} \right\|_\infty \right\} \cdots \right\}\right.\right.\right.$$

3 ROBUST MPC AS A MULTI-PARAMETRIC OPTIMIZATION

The robust model predictive control problem formulated before is based on the on-line solving of the associated min-max optimization problem:

$$\begin{aligned} & \min_{\mathbf{k_u}} \max_{\mathbf{k}_v} J(x_t, \mathbf{k_u}, \mathbf{k}_v) \\ & \text{subj. to } F_{in}\mathbf{k_u} + G_{in}\mathbf{k}_v \leqslant h_{in} + H_{in} x_t \end{aligned} \tag{7}$$

with $\mathbf{k_u} = \left\{u_t, .., u_{t+N_u-1}\right\}$, $\mathbf{k}_v = \{v_t, .., v_{t+N-1}\}$ and a convex cost function $J(x_t, \mathbf{k_u}, \mathbf{k}_v)$ based on a sum of ∞-norm terms. $F_{in}, G_{in}, h_{in}, H_{in}$ translate in a compact form the set of constraints in (5). Both the cost function and the set of constraints depend on the current state vector x_t which plays the role of a parameter. This parameterization of the optimization problem to be solved at each sampling time transforms the on-line location of the minimum argument in a computationally prohibitive task. The alternative solution is to explicitly formulate off-line the optimal solution $\mathbf{k}_\mathbf{u}^*(x_t)$ in terms of a piecewise linear function and further evaluate this function on-line.

3.1 The inner optimization

The influence of the disturbances in the form (7) can be examined by the reconsideration of the extremal possible combination of vertices in V for each prediction stage completing the sequence $\mathbf{k}_v$.

$$v_t \in V \subset \mathbb{R}^p \Rightarrow \mathbf{k}_v \in V^N \subset \mathbb{R}^{N \times p} \tag{8}$$

Remark: For the inner optimization, the set of constraints is constituted only by the inequalities defining the polyhedral domain as in (3) and the constraints imposed by the system dynamics in (1). This fact is transparent from the definition of the predictive control law, which allows any combination of disturbances satisfying (3). If one of these combinations is not allowed by the set of constraints in (7), it means in fact that the MPC law is infeasible.

Taking into account the convexity of the objective function and the previous remark, it can be concluded that the optimum for the inner optimization in (7) is on the border of the feasible domain, more precisely on one of the vertices of V^N as long as it is defined as a polytope. Thus (7) becomes:

$$\begin{aligned} &\min_{\mathbf{k_u}} \max_{\mathbf{v_{k_\upsilon}}} J(x_t, \mathbf{k_u}, \mathbf{k}_{\upsilon_l}) \\ &\text{subj. to } F_{in}\mathbf{k_u} + G_{in}\mathbf{k}_{\upsilon_l} \leqslant h_{in} + H_{in}x_t \\ &l \in L, \mathbf{k}_{\upsilon_l} \in V^N \end{aligned} \tag{9}$$

with $L = \{1, 2, \ldots, N_v\}$ and N_v the number of vertices in V^N.

This means that the inner optimization in (7) will act only on the set of vertices in V^N. Further this may be written as:

$$\begin{aligned} &\min_{\mathbf{k_u}, \mu} \mu \\ &\text{subj. to } F_{in}\mathbf{k_u} + G_{in}\mathbf{k}_{\upsilon_l} \leqslant h_{in} + H_{in}x_t \\ &J(x_t, \mathbf{k_u}, \mathbf{k}_{\upsilon_l}) \leqslant \mu \\ &l \in L, \mathbf{k}_{\upsilon_l} \in V^N \end{aligned} \tag{10}$$

3.2 The outer optimization problem

An impediment in finding the explicit solution for (7) is the expression of the cost function, given as a collection of ∞-norm terms. In order to avoid the inherent difficulty of handling it, an equivalent linear program (LP) (Kerrigan, 2004) formulation must be achieved based on the idea that each ∞-norm term can be bounded. The optimization problem is equivalent with the minimization of the sum of these bounds. This is resumed by the following result where the cost function is considered as a sum of ∞-norm terms linear in the vector of unknowns $\mathbf{x}$ and parameters $\mathbf{p}$ (to identify them, one can observe that for a fix sequence $\mathbf{k_v} = ct$ and noting $\mathbf{x} = \mathbf{k_u}$ and $\mathbf{p} = x_t$ in (7), the cost function is a sum of $\|S_i\mathbf{x} + P_i\mathbf{p} + s_i\|_\infty$ terms, with S_i, P_i, s_i defined after case).

Proposition 1. The formulations (1) and (2) are equivalent:

$$(1) \quad K(\mathbf{p}) = \min_{\mathbf{x}} J(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{x}} \sum_{i=1}^{n} \|S_i\mathbf{x} + P_i\mathbf{p} + s_i\|_\infty$$
$$\text{subject to } A_{in}\mathbf{x} \leqslant b_{in} + B_{in}\mathbf{p}$$

$$K(\mathbf{p}) = \min_{\rho, \{\sigma_i\}, \mathbf{x}} \rho$$
$$(2) \quad \text{subject to } \begin{cases} -\mathbf{1}\sigma_i \leqslant S_i\mathbf{x} + P_i\mathbf{p} + s_i \leqslant \mathbf{1}\sigma_i, 1 \leqslant i \leqslant n \\ \sum_{i=1}^{n} \sigma_i \leqslant \rho \\ A_{in}\mathbf{x} \leqslant b_{in} + B_{in}\mathbf{p} \end{cases}$$

where $\sigma_i, \rho \in \mathbb{R}$ and $\mathbf{1}$ is a vector with unit entries.

3.3 RMPC multi-parametric optimization problem

With the previous two transformations, the optimization (7) can be rewritten as:

$$\mathbf{k_u} * (x_t) = \min_{\rho, \mathbf{k_u}, \{\sigma_i^j\}} \rho$$
$$\begin{cases} -\mathbf{1}\sigma_i^j \leqslant S_i\mathbf{k_u} + P_i x_t + W_i\mathbf{k}_{\upsilon_l} + s_i \leqslant \mathbf{1}\sigma_i^j, \\ \qquad 1 \leqslant i \leqslant n, 1 \leqslant l \leqslant N_v \\ \begin{bmatrix} \sum_{i=1}^{n} \sigma_i^1 \\ \vdots \\ \sum_{i=1}^{n} \sigma_i^{N_v} \end{bmatrix} \leqslant \mathbf{1}\rho \\ F_{in}\mathbf{k_u} + G_{in}\mathbf{k}_{\upsilon_l} \leqslant h_{in} + H_{in}x_t, \\ \qquad 1 \leqslant l \leqslant N_v \end{cases} \tag{11}$$

Example 1: To illustrate these transformations, consider the parameter-free optimization (Fig. 1):

$$\min_{x_1} \max_{x_2} \left\| \begin{matrix} 2x_1 + x_2 - 3 \\ x_1 - x_2 + 1 \end{matrix} \right\|_\infty + \left\| \begin{matrix} x_1 - 2x_2 + 1 \\ 2x_1 + 3x_2 - 7 \end{matrix} \right\|_\infty$$
$$\text{subject to } \begin{cases} x_2 \in [-1,1] \\ x_1 \in [0, 6] \end{cases}$$

equivalent with:

$$\min_{x_1, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \rho} \rho$$
$$\text{s.t.} -\begin{bmatrix} \sigma_1 \\ \sigma_1 \end{bmatrix} \leqslant \begin{bmatrix} 2x_1 - 2 \\ x_1 \end{bmatrix} \leqslant \begin{bmatrix} \sigma_1 \\ \sigma_1 \end{bmatrix};$$
$$-\begin{bmatrix} \sigma_2 \\ \sigma_2 \end{bmatrix} \leqslant \begin{bmatrix} 2x_1 - 4 \\ x_1 + 2 \end{bmatrix} \leqslant \begin{bmatrix} \sigma_2 \\ \sigma_2 \end{bmatrix};$$
$$-\begin{bmatrix} \sigma_3 \\ \sigma_3 \end{bmatrix} \leqslant \begin{bmatrix} x_1 - 1 \\ 2x_1 - 4 \end{bmatrix} \leqslant \begin{bmatrix} \sigma_3 \\ \sigma_3 \end{bmatrix};$$
$$-\begin{bmatrix} \sigma_4 \\ \sigma_4 \end{bmatrix} \leqslant \begin{bmatrix} x_1 + 3 \\ 2x_1 - 10 \end{bmatrix} \leqslant \begin{bmatrix} \sigma_4 \\ \sigma_4 \end{bmatrix};$$
$$\sigma_1 + \sigma_3 \leqslant \rho; \sigma_2 + \sigma_4 \leqslant \rho; x_1 \in [0, 6]$$

which can be tackled by any LP solver with solution:

$$[x_1\, \sigma_1\, \sigma_2\, \sigma_3\, \sigma_4\, \rho]^* = [2.33\ 4.33\ 5.33\ 1.33\ 2.66\ 9.66]$$

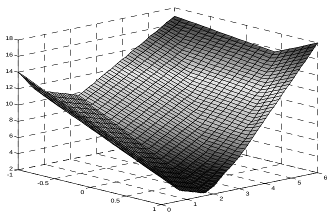

Figure 1: Cost function for example 1.

4 THE EXPLICIT SOLUTION

In the following, the closed form of the RMPC law is the main objective. It can be expressed as a function of parameters if a procedure of describing the explicit solution of multi-parametric linear programs (MPLP) is available. The literature on MPLP contains the works of Gal and Nedoma (Gal and Nedoma, 1972) and further developments to linear, quadratic, non-linear or mixed-integer solvers (Borelli, 2003). Another procedure will be proposed here focusing on the set of constraints and its geometrical representation. The feasible domain will be expressed as a parametrized polyhedron. Due to the reformulation of the optimization problem, the use of mixed variables is avoided. Thus the resulting algorithm differs from the solutions based on branch and bound methods or other mixed integer linear solvers, being mainly focused on the enumeration of the edges of an augmented dimension polyhedron.

4.1 Parameterized polyhedra

A system of linear constraints define a polyhedron:

$$P = \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{A}_{eq}\,\mathbf{x} = \mathbf{b}_{eq}; \mathbf{A}_{in}\mathbf{x} \leqslant \mathbf{b}_{in}\} \tag{12}$$

by dual Minkowski representation of generators (Schrijver, 1986):

$$\begin{aligned} P = {} & conv.hull\,\{\mathbf{x}_1, \ldots, \mathbf{x}_v\} + \\ & + cone\,\{\mathbf{y}_1, \ldots, \mathbf{y}_r\} + lin.space\mathbf{Z} \end{aligned} \tag{13}$$

where $conv.hullX$ denotes the set of convex combinations of points in X, $coneY$ denotes nonnegative combinations of unidirectional rays and $lin.spaceZ$ represents a linear combination of bidirectional rays. It can be rewritten as:

$$\begin{gathered} P = \left\{ \mathbf{x} |\, \mathbf{x} = \sum_{i=1}^{v} \lambda_i \mathbf{x}_i + \sum_{i=1}^{r} \gamma_i \mathbf{y}_i + \sum_{i=1}^{l} \mu_i \mathbf{z}_i \right\} \\ 0 \leqslant \lambda_i \leqslant 1,\ \sum_{i=1}^{v} \lambda_i = 1\ ,\ \gamma_i \geqslant 0\ ,\ \forall \mu_i \end{gathered} \tag{14}$$

Remark: The generators saturate all the equalities, the lines saturate all the constraints and only the rays and the vertices can verify but not saturate a part of the inequalities.

The geometrical computations might be burdened by the differences that have to be taken into consideration between rays and lines. These problems are overcome with an *homogenous* representation (Wilde, 1993):

$$D = \left\{ \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi \end{array} \right) \in \mathbb{R}^{n+1} \left| \begin{array}{l} \hat{\mathbf{A}}_{eq} \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi \end{array} \right) = 0 \\ \hat{\mathbf{A}}_{in} \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi \end{array} \right) \geqslant 0 \end{array} \right. \right\} \tag{15}$$

$$\hat{\mathbf{A}}_{eq} = \left[\ \mathbf{A}_{eq} | {-\mathbf{b}_{eq}}\ \right] \quad \hat{\mathbf{A}}_{in} = \left[\begin{array}{c|c} \mathbf{A}_{in} & -\mathbf{b}_{in} \\ \hline 0 \cdots 0 & 1 \end{array} \right] \tag{16}$$

The original polyhedron P is found intersecting D with the hyper-plane of equation $\xi = 1$. Following the same change of dimension, the rays, vertices and lines have a similar unified homogenous description:

$$\hat{\mathbf{Y}} = \left[\begin{array}{c|c} \mathbf{Y} & \mathbf{X} \\ \hline 0 \cdots 0 & 1 \cdots 1 \end{array} \right]; \hat{\mathbf{Z}} = \left[\begin{array}{c} \mathbf{Z} \\ \hline 0 \cdots 0 \end{array} \right] \tag{17}$$

and the generators representation will be:

$$D = \left\{ \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi \end{array} \right) \middle| \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi \end{array} \right) = \hat{\mathbf{Y}}\,\lambda' + \hat{\mathbf{Z}}\,\mu; \lambda' \geqslant 0 \right\} \tag{18}$$

A parameterized polyhedron is defined in the implicit form by a finite number of inequalities and equalities but the affine part depends linearly on a parameter vector p for both equalities and inequalities:

$$\begin{aligned} P'(\mathbf{p}) &= \left\{ \mathbf{x} \in \Re^n \,|\mathbf{A}_{eq}\ \mathbf{x} = \mathbf{B}_{eq}\mathbf{p} + \mathbf{b}_{eq}; \mathbf{A}_{in}\mathbf{x} \leqslant \mathbf{B}_{in}\mathbf{p} + \mathbf{b}_{in} \right\} \\ &= \left\{ \mathbf{x}(\mathbf{p}) |\ \mathbf{x}(\mathbf{p}) = \sum_{i=1}^{v} \lambda_i(\mathbf{p}) \mathbf{x}_i(\mathbf{p}) + \sum_{i=1}^{r} \gamma_i \mathbf{y}_i + \sum_{i=1}^{l} \mu_i \mathbf{z}_i \right\} \\ & 0 \leqslant \lambda_i(\mathbf{p}) \leqslant 1,\ \sum_{i=1}^{v} \lambda_i(\mathbf{p}) = 1\ ,\ \gamma_i \geqslant 0\ ,\ \forall \mu_i \end{aligned} \tag{19}$$

where $\mathbf{z}_i$ are the lines, $\mathbf{y}_i$ are the rays, $\mathbf{x}_i$ are the vertices and $\mu_i, \gamma_i, \lambda_i$ the corresponding coefficients.

Remark: Only the vertices are concerned by the parameterization of the polyhedron (*parameterized vertices* $\mathbf{x}_i(\mathbf{p})$), the rays and the lines do not change with the parameters' variation.

The parameterized polyhedron $P'(\mathbf{p})$ can be written as a non-parameterized polyhedron in an augmented space as:

$$\tilde{P}' = \left\{ \left(\begin{array}{c} \mathbf{x} \\ \mathbf{p} \end{array} \right) \in \mathbb{R}^{n+m} \left| \begin{array}{l} [\mathbf{A}_{eq} | -\mathbf{B}_{eq}] \left(\begin{array}{c} \mathbf{x} \\ \mathbf{p} \end{array} \right) = \mathbf{b}_{eq} \\ [\mathbf{A}_{in} | -\mathbf{B}_{in}] \left(\begin{array}{c} \mathbf{x} \\ \mathbf{p} \end{array} \right) = \mathbf{b}_{in} \end{array} \right. \right\} \tag{20}$$

with a homogenous representation given by:

$$\begin{aligned} \tilde{D} &= \left\{ \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi\,\mathbf{p} \\ \xi \end{array} \right) \left| \begin{array}{l} \tilde{\mathbf{A}}_{eq} \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi\,\mathbf{p} \\ \xi \end{array} \right) = 0 \\ \tilde{\mathbf{A}}_{in} \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi\,\mathbf{p} \\ \xi \end{array} \right) \geqslant 0 \end{array} \right. \right\} = \\ &= \left\{ \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi\,\mathbf{p} \\ \xi \end{array} \right) \middle| \left(\begin{array}{c} \xi\,\mathbf{x} \\ \xi\,\mathbf{p} \\ \xi \end{array} \right) = \tilde{\mathbf{Z}}\tilde{\lambda} + \tilde{\mathbf{Y}}\tilde{\mu};\ \tilde{\mu} \geqslant 0 \right\} \end{aligned} \tag{21}$$

where $\tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}$ are as in (17), the matrices:

$$\begin{aligned} \tilde{\mathbf{A}}_{eq} &= [\mathbf{A}_{eq} |\ {-\mathbf{B}_{eq}} | -\mathbf{b}_{eq}]\ ; \\ \tilde{\mathbf{A}}_{in} &= \left[\begin{array}{c|c|c} \mathbf{A}_{in} & -\mathbf{B}_{in} & -\mathbf{b}_{in} \\ \hline 0 \cdots 0 & 0 \cdots 0 & 1 \end{array} \right] \end{aligned}$$

and $\tilde{\lambda},\ \tilde{\mu}$ are free-valued column vectors.

The form (19) faces an important difficulty as it contains unknown parts, i.e. the parameterized vertices $\mathbf{x}_i(\mathbf{p})$.

The parameterized vertices correspond to m-polyhedra in the augmented (data($\mathbb{R}^n$)+parameter($\mathbb{R}^m$)) space as in (20); consequently the original vertices are:

$$\mathbf{x}_i(\mathbf{p}) = \text{Proj}_n\left(F_i^m(\tilde{P}') \cap S(\mathbf{p})\right) \quad (22)$$

where $\text{Proj}_x(.)$ projects the combined space $\mathbb{R}^{n+m}$ onto the original space $\mathbb{R}^n$ and $S(\mathbf{p})$ is the affine subspace:

$$S(\hat{\mathbf{p}}) = \left\{\left(\begin{array}{c}\mathbf{x}\\ \mathbf{p}\end{array}\right) \in \mathbb{R}^{n+m} \middle| \, \mathbf{p} = \hat{\mathbf{p}}\right\} \quad (23)$$

and $F_i^m(\tilde{P}')$ is a m-face of $\tilde{P}'$ found as the intersection between $\tilde{P}'$ and the supporting hyperplanes (Loechner and Wilde, 1997).

For each face of the polyhedron $\tilde{P}'$, a set of active constraints is well defined, resulting in the fact that each point $\left(\begin{array}{cc}\mathbf{x}_i(\mathbf{p})^{\mathrm{T}} & \mathbf{p}^{\mathrm{T}}\end{array}\right)^{\mathrm{T}} \in F_i^m(\tilde{P}')$ lies in a subspace of dimension m and thus x and p are related by:

$$\left[\begin{array}{c}\mathbf{A}_{eq}\\ \bar{\mathbf{A}}_{in_i}\end{array}\right]\mathbf{x} = \left[\begin{array}{c}\mathbf{A}'_{eq}\\ \bar{\mathbf{A}}'_{in_i}\end{array}\right]\mathbf{p} + \left[\begin{array}{c}\mathbf{b}_{eq}\\ \bar{\mathbf{b}}_{in_i}\end{array}\right] \quad (24)$$

where $\bar{\mathbf{A}}_{in_i}, \bar{\mathbf{A}}'_{in_i}, \bar{\mathbf{b}}_{in_i}$ are the subset of the inequalities defined previously, satisfied by saturation. If the matrix $\left[\begin{array}{cc}\mathbf{A}_{eq}^{\mathrm{T}} & \bar{\mathbf{A}}_{in_i}^{\mathrm{T}}\end{array}\right]^T$ is not invertible, it corresponds to faces $F_i^m(\tilde{P}')$where for one given p more than one point $\mathbf{x} \in \mathbb{R}^n$ is feasible and such combinations do not match a vertex of $P'(\mathbf{p})$. In fact this case corresponds to the zones where $P'(\mathbf{p})$ changes its shape.

In the invertible case, the dependencies could be rewritten:

$$\begin{aligned}\mathbf{x}_i(\mathbf{p}) = &\left[\begin{array}{c}\mathbf{A}_{eq}\\ \bar{\mathbf{A}}_{in_i}\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{A}'_{eq}\\ \bar{\mathbf{A}}'_{in_i}\end{array}\right]\mathbf{p} +\\ &+\left[\begin{array}{c}\mathbf{A}_{eq}\\ \bar{\mathbf{A}}_{in_i}\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{b}_{eq}\\ \bar{\mathbf{b}}_{in_i}\end{array}\right]\end{aligned} \quad (25)$$

For the implementation of these theoretical results, an enumeration of the m-faces must be available together with the $k(> \ m)$ generators of each face $F_i^m(\tilde{D})$ in a homogenous representation. If the projections:

$$Pr_n\left(\begin{array}{c}\xi\,\mathbf{x}_i(\mathbf{p})\\ \xi\,\mathbf{p}\\ \xi\end{array}\right) = \left(\begin{array}{c}\xi\,\mathbf{x}_i(\mathbf{p})\\ \xi\end{array}\right); \quad (26)$$

$$Pr_m\left(\begin{array}{c}\xi\,\mathbf{x}_i(\mathbf{p})\\ \xi\,\mathbf{p}\\ \xi\end{array}\right) = \left(\begin{array}{c}\xi\,\mathbf{p}\\ \xi\end{array}\right) \quad (27)$$

are defined, then (22) could be rewritten as:

$$\begin{aligned}&\left(\begin{array}{c}\xi\,\mathbf{x}_i(\mathbf{p})\\ \xi\end{array}\right) = \Pr_n(F_i)\Pr_m(F_i)^{-1}\left(\begin{array}{c}\mathbf{p}\\ \xi\end{array}\right);\\ &F_i = \left[\left(\begin{array}{c}\xi\,\mathbf{x}_{ij}(\mathbf{p})\\ \xi\,\mathbf{p}\\ \xi\end{array}\right)\right], j = 1..k\end{aligned} \quad (28)$$

The case when the right inverse $Pr_m(F_i)^{-1}$ does not exist results in the already mentioned conditions of an m-face that does not define a unique vertex of $P'(\mathbf{p})$.

Remark: Numerical methods (Leverge, 1994) exist for implementing the double description of polyhedra. The polyhedral duality allows both transformations, from constraints to generators and conversely ((Leverge, 1994), (Loechner and Wilde, 1997), (Motzkin et al., 1953), (Schrijver, 1986), (Wilde, 1993)).

4.2 Explicit solution of LP

Recalling the problem to be solved similar to (11):

$$\begin{aligned}&x^*(\mathbf{p}) = \min_{\mathbf{x}} f^T\mathbf{x}\\ &\text{subject to } A_{in}\mathbf{x} \leqslant B_{in}\mathbf{p} + b_{in}\end{aligned} \quad (29)$$

with the optimal solution as a piecewise affine function of the parameter.

Consider now a fixed parameter $\mathbf{p}_{ct}$. When analyzing the optimization problem (29) corresponding to this value, a geometrical point of view can be used, as in Chernikova algorithm (Leverge, 1994).

Proposition 2. For a linear problem three cases may arise:

a) If the associated polyhedron $P = \{x | A_{in}\mathbf{x} \leqslant B_{in}\mathbf{p}_{ct} + b_{in}\}$ is empty, the problem is infeasible;

b) If there exists a bidirectional ray $\mathbf{z}$ such that $f^T\mathbf{z} \neq 0$ or there exists a unidirectional ray $\mathbf{y}$ such that $f^T\mathbf{y} \leqslant 0$, then the minimum is unbounded;

c) If all bidirectional rays $\mathbf{z}$ are such that $f^T\mathbf{z} = 0$ and all unidirectional rays $\mathbf{y}$ are such that $f^T\mathbf{y} \geqslant 0$, then the minimum is defined by: $\min\left\{f^T\mathbf{x}_i | \mathbf{x}_i \text{ vertex of } P\right\}$ and the solution is:

$$\begin{aligned}&S = conv.hull\left\{\mathbf{x}'_1, \ldots, \mathbf{x}'_s\right\} +\\ &+cone\left\{\mathbf{y}'_1, \ldots, \mathbf{y}'_r\right\} + lin.spaceP\end{aligned}$$

where $\mathbf{x}'_i$ are the vertices attaining the minimum and y'_i are such that $f^T\mathbf{y}'_i = 0$.

Now extending this perspective to the multi-parametric case for each $\mathbf{p} \in \mathbb{R}^n$, a similar result can be established.

Proposition 3. The solution of a multi-parametric linear optimization problem is characterized by the followings:

a) If there exists a bidirectional ray $\mathbf{z}$ such that $f^T\mathbf{z} \neq 0$ or there exists a unidirectional ray $\mathbf{y}$ such that $f^T\mathbf{y} \leqslant 0$, then the minimum is unbounded;

b) For the sub domains of the parameter space $D_{ifez} \in \mathbb{R}^n$ with the associated polyhedron $P = \{\mathbf{x} | A_{in}\mathbf{x} \leqslant B_{in}\mathbf{p} + b_{in}\}$ empty while $\mathbf{p} \in D_{ifez}$, the problem is infeasible (this can be restated in terms of parameterized vertices: "for the sub domains where no parameterized vertex is available, the problem is infeasible");

c) If all bidirectional rays $\mathbf{z}$ are such that $f^T\mathbf{z} = 0$ and all unidirectional rays $\mathbf{y}$ are such that $f^T\mathbf{y} \geqslant 0$, then the sub domains D_k can be defined such that the minimum:

$$\min\left\{f^T\mathbf{x}_i(\mathbf{p})|\mathbf{x}_i(\mathbf{p}) \text{ vertex of } P(\mathbf{p})\right\}$$

is attained by the same subset of vertices of . The complete solution for this sub domain is:

$$S_k(\mathbf{p}) = conv.hull\left\{\mathbf{x}'_{1k}(\mathbf{p}), \ldots, \mathbf{x}'_{sk}(\mathbf{p})\right\} +$$
$$+cone\left\{\mathbf{y}'_1, \ldots, \mathbf{y}'_r\right\} + lin.space P(\mathbf{p})$$

where $\mathbf{x}'_i$ are the vertices corresponding to the minimum and $\mathbf{y}'_i$ are such that $f^T\mathbf{y}'_i = 0$.

One has to observe that our goal is to find the explicit solution for the LP derived from the optimization problem in robust MPC which has some particularities:

- The linearity space is empty since the cost function is positive convex.
- There is no unidirectional ray such that because this will imply that the cost function is not convex.
- A single value in $S_k(\mathbf{p})$ is to be used on-line in MPC.

Proposition 4. The solution of a multi-parametric linear optimization problem within robust MPC satisfies:

a) The problem is infeasible for the sub domains $D_{ifez} \in \mathbb{R}^n$ where no parameterized vertex is available;

b) Sub domains D_k are defined as the zones for which the solution $S_k(\mathbf{p}) = conv.hull\left\{\mathbf{x}'_{1k}, \ldots, \mathbf{x}'_{sk}\right\}$ is given by the same set of parameterized vertices satisfying:

$$f^T\mathbf{x}'_{1k} = \ldots = f^T\mathbf{x}'_{sk} =$$
$$= \min\left\{f^T\mathbf{x}_i(\mathbf{p})|\mathbf{x}_i(\mathbf{p}) \text{ vertex of } P(\mathbf{p})\right\}$$

Remark: As the parameters in (29) vary inside the parameter space, the vertices of the optimization domain may split, shift or merge. The global optimum will follow this evolution within the parameter space as the optimum is a continuous function of parameter.

From a practical point of view the implementation of this result is direct and follows the steps:

1. Find the expression of the parameterized feasible domain in the augmented data+parameter space:
$$A_{in}\mathbf{x} \leqslant B_{in}\mathbf{p}+b_{in} \Leftrightarrow \left[\begin{array}{c} \mathrm{A_{in}}|-B_{in} \end{array}\right]\left[\begin{array}{c}\mathbf{x}\\ \mathbf{p}\end{array}\right] \leqslant b_{in}$$
2. Find the m-vertices where n is the dimension of the parameter space.
3. Retain only those corresponding to parameterized vertices by ignoring those with non-invertible projection on the parameter space
4. Compute validity domain D_k for each parameterized vertex
5. Compare each pair of vertices. In the case of a non-empty intersection of their validity domains, split them using the linear cost function. The final expression will be a union of regions corresponding to the parameterized vertices containing the optimum.

A special attention must be given to the step 5 with the iterative comparison of the vertices and their validity domains. A possible routine may be based on the following procedure.

```
 procedure CutDomains (VD: the set of
all validity domains)

n=cardinal (VD)
i=1; j=2
while i<n+1
while j<n+1
if VD_j ∩ VD_i ≠ ∅
if f^T x_i ⩽ f^T x_j then VD_j = VD_j − VD_i
if f^T x_j ⩽ f^T x_i then VD_i = VD_i − VD_j
j=j+1
endif
end
i=i+1
end
```

Remark: The procedure is initialized with the set of validity domains obtained after the edges' enumeration (step 2).

Remark: The difference of two convex domains is not a close operation and thus the output of the procedure is a union of convex sub domains of the parameters space which do not necessarily cover the entire $\mathbb{R}^m$ (step 4).

From the RMPC point of view, the difference:

$$\aleph = \mathbb{R}^m \backslash \left\{\cup D_k;\ k = 1..n_D\right\} \qquad (30)$$

describes the regions of infeasible parameters.

Once the set of parameter space sub domains D_k created, it can be used in an on-line optimization finding the control sequence for robust MPC.

Algorithm 2 (on-line solver)

1. Find the appurtenance set $D_k;\ k = 1..n_D$ for the current parameter p. Return infeasible if no D_k is found.
2. Compute $k_{u_{MPC}} = x_k(\mathbf{p})$ using the piecewise formulation of the parameterized vertices as in (25) and effectively apply the first component.
3. Restart from 1 with the new p.

5 EXAMPLE

Consider the model (Scokaert and Mayne, 1998):

$$x_{t+1} = x_t + u_t + v_t$$

In order to illustrate the ideas of RMPC presented earlier, a two step prediction is considered and thus the following optimization problem is to be solved at each sampling time:

$$V(x_t) = \min_{u_t, u_{t+1}} \sum_{k=0}^{1} \left|x_{t+k|t}\right| + 10\left|u_{t+k}\right|$$
$$s.t. \begin{cases} \text{- } 1.2 \leqslant x_{t+k|t} \leqslant 2, k = 0,1,2 \\ \text{- } 1 \leqslant x_{t+2|t} \leqslant 1, \\ \text{- } 1 \leqslant v_{t+k} \leqslant 1, k = 0,1 \end{cases} \tag{31}$$

Ignoring the disturbances, the explicit solution of the problem can be found using the geometrical approach presented in the previous section by inspecting the 22 parameterized vertices. After the stage of discrimination of the validity domains, the explicit RMPC law is found as:

Affine control law	Validity domain
$u_t = -x_t - 1$	$-1.2 \leqslant x_t \leqslant -1$
0	$-1 \leqslant x_t \leqslant 0$
0	$0 \leqslant x_t \leqslant 1$
$u_t = -x_t + 1$	$1 \leqslant x_t \leqslant 2$

It can be observed that there are two domains with the same control law due to the fact that the cost function changes its slope and thus the maximum lies on different parameterized vertices in the augmented space. In this case, as their union is a convex set, they can be collated in a single set. In the general case, this operation can be done using tools of convex recognition of union of polyhedra (see (Bemporad et al., 2002a) for details).

Simulating this control law for an initial condition $x_0 = -1.2$ proves to keep the system trajectory inside the constraints in the disturbance free case (Figure 2a). If the same controller is used with $v_k = -1/k,\ k \geqslant 1$, the trajectory will violate the constraints (Figure 2b).

Further if the robust MPC explicit formulation is to be achieved then the min-max version of (31) is to be solved:

$$V(x_t) = \min_{u_t, u_{t+1}} \max_{v_t, v_{t+1}} \sum_{k=0}^{1} \left|x_{t+k|t}\right| + 10\left|u_{t+k}\right|$$
$$s.t. \begin{cases} \text{- } 1.2 \leqslant x_{t+k|t} \leqslant 2, k = 0,1,2 \\ \text{- } 1 \leqslant x_{t+2|t} \leqslant 1, \\ \text{- } 1 \leqslant v_{t+k} \leqslant 1, k = 0,1 \end{cases} \tag{32}$$

In this form, there is no solution as the optimization is infeasible. In fact there is no control law at first sampling time:

$$u_{t|t} = a_1 x_t + b_1$$
$$u_{t+1|t} = a_2 x_t + b_2 u_{t|t} + c_2$$

which can keep robustly the system trajectory within the constraints. This fact is obvious as long as an

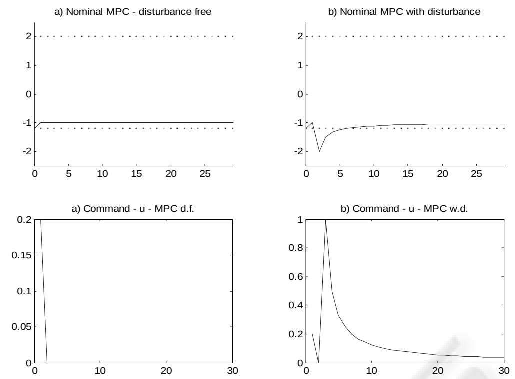

Figure 2: a) Left: Nominal MPC - disturbance-free case; b) Right: Nominal MPC for the system affected by disturbances.

"open-loop" type of RMPC is considered, where the cumulative damage of the disturbances can not be mitigated. When writing explicitly the end-point constraints in (32) for the extremal combinations of disturbances, this becomes evident as:

$$\begin{bmatrix} \nu_t \\ \nu_{t+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$$
$$-1 \leqslant x_t + u_{t|t} + u_{t+1|t} + 2 \leqslant 1 \Rightarrow$$
$$-3 \leqslant x_t + u_{t|t} + u_{t+1|t} \leqslant -1;$$
$$\begin{bmatrix} \nu_t \\ \nu_{t+1} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Rightarrow$$
$$-1 \leqslant x_t + u_{t|t} + u_{t+1|t} - 2 \leqslant 1 \Rightarrow$$
$$1 \leqslant x_t + u_{t|t} + u_{t+1|t} \leqslant 3$$

which means that there is no control combination to maintain the law feasible without a prior knowledge of disturbances. However the so called "closed loop" formulation provides the necessary degrees of freedom in this sense. One has to solve:

$$V(x_t) = \min_{u_t} \max_{v_t} \min_{u_{t+1}} \max_{v_{t+1}} \sum_{k=0}^{1} \left|x_{t+k|t}\right| + 10\left|u_{t+k}\right|$$
$$s.t. \begin{cases} \text{- } 1.2 \leqslant x_{t+k|t} \leqslant 2, k = 0,1,2 \\ \text{- } 1 \leqslant x_{t+2|t} \leqslant 1, \\ \text{- } 1 \leqslant v_{t+k} \leqslant 1, k = 0,1 \end{cases} \tag{33}$$

Following the theoretical result in section 4, the explicit solution can be achieved by solving the inner minimization:

$$V(x_t, u_t, v_t) = \min_{u_{t+1}} \max_{v_{t+1}} |x_t| +$$
$$+ |x_t + u_t + \nu_t| + 10|u_t| + 10|u_{t+1}|$$
$$s.t. \begin{cases} \text{- } 1.2 \leqslant x_{t+k|t} \leqslant 2, k = 0,1,2 \\ \text{- } 1 \leqslant x_{t+2|t} \leqslant 1, \\ \text{- } 1 \leqslant v_{t+k} \leqslant 1, k = 0,1 \end{cases} \tag{34}$$

The solution using the geometrical approach is immediate as there are exactly 2 parameterized vertices on which the minimum lies and associated control law is:

$u_{t+1|t} = -(x_t + u_{t|t} + \nu_t) = -x_{t+1}$ for - 1.2 $\leqslant x_t \leqslant 2$
Notice that the control law uses the additional information available in comparison with (32). With this result, for the outer optimization problem:

$$\min_{u_t} \max_{v_t} \ |x_t| + \left|11x_{t+1|t}\right| + |10u_t|$$
$$s.t. \begin{cases} -1.2 \leqslant x_{t+k|t} \leqslant 2, k = 0, 1 \\ -1 \leqslant v_t \leqslant 1 \end{cases} \quad (35)$$

the explicit solution is once more immediate as there are only two non-degenerate parameterized vertices describing the geometric locus of the minimum. Applying this RMPC law:

$$u_t = -x_t \text{ for - 1.2} \leqslant x_t \leqslant 2$$

the system affected by disturbances is regulated to the origin (Figure 3). The solutions of the optimization

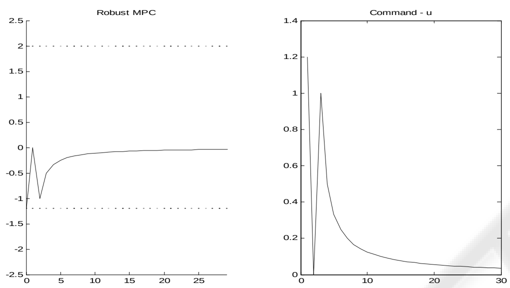

Figure 3: System trajectory with robust MPC law.

problems in (31), (34), (35) were obtained using parameterized polyhedra routines in 2, 0.39 and 0.91 seconds respectively. However for complex system the computational time may explode as the number of parameterized vertices has an exponential dependence on the number of constraints added during the transformation stages.

6 CONCLUSION

The paper used a unified approach for the constraints handling in the context of RMPC confirming the formulation of the optimal sequence as a multi-parametric quadratic problem. The explicit solution of this problem was synthesized by means of parameterized polyhedra. This geometrical approach proposes an alternative to the recent methods presented in the literature. Its advantages might be the fact that optimum lies on the parameterized vertices providing a natural constant linear affine dependence in the context parameters. An aspect which may receive further attention is the enumeration of faces for the parameterized polyhedra which may turn to be a computationally demanding task.

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