EVOLUTIONARY COMPUTATION FOR DISCRETE AND CONTINUOUS TIME OPTIMAL CONTROL PROBLEMS

Yechiel Crispin
Department of Aerospace Engineering
Embry-Riddle University
Daytona Beach, FL 32114

Keywords: Optimal Control, Rocket Dynamics, Goddard's Problem, Evolutionary Computation, Genetic Algorithms.

Abstract: Nonlinear discrete time and continuous time optimal control problems with terminal constraints are solved using a new evolutionary approach which seeks the control history directly by evolutionary computation. Unlike methods that use the first order necessary conditions to determine the optimum, the main advantage of the present method is that it does not require the development of a Hamiltonian formulation and consequently, it eliminates the requirement to solve the adjoint problem which usually leads to a difficult two-point boundary value problem. The method is verified on two benchmark problems. The first problem is the discrete time velocity direction programming problem with the effects of gravity, thrust and drag and a terminal constraint on the final vertical position. The second problem is a continuous time optimal control problem in rocket dynamics, the Goddard's problem. The solutions of both problems compared favorably with published results based on gradient methods.

1 INTRODUCTION

An optimal control problem consists of finding the time histories of the controls and the state variables such as to maximize an integral performance index over a finite period of time, subject to dynamical constraints in the form of a system of ordinary differential equations (Bryson, 1975). In a discrete-time optimal control problem, the time period is divided into a finite number of time intervals of equal duration ΔT. The controls are kept constant over each time interval. This results in a considerable simplification of the continuous time problem, since the ordinary differential equations can be reduced to difference equations and the integral performance index can be reduced to a finite sum over the discrete time counter (Bryson, 1999). In some problems, additional constraints may be prescribed on the final states of the system.

Modern methods for solving the optimal control problem are extensions of the classical methods of the calculus of variations (Fox, 1950). These methods are known as indirect methods and are based on the maximum principle of Pontryagin, which is a statement of the first order necessary conditions for optimality, and results in a two-point boundary value problem (TPBVP) for the state and adjoint variables (L.S. Pontryagin and Mishchenko, 1962).

It has been known, however, that the TPBVP is much more difficult to solve than the initial value problem (IVP). As a consequence, a second class of solutions, known as the direct method has evolved. For example, attempts have been made to recast the original dynamic optimization problem as a static optimization problem by direct transcription (Betts, 2001) or some other discretisation method, eventually reformulating the original problem as a nonlinear programming (NLP) problem. This is often achieved by parameterisation of the state variables or the controls, or both. The original differential equations or difference equations are reduced to algebraic equality constraints. A significant advantage of this method is that the Hamiltonian formulation is completely avoided, which can be advantageous to practicing engineers who have not been exposed to the theoretical framework of optimal control. However, there are some problems with this approach. First, it might result in a large scale NLP problem which might suffer from numerical stability and convergence problems and might require excessive computing time. Also, the parameterisation might introduce spurious local minima which are not present in the original problem.

With the advent of computing power and the

Crispin Y. (2005).
EVOLUTIONARY COMPUTATION FOR DISCRETE AND CONTINUOUS TIME OPTIMAL CONTROL PROBLEMS.
In *Proceedings of the Second International Conference on Informatics in Control, Automation and Robotics*, pages 45-54
DOI: 10.5220/0001171200450054

progress made in methods that are based on optimization analogies from nature, it became possible to achieve a remedy to some of the above mentioned disadvantages through the use of global methods of optimization. These include stochastic methods, such as simulated annealing (Laarhoven and Aarts, 1989), (Kirkpatrick and Vecchi, 1983) and evolutionary computation methods (Fogel, 1998), (Schwefel, 1995) such as genetic algorithms (GAs) (Michalewicz, 1992), see also (Z. Michalewicz and Krawczyk, 1992) for an interesting treatment of the linear discrete-time problem.

Genetic algorithms provide a powerful mechanism towards a global search for the optimum, but in many cases, the convergence is very slow. However, as will be shown in this paper, if the GA is supplemented by problem specific heuristics, the convergence can be accelerated significantly. It is well known that GAs are based on a guided random search through the genetic operators and evolution by artificial selection. This process is inherently very slow, because the search space is very large and evolution progresses step by step, exploring many regions with solutions of low fitness. However, it is often possible to guide the search further, by incorporating qualitative knowledge about potential good solutions. In many problems, this might involve simple heuristics, which when combined with the genetic search, provide a powerful tool for finding the optimum very quickly.

The purpose of the present work is to incorporate problem specific heuristic arguments, which when combined with a modified hybrid GA, can solve the discrete-time optimal control problem very easily. There are significant advantages to this approach. First, the need to solve a difficult two-point boundary value problem (TPBVP) is completely avoided. Instead, only initial value problems (IVP) need to be solved. Second, after finding an optimal solution, we verify that it approximately satisfies the first-order necessary conditions for a stationary solution, so the mathematical soundness of the traditional necessary conditions is retained. Furthermore, after obtaining a solution by direct genetic search, the static and dynamic Lagrange multipliers, i.e., the adjoint variables, can be computed and compared with the results from a gradient method. All this is achieved without directly solving the TPBVP. There is a price to be paid, however, since, in the process, we are solving many initial value problems (IVPs). This might present a challenge in more advanced and difficult problems, where the dynamics are described by higher order systems of ordinary differential equations, or when the equations are difficult to integrate over the required time interval and special methods of numerical integration are required. On the other hand, if the system is described by discrete-time difference equations that are relatively well behaved and easy to iterate, the need to solve the initial value problem many times does not represent a serious problem. For instance, the example problem presented here , the discrete velocity programming problem (DVDP) with the combined effects of gravity, thrust and drag, together with a terminal constraint (Bryson, 1999), runs on a 1.6 GHz pentium 4 processor in less than one minute CPU time.

In the next section, a mathematical formulation of the discrete time optimal control problem is given. This formulation is used to study a specific example of a discrete time problem, namely the velocity direction programming of a body moving in a viscous fluid. Details of this problem are given in Section 3. The evolutionary computation approach to the solution is then described in Section 4 where results are presented and compared with the results of an indirect gradient method developed by Bryson (Bryson, 1999). In Section 5, a mathematical formulation of the continuous time optimal control problem for nonlinear dynamical systems is presented. A specific illustrative example of a continuous time optimal control problem is described in Section 6, where we study the Goddard's problem of rocket dynamics using the proposed evolutionary computation method. Finally conclusions are summarized in Section 7.

2 OPTIMAL CONTROL OF DISCRETE TIME NONLINEAR SYSTEMS

In this section, a formulation is developed for the nonlinear discrete-time optimal control problem subject to terminal constraints. Consider the nonlinear discrete-time dynamical system described by difference equations with initial conditions

$$\boldsymbol{x}(i+1) = \boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] \tag{2.1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x_0} \tag{2.2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of state variables, $\boldsymbol{u} \in \mathbb{R}^p$, $p < n$ is the vector of control variables and $i \in [0, N-1]$ is a discrete time counter. The function $\boldsymbol{f}$ is a nonlinear function of the state vector, the control vector and the discrete time i, i.e., $\boldsymbol{f} : \mathbb{R}^n$ x $\mathbb{R}^p$ x $\mathbb{R} \mapsto \mathbb{R}^n$. Next, define a performance index

$$J[\boldsymbol{x}(i), \boldsymbol{u}(i), i, N] = \phi[\boldsymbol{x}(N)] + \Sigma_{i=0}^{M} L[\boldsymbol{x}(i), \boldsymbol{u}(i), i] \tag{2.3}$$

where $M = N - 1$, $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, $L : \mathbb{R}^n$ x $\mathbb{R}^p$ x $\mathbb{R} \mapsto \mathbb{R}$

Here L is the Lagrangian function and $\phi[\boldsymbol{x}(N)]$ is a function of the terminal value of the state vector

$\boldsymbol{x}(N)$. In some problems, additional terminal constraints can be prescribed through the use of functions ψ of the state variables $\boldsymbol{x}(N)$

$$\psi[\boldsymbol{x}(N)] = 0, \qquad \psi : \mathbb{R}^n \mapsto \mathbb{R}^k \quad k \leq n \tag{2.4}$$

The optimal control problem consists of finding the control sequence $\boldsymbol{u}(i)$ such as to maximize (or minimize) the performance index defined by (2.3), subject to the dynamical equations (2.1) with initial conditions (2.2) and terminal constraints (2.4). This formulation is known as the Bolza problem in the calculus of variations. In an alternative formulation, due to Mayer, the state vector $x_j,\ j \in [1,n]$ is augmented by an additional state variable x_{n+1} which satisfies the initial value problem:

$$x_{n+1}(i+1) = x_{n+1}(i) + L[\boldsymbol{x}(i), \boldsymbol{u}(i), i] \tag{2.5}$$

$$x_{n+1}(0) = 0 \tag{2.6}$$

The performance index can then be written in the following form

$$J(N) = \phi[\boldsymbol{x}(N)] + x_{n+1}(N) \equiv \phi_a[\boldsymbol{x_a}(N)] \tag{2.7}$$

where $\boldsymbol{x_a} = [\boldsymbol{x} \quad x_{n+1}]^T$ is the augmented state vector and ϕ_a the augmented performance index. In this paper, the Meyer formulation is used.

We next define an augmented performance index with adjoint constraints ψ and adjoint dynamical constraints $\boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] - \boldsymbol{x}(i+1) = 0$, with static and dynamical Lagrange multipliers ν and λ, respectively, in the following form:

$$J_a = \phi + \nu^T \psi + \lambda^T(0)[\boldsymbol{x_0} - \boldsymbol{x}(0)] +$$

$$+ \Sigma_{i=0}^{M} \lambda^T(i+1)\{\boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] - \boldsymbol{x}(i+1)\} \tag{2.8}$$

Define a Hamiltonian function as

$$H(i) = \lambda^T(i+1)\boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] \tag{2.9}$$

Rewriting the augmented performance index in terms of the Hamiltonian function, we get

$$J_a = \phi + \nu^T \psi - \lambda^T(N)\boldsymbol{x}(N) + \lambda^T(0)\boldsymbol{x_0} +$$

$$+ \Sigma_{i=0}^{M}[H(i) - \lambda^T(i)\boldsymbol{x}(i)] \tag{2.10}$$

A first order necessary condition for J_a to reach a stationary solution is given by the discrete version of the Euler-Lagrange equations

$$\lambda^T(i) = H_{\boldsymbol{x}}(i) = \lambda^T(i+1)\boldsymbol{f_x}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] \tag{2.11}$$

with final conditions

$$\lambda^T(N) = \phi_x + \nu^T \psi_{\boldsymbol{x}} \tag{2.12}$$

The control $\boldsymbol{u}(i)$ satisfies the optimality condition:

$$H_{\boldsymbol{u}}(i) = \lambda^T(i+1)\boldsymbol{f_u}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] = 0 \tag{2.13}$$

If we define an augmented function Φ as

$$\Phi = \phi + \nu^T \psi \tag{2.14}$$

then the final conditions can be written in terms of the augmented function Φ in a similar way to the problem without terminal constraints

$$\lambda^T(N) = \Phi_{\boldsymbol{x}} = \phi_{\boldsymbol{x}} + \nu^T \psi_{\boldsymbol{x}} \tag{2.15}$$

The indirect approach to optimal control uses the necessary conditions for an optimum to obtain a solution. In this approach, the state equations (2.1) with initial conditions (2.2) need to be solved together with the adjoint equations (2.11) and the final conditions (2.15), where the control sequence $\boldsymbol{u}(i)$ is to be determined from the optimality condition (2.13). This represents a coupled system of nonlinear difference equations with part of the boundary conditions specified at the initial time $i = 0$ and the rest of the boundary conditions specified at the final time $i = N$. This is a nonlinear two-point boundary value problem (TPBVP) in difference equations. Except for some special simplified cases, it is usually very difficult to obtain solutions for such nonlinear TPBVPs in closed form. Therefore, many numerical methods have been developed to tackle this problem.

Several gradient based methods have been proposed for solving the discrete-time optimal control problem (Mayne, 1966). For example, Murray and Yakowitz (Murray and Yakowitz, 1984) and (Yakowitz and Rutherford, 1984) developed a differential dynamic programming and Newton's method for the solution of discrete optimal control problems, see also the book of Jacobson and Mayne (Jacobson and Mayne, 1970), (Ohno, 1978), (Pantoja, 1988) and (Dunn and Bertsekas, 1989). Similar methods have been further developed by Liao and Shoemaker (Liao and Shoemaker, 1991). Another method, the trust region method, was proposed by Coleman and Liao (Coleman and Liao, 1995) for the solution of unconstrained discrete-time optimal control problems. Although confined to the unconstrained problem, this method works for large scale minimization problems.

In contrast to the indirect approach, in the present proposed approach, the optimality condition (2.13) and the adjoint equations (2.11) together with their final conditions (2.15) are not used in order to obtain the optimal solution. Instead, the optimal values of the control sequence $\boldsymbol{u}(i)$ are found by a modified genetic search method starting with an initial population of solutions with values of $\boldsymbol{u}(i)$ randomly distributed within a given domain of admissible controls. During the search, approximate, not necessarily optimal values of the solutions $\boldsymbol{u}(i)$ are found for each generation. With these approximate values known, the state equations (2.1) together with their initial conditions (2.2) are very easy to solve as an initial value problem, by a straightforward iteration of the difference equations from $i = 0$ to $i = N - 1$. At the end of this iterative process, the final values $\boldsymbol{x}(N)$ are obtained, and the fitness function can be determined. The search than seeks to maximize the fitness function F such as to fulfill the goal of the evolution, which is to maximize $J(N)$, as given by the following Eq.(2.16), subject to the terminal constraints as defined by Eq.(2.17).

$$\text{maximize } J(N) = \phi[\boldsymbol{x}(N)] \tag{2.16}$$

subject to the dynamical equality constraints, Eqs. (2.1-2.2) and to the terminal constraints (2.4), which are repeated here for convenience as Eq.(2.17)

$$\psi[\boldsymbol{x}(N)] = 0$$

$$\psi : \mathbb{R}^n \mapsto \mathbb{R}^k \quad k \leq n \tag{2.17}$$

Since we are using a direct search method, condition (2.17) can also be stated as a search for a maximum, namely we can set a goal which is equivalent to (2.17) in the form

$$\text{maximize } J_1(N) = -\psi^{\mathrm{T}}[\boldsymbol{x}(N)]\psi[\boldsymbol{x}(N)] \tag{2.18}$$

The fitness function F can now be defined by

$$F(N) = \alpha J(N) + (1-\alpha)J_1(N) =$$

$$= \alpha\phi[\boldsymbol{x}(N)] - (1-\alpha)\psi^{\mathrm{T}}[\boldsymbol{x}(N)]\psi[\boldsymbol{x}(N)] \tag{2.19}$$

with $\alpha \in [0, 1]$ and $\boldsymbol{x}(N)$ determined from a solution of the original initial value problem for the state variables:

$$\boldsymbol{x}(i+1) = \boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i], \quad i \in [0, N-1] \tag{2.20}$$

$$\boldsymbol{x}(0) = \boldsymbol{x_0} \tag{2.21}$$

3 VELOCITY DIRECTION CONTROL OF A BODY IN A VISCOUS FLUID

In this section, we treat the case of controlling the motion of a particle moving in a viscous fluid medium by varying the direction of a thrust vector of constant magnitude. We describe the motion in a cartesian system of coordinates in which x is pointing to the right and y is positive downward. The constant thrust force $\boldsymbol{F}$ is acting along the path, i.e. in the direction of the velocity vector $\boldsymbol{V}$ with magnitude $F = amg$. The acceleration of gravity g is acting downward in the positive y direction. The drag force is proportional to the square of the speed and acts in a direction opposite to the velocity vector $\boldsymbol{V}$. The motion is controlled by varying the angle γ, which is positive downward from the horizontal. The velocity direction γ is to be programmed such as to achieve maximum range and fulfill a prescribed terminal constraint on the vertical final location y_f. Newton's second law of motion for a particle of mass m can be written as

$$mdV/dt = mg(a + sin\gamma) - \frac{1}{2}\rho V^2 C_D S \tag{3.1}$$

where ρ is the fluid density, C_D is the coefficient of drag and S is a typical cross-section area of the body. For example, if the motion of the center of gravity of a spherical submarine vehicle is considered, then S is the maximum cross-section area of the vehicle and C_D would depend on the Reynolds number $Re = \rho V d/\mu$, where μ is the fluid viscosity and d the diameter of the vehicle. Dividing (3.1) by the mass m, we obtain

$$dV/dt = g(a + sin\gamma) - V^2/L_c \tag{3.2}$$

The length $L_c = 2m/(\rho S C_D)$ is a typical hydrodynamic length. The other equations of motion are:

$$dx/dt = V cos\gamma \tag{3.3}$$

$$dy/dt = V sin\gamma \tag{3.4}$$

with initial conditions and final constraint

$$V(0) = 0, \quad x(0) = 0, \quad y(0) = 0 \tag{3.5}$$

$$y(t_f) = y_f \tag{3.6}$$

In order to rewrite the equations in nondimensional form, we introduce the following nondimensional variables, denoted by primes:

$$t = (L_c/g)^{1/2} t\prime, \;\; V = (gL_c)^{1/2} V\prime$$

$$x = L_c x\prime, \;\; y = L_c y\prime \tag{3.7}$$

where we have chosen L_c as the characteristic length. Substituting the nondimensional variables (3.7) in the equations of motion (3.2-3.4) and omitting the prime notation, we obtain the nondimensional state equations

$$dV/dt = a + sin\gamma - V^2 \tag{3.8}$$

$$dx/dt = V cos\gamma \tag{3.9}$$

$$dy/dt = V sin\gamma \tag{3.10}$$

In order to formulate a discrete time version of this problem, we first rewrite (3.8) in separated variables form as $dV/(a + sin\gamma - V^2) = dt$. Integrating and using the condition $V(0) = 0$, we get

$$(1/b)argtanh(V/b) = t \tag{3.11}$$

Solving for the speed, we obtain

$$V = btanh(bt) \tag{3.12}$$

$$b = (a + sin\gamma)^{1/2} \tag{3.13}$$

We now develop a discrete time model by dividing the trajectory into a finite number N of straight line segments of fixed duration $\Delta T = t_f/N$ along which the control γ is kept constant. The time at the end of each segment is given by $t(i) = i\Delta T$, with i a time step counter at point i. The time is normalized by $(L_c/g)^{1/2}$, so the nondimensional final time is $t\prime_f = t_f/(L_c/g)^{1/2}$ and the nondimensional time at step i is $t\prime(i) = it\prime_f/N$. The nondimensional time interval is $(\Delta T)\prime = t\prime_f/N$. Writing (3.12) at $t(i+1) = (i+1)\Delta T$, we obtain the velocity at the point $(i+1)$ along the trajectory

$$V(i+1) = b(i)tanh[b(i)(i+1)t\prime_f/N] \tag{3.14}$$

$$b(i) = (a + sin\gamma(i))^{1/2} \tag{3.15}$$

Similarly, substituting the time $t\prime(i) = it\prime_f/N$ in (3.11), the following expression is obtained, which we define as the function $G_0(i)$.

$$ib(i)t\prime_f/N = argtanh[V(i)/b(i)] \equiv G_0(i) \tag{3.16}$$

Introducing a second function $G_1(i)$ defined by

$$G_1(i) = G_0(i) + b(i)t\prime_f/N \tag{3.17}$$

Eq.(3.14) can be written as

$$V(i+1) = b(i)tanh[G_1(i)] \tag{3.18}$$

We now determine the coordinates x and y as a function of time. Using the state equation (3.9) together with the result (3.12) and defining $\theta = bt$, we obtain

$$dx = V cos\gamma dt = bcos\gamma tanh(bt)dt =$$

$$= cos\gamma tanh\theta d\theta \tag{3.19}$$

Integrating along a straight line segment between points i and $i+1$, we get

$$x(i+1) = x(i) +$$

$$+ cos\gamma(i)[logcosh\theta(i+1) - logcosh\theta(i)] \tag{3.20}$$

$$\theta(i) = ib(i)t\prime_f/N = G_0(i) \tag{3.21}$$

$$\theta(i+1) = b(i)(i+1)t\prime_f/N = ib(i)t\prime_f/N +$$

$$+ b(i)t\prime_f/N = G_0(i) + b(i)t\prime_f/N = G_1(i) \tag{3.22}$$

Substituting (3.21-3.22) in (3.20), we obtain the following discrete-time state equation (3.24) for the location $x(i+1)$. The equation for the coordinate $y(i+1)$ can be developed in a similar way to $x(i+1)$, with $cos\gamma(i)$ replaced by $sin\gamma(i)$. Adding the state equation (3.18) for the velocity $V(i+1)$, which is repeated here as Eq.(3.23), the state equations become:

$$V(i+1) = b(i)tanh[G_1(i)] \tag{3.23}$$

$$x(i+1) = x(i) +$$

$$+ cos\gamma(i)\, log[coshG_1(i)/coshG_0(i)] \tag{3.24}$$

$$y(i+1) = y(i) +$$

$$+ sin\gamma(i) log[coshG_1(i)/coshG_0(i)] \tag{3.25}$$

with initial conditions and terminal constraint

$$V(0) = 0, \quad x(0) = 0, \quad y(0) = 0 \tag{3.26}$$

$$y(N) = y\prime_f = y_f/L_c \tag{3.27}$$

The optimal control problem now consists of finding the sequence $\gamma(i)$ for $i \in [0, N-1]$ such as to maximize the range $x(N)$, subject to the state equations (3.23-3.25), the initial conditions (3.26) and the terminal constraint (3.27), where $y\prime_f$ is in units of L_c and the final time $t\prime_f$ in units of $(L_c/g)^{1/2}$.

4 EVOLUTIONARY APPROACH TO OPTIMAL CONTROL

We now describe the proposed direct approach which is based on a genetic search method. As was previously mentioned, an important advantage of this approach is that there is no need to solve the two-point boundary value problem described by the state equations (2.1) and the adjoint equations (2.11), together with the initial conditions (2.2), the final conditions (2.15), the terminal constraints (2.4) and the optimality condition (2.13) for the optimal control $u(i)$.

Instead, the direct evolutionary computation method allows us to evolve a population of solutions such as to maximize the objective function or fitness function $F(N)$. The initial population is built by generating a random population of solutions $\gamma(i)$, $i \in [0, N-1]$, uniformly distributed within a domain $\gamma \in [\gamma_{\text{min}}, \gamma_{\text{max}}]$. Typical values are $\gamma_{\text{max}} = \pi/2$ and either $\gamma_{\text{min}} = -\pi/2$ or $\gamma_{\text{min}} = 0$ depending on the problem. The genetic algorithm evolves this initial population using the operations of selection, mutation and crossover over many generations such as to maximize the fitness function:

$$F(N) = \alpha J(N) + (1-\alpha) J_1(N) =$$

$$= \alpha\phi[\xi(N)] - (1-\alpha)\psi^{\text{T}}[\xi(N)]\psi[\xi(N)] \quad (4.1)$$

with $\alpha \in [0, 1]$ and $J(N)$ and $J_1(N)$ given by:

$$J(N) = \phi[\xi(N)] = x(N) \quad (4.2)$$

$$J_1(N) = \psi^2[\xi(N)] = (y(N) - y_f)^2 \quad (4.3)$$

For each member in the population of solutions, the fitness function depends on the final values $x(N)$ and $y(N)$, which are determined by solving the initial value problem defined by the state equations (3.23-3.25) together with the initial conditions (3.26). This process is repeated over many generations. Here, we run the genetic algorithm for a predetermined number of generations and then we check if the terminal constraint (3.27) is fulfilled. If the constraint is not fulfilled, we can either increase the number of generations or readjust the weight $\alpha \in [0, 1]$.

We now present results obtained by solving this problem using the proposed approach. We first treat the case where $x(N)$is maximized with no constraint placed on y_f. We solve an example where the value of the thrust is $a = 0.05$ and the final time is $t_f = 5$.

The evolution of the solution over 50 generations is shown in Fig.1. The control sequence $\gamma(i)$, the optimal trajectory and the velocity $V^2(i)$ are displayed in Fig.2. The sign of y is reversed for plotting. It can be seen from the plots that the angle varies at the beginning and at the end of the motion, but remains constant in the middle of the maneuver, resulting in a dive along a straight line along a considerable portion of the trajectory. This finding agrees well with the results obtained by Bryson (Bryson, 1999) using a gradient method.

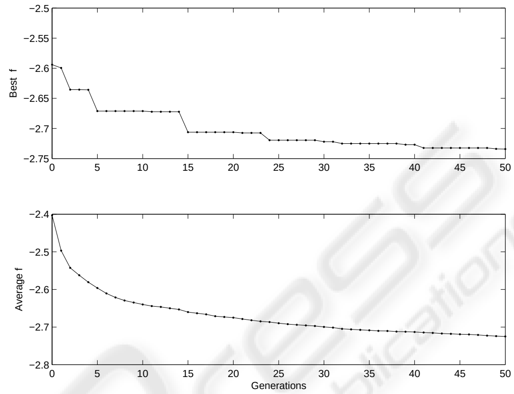

Figure 1: Convergence of the DVDP solution with gravity, thrust a=0.05 and drag, with no terminal constraint on y_f. $x(N)$ is maximized. With final time $t_f = 5$.

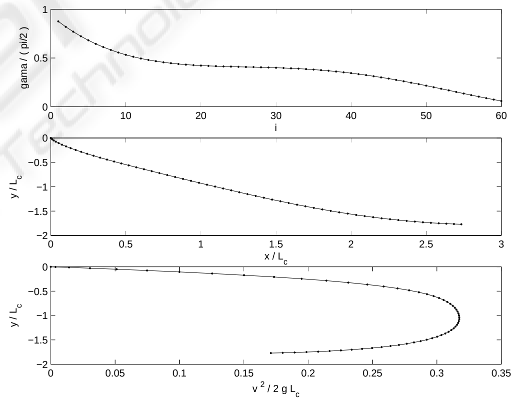

Figure 2: The control sequence, the optimal trajectory and the velocity $V^2(i)$ for the DVDP problem with gravity, thrust a=0.05 and drag. No terminal constraint on y_f. Final time $t_f = 5$. The sign of y is reversed for plotting.

5 NONLINEAR CONTINUOUS TIME OPTIMAL CONTROL

In this section, a formulation is developed for the nonlinear continuous time optimal control problem subject to terminal constraints. Consider the continuous time nonlinear problem described by a system of ordinary differential equations with initial conditions

$$d\boldsymbol{x}/dt = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{5.1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x_0} \tag{5.2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of state variables, $\boldsymbol{u} \in \mathbb{R}^p$, $p < n$ is the vector of control variables and $t \in [0, t_f]$ is the continuous time. The function $\boldsymbol{f}$ is a nonlinear function of the state vector, the control vector and the time t, i.e., $\boldsymbol{f} : \mathbb{R}^n \text{ x } \mathbb{R}^p \text{ x } \mathbb{R} \mapsto \mathbb{R}^n$. Next, define a performance index

$$J[\boldsymbol{x}(t), \boldsymbol{u}(t), t_f] = \phi[\boldsymbol{x}(t_f)] + \int_0^{tf} L[\boldsymbol{x}(t), \boldsymbol{u}(t), t]dt \tag{5.3}$$

$$\phi : \mathbb{R}^n \mapsto \mathbb{R}, \quad L : \mathbb{R}^n \text{x} \mathbb{R}^p \text{x} \mathbb{R} \mapsto \mathbb{R}$$

Here L is the Lagrangian function and $\phi[\boldsymbol{x}(t_f)]$ is a function of the terminal value of the state vector $\boldsymbol{x}(t_f)$. In some problems, additional terminal constraints can be prescribed through the use of functions ψ of the state variables $\boldsymbol{x}(t_f)$

$$\psi[\boldsymbol{x}(t_f)] = 0, \quad \psi : \mathbb{R}^n \mapsto \mathbb{R}^k, \ \ k \le n \tag{5.4}$$

The formulation of the optimal control problem according to Bolza consists of finding the control $\boldsymbol{u}(t)$ such as to maximize the performance index defined by (5.3), subject to the state equations (5.1) with initial conditions (5.2) and terminal constraints (5.4). In the alternative formulation, due to Mayer, the state vector x_j, $j \in [1, n]$ is augmented by an additional state variable x_{n+1} which satisfies the following initial value problem:

$$dx_{n+1}/dt = L[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{5.5}$$

$$x_{n+1}(0) = 0 \tag{5.6}$$

The performance index can then be written as

$$J(t_f) = \phi[\boldsymbol{x}(t_f)] + x_{n+1}(t_f) \equiv \phi_a[\boldsymbol{x_a}(t_f)] \tag{5.7}$$

where $\boldsymbol{x_a} = [\boldsymbol{x} \ \ x_{n+1}]^T$ is the augmented state vector and ϕ_a the augmented performance index. We next define an augmented performance index with adjoint constraints ψ and adjoint dynamical constraints $\boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] - d\boldsymbol{x}/dt = 0$, with static and dynamical Lagrange multipliers ν and λ as:

$$J_a(t_f) = \phi[\boldsymbol{x}(t_f)] + \nu^T \psi[\boldsymbol{x}(t_f)] + \lambda^T(0)[\boldsymbol{x_0} - \boldsymbol{x}(0)]$$

$$+ \int_0^{tf} \lambda^T(t)[\boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] - d\boldsymbol{x}/dt]dt \tag{5.8}$$

Introducing a Hamiltonian function

$$H(t) = \lambda^T(t)\boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{5.9}$$

and rewriting the augmented performance index in terms of the Hamiltonian, we get

$$J_a(t_f) = \phi + \nu^T \psi - \lambda^T(t_f)\boldsymbol{x}(t_f) + \lambda^T(0)\boldsymbol{x_0} +$$

$$+ \int_0^{tf} [H(t) - \lambda^T(t)\boldsymbol{x}(t)]dt \tag{5.10}$$

A first order necessary condition for J_a to reach a stationary solution is given by the Euler-Lagrange equations

$$d\lambda^T/dt = -H_{\boldsymbol{x}}(t) = -\lambda^T(t)\boldsymbol{f_x}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{5.11}$$

with final conditions

$$\lambda^T(t_f) = \phi_x[\boldsymbol{x}(t_f)] + \nu^T \psi_{\boldsymbol{x}}[\boldsymbol{x}(t_f)] \tag{5.12}$$

The control $\boldsymbol{u}(t)$ satisfies the optimality condition:

$$H_{\boldsymbol{u}}(t) = \lambda^T(t)\boldsymbol{f_u}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] = 0 \tag{5.13}$$

If we define an augmented function $\Phi[\boldsymbol{x}(t_f)]$ as

$$\Phi[\boldsymbol{x}(t_f)] = \phi[\boldsymbol{x}(t_f)] + \nu^T \psi[\boldsymbol{x}(t_f)] \tag{5.14}$$

then the final conditions can be written in terms of the augmented function Φ in a similar way to the problem without terminal constraints

$$\lambda^T(t_f) = \Phi_{\boldsymbol{x}}[\boldsymbol{x}(t_f)] = \phi_{\boldsymbol{x}}[\boldsymbol{x}(t_f)] + \nu^T \psi_{\boldsymbol{x}}[\boldsymbol{x}(t_f)] \tag{5.15}$$

In the indirect approach to optimal control, the necessary conditions are used to obtain an optimal solution: the state equations (5.1) with initial conditions (5.2) have to be solved together with the adjoint equations (5.11) and the final conditions (5.15). The control history $u(t)$ is determined from the optimality condition (5.13). Consequently, this approach leads to a coupled system of nonlinear ordinary differential equations with the boundary conditions for the state variables specified at the initial time $t = 0$ and the boundary conditions for the adjoint variables specified at the final time $t = t_f$. This is a nonlinear two-point boundary value problem (TPBVP) in ordinary differential equations. Except for some special simplified cases, it is usually very difficult to obtain solutions for such nonlinear TPBVPs analytically. Many numerical methods have been developed in order to obtain approximate solutions to this problem.

6 GODDARD'S OPTIMAL CONTROL PROBLEM IN ROCKET DYNAMICS

We now illustrate the above approach with a continuous time optimal control example. We apply the optimal control formulation described in the previous section for continuous time dynamical systems to the study of the vertical climb of a single stage sounding rocket launched vertically from the ground. This is known in the literature as the Goddard's problem. The problem is to control the thrust of the rocket such as to maximize the final velocity or the final altitude. There are two versions to this problem: in the first version, the final mass of the rocket is prescribed and the final time is free. In the second version, the final time is prescribed and the final mass is free. The second version of this problem will be presented.

Let $h(t)$ denote the altitude of the rocket as measured from sea level and $v(t)$ and $m(t)$ the velocity and the mass of the rocket, respectively. Here the time t is continuous. The trajectory of the rocket is a vertical straight line. The forces acting on the rocket are the thrust $T(t)$, which is used as the control variable or control history, the aerodynamic drag force $D(h, v)$, which is a function of altitude and speed and the weight of the rocket $m(t)g$, where $m(t)$ is the mass and g is the acceleration of gravity, assumed constant. The equations of motion are:

$$dh/dt = v \tag{6.1}$$

$$mdv/dt = T - D - mg \tag{6.2}$$

$$dm/dt = -T/c \tag{6.3}$$

where the drag force is given by

$$D(h, v) = D_0\, v^2\, exp(-h/h_r) \tag{6.4}$$

Here $h_r = 23800$ ft is a characteristic altitude and D_0 is a characteristic drag force given by

$$D_0 = 0.711 T_M/c^2 \tag{6.5}$$

where T_M is the maximum thrust developed by the rocket. The speed c is the propellant jet exhaust speed. An important parameter in rocket dynamics is the thrust to weight ratio $\tau = T_M/(m_0 g)$, where m_0 is the initial mass of the vehicle and $g = 32.2\ ft/s^2$. In this example, a ratio of 2 is chosen:

$$\tau = T_M/(m_0 g) = 2 \tag{6.6}$$

Here we take a value of $m_0 = 3$ slugs for a small experimental rocket. A typical value of the exhaust speed c and the specific impulse I_{sp} for an early rocket such as the one tested by Goddard is given by

$$c = (3.264 g h_r)^{1/2} = 1581 ft/s$$

$$I_{sp} = c/g = 49.14 sec \tag{6.7}$$

The initial conditions are

$$h(0) = 0, \quad v(0) = 0, \quad m(0) = m_0 = 3\ \ slugs \tag{6.8}$$

The optimal control problem is to find the control history $T(t)$ such as to maximize the final altitude (or the altitude at burnout) $h(t_f)$ in a given time t_f, where a value $t_f = 18$ sec was used in this example. The state equations (6.1-6.3) with the initial conditions (6.8) are to be solved in the optimization process. Before solving this problem, we first restate the problem in non-dimensional form. Choosing the characteristic speed $(gh_r)^{1/2} = 875$ ft/s and the characteristic time $(h_r/g)^{1/2} = 27.2$ sec, we introduce nondimensional variables, denoted here by primes:

$$h = h_r h\prime, \quad v = (gh_r)^{1/2}\, v\prime, \quad t = (h_r/g)^{1/2}\, t\prime$$

$$m = m_0 m\prime, \quad T = T_M T\prime, \quad D = T_M D\prime \tag{6.9}$$

Introducing the variables from Eq.(6.9) in the state equations (6.1-6.3) and simplifying, the following system of non-dimensional equations is obtained:

$$dh/dt = v \tag{6.10}$$

$$mdv/dt = \tau T - \tau\sigma^2\, v^2\, exp(-h) - m \tag{6.11}$$

$$dm/dt = -1.186\, \sigma\tau T \tag{6.12}$$

In this system of equations (6.10-6.12) all the variables are non-dimensional and the prime notation has been omitted. Two independent non-dimensional parameters characterizing this problem are obtained: the thrust to weight ratio τ, introduced before and a ratio of two speeds σ defined by

$$\tau = T_M/(m_0 g) = 2, \quad T_M = 193\ \ lbs$$

$$\sigma = (0.711 g h_r)^{1/2}/c = 0.467 \tag{6.13}$$

The non-dimensional initial conditions are:

$$h(0) = 0, \quad v(0) = 0, \quad m(0) = 1 \tag{6.14}$$

The optimal control problem is to find the control history $T(t)$ such as to maximize the final altitude (the altitude at burnout) $h(t_f)$ in a given normalized time

$$t\prime_f = t_f/(h_r/g)^{1/2} = 0.662$$

subject to the state equations (6.10-6.12) and the initial conditions (6.14). The results of this continuous time optimal control problem are given in Figures (3-4). Figure 3 shows the control history of the thrust as a function of the time in seconds. In the genetic search, the search range for the normalized thrust was between a lower bound of $(T/T_{max})_L = 0.1$ and an upper bound $(T/T_{max})_U = 1$. It can be seen that the thrust increases sharply during the first two seconds of the flight and remains closer to 1 afterwards. Figure 4 displays the state variables, the altitude, the velocity and the mass of the rocket as a function of time. The mass of the rocket decreases almost linearly during the flight due to the optimal control which requires an almost constant thrust. This is in agreement with the results of Betts (J. Betts and Huffman, 1993) and (A.S. Bondarenko and More, 1999) who used a nonlinear programming method. See also (Dolan and More, 2000).

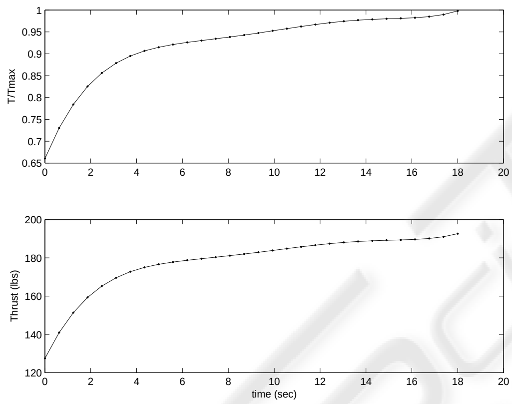

Figure 3: The thrust control history. The upper graph displays the value of the thrust normalized by maximum thrust. The lower graph shows the actual thrust in lbs.

7 CONCLUSIONS

A new method for solving both discrete time and continuous time nonlinear optimal control problems with terminal constraints has been presented. Unlike other methods that use the first-order necessary conditions to find the optimum, the present method seeks the best control history directly by a modified genetic search. As a consequence of this direct search approach, there is no need to develop a Hamiltonian formulation and therefore there is no need to solve a difficult two-point boundary value problem for the state and adjoint variables. This has a significant advantage in more advanced and higher order problems where it is difficult to solve the two point boundary value problem (TPBVP) with large systems of ordinary differential equations. There is a computational price to be paid, however, since the method involves repetitive solutions of the initial value problem (IVP) for the state variables during the evolutionary process.

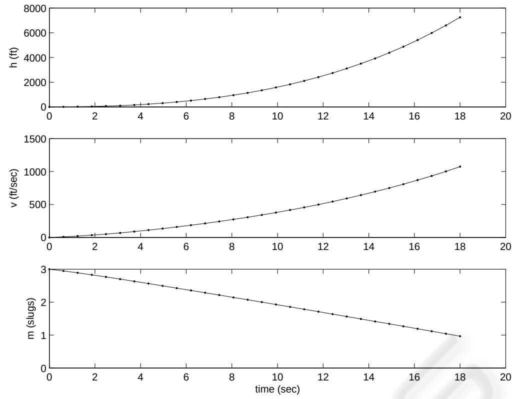

Figure 4: The state variables, the altitude, velocity and mass of the rocket as a function of time

The method was demonstrated by solving a discrete-time optimal control problem, namely, the discrete velocity direction programming problem (DVDP) of a body with the effects of gravity, thrust and hydrodynamic drag. Benchmark problems of this kind were pioneered by Bryson who used analytical and gradient methods. This discrete time problem was solved easily using the proposed approach and the results compared favorably with the results of Bryson.

The method was also applied to a continuous time nonlinear optimal control problem, the Goddard's problem of rocket dynamics. The results compared favorably with published results obtained by a nonlinear programming method.

REFERENCES

A.S. Bondarenko, D. B. and More, J. (1999). Cops: Large scale nonlinearly constrained optimization problems. In *Argonne National Laboratory Technical Report ANL/MCS-TM-237*.

Betts, J. (2001). *Practical Methods for Optimal Control Using Nonlinear Programming*. SIAM, Society for Industrial and Applied Mathematics, Philadelphia, PA.

Bryson, A. (1999). *Dynamic Optimization*. Addison-Wesley Longman, Menlo Park, CA.

Coleman, T. and Liao, A. (1995). An efficient trust region method for unconstrained discrete-time optimal control problems. In *Computational Optimization and Applications, 4:47-66*.

Dolan, E. and More, J. (2000). Benchmarking optimization software with cops. In *Argonne National Labo-ratory Technical Report ANL/MCS-TM-246*.

Dunn, J. and Bertsekas, D. (1989). Efficient dynamic programming implementations of newton's method for unconstrained optimal control problems. In *J. of Optimization Theory and Applications,63, pp. 23-38*.

Fogel, D. (1998). *Evolutionary Computation, The Fossil Record*. IEEE Press, New York.

Fox, C. (1950). *An Introduction to the Calculus of Variations*. Oxford University Press, London.

J. Betts, S. E. and Huffman, W. (1993). Sparse nonlinear programming test problems. In *Technical Report BCSTECH-93-047*. Boeing Computer Services, Seattle, Washington.

Jacobson, D. and Mayne, D. (1970). *Differential Dynamic Programming*. Elsevier Science Publishers, Amsterdam, Netherland.

Kirkpatrick, G. and Vecchi (1983). Optimization by simulated annealing. In *Science, 220: 671-680*. AAAS.

Laarhoven, P. and Aarts, E. (1989). *Simulated Annealing: Theory and Applications*. Kluwer, Amsterdam.

Liao, L. and Shoemaker, C. (1991). Convergence in unconstrained discrete-time differential dynamic programming. In *IEEE Transactions on Automatic Control, 36, pp. 692-706*. IEEE.

L.S. Pontryagin, V.G. Boltyanskii, R. G. and Mishchenko, E. (1962). *The Mathematical Theory of Optimal Processes*. translated by K.N. Trirogoff, L.W. Neustadt (Ed.), Moscow, interscience, new york edition.

Mayne, D. (1966). A second-order gradient method for determining optimal trajectories of nonlinear discrete time systems. In *International Journal of Control, 3, pp. 85-95*.

Michalewicz, Z. (1992). *Genetic Algorithms + Data Structures= Evolution Programs*. Springer-Verlag, Berlin.

Murray, D. and Yakowitz, S. (1984). Differential dynamic programming and newton's method for discrete optimal control problems. In *J. of Optimization Theory and Applications, 43:395-414*.

Ohno, K. (1978). A new approach of differential dynamic programming for discrete time systems. In *IEEE Transactions on Automatic Control, 23, pp. 37-47*. IEEE.

Pantoja, J. (1988). Differential dynamic programming and newton's method. In *International Journal of Control, 53: 1539-1553*.

Schwefel, H. (1995). *Evolution and Optimum Seeking*. Wiley, New York.

Yakowitz, S. and Rutherford, B. (1984). Computational aspects of discrete-time optimal control. In *Appl. Math. Comput., 15, pp. 29-45*.

Z. Michalewicz, C. J. and Krawczyk, J. (1992). A modified genetic algorithm for optimal control problems. In *Computers Math. Appl., 23(12),8394*.