A GENERAL SOLUTION TO THE OUTPUT-ZEROING PROBLEM FOR DISCRETE-TIME MIMO LTI SYSTEMS
Signal Processing, Systems Modelling and Control

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Abstract: The problem of zeroing the output in an arbitrary linear discrete-time system S(A,B,C,D) with a nonvanishing transfer-function matrix is discussed and necessary conditions for output-zeroing inputs are formulated. All possible real-valued inputs and real initial conditions which produce the identically zero system response are characterized. Strictly proper and proper systems are discussed separately.

1 INTRODUCTION

The problem of zeroing the system output is strictly related to the notion of multivariable zeros. These zeros, however, are defined in many, not necessarily equivalent, ways (for a survey of these definitions see MacFarlane and Karcanias, 1976; Schrader and Sain, 1989; see also Bourles and Fliess, 1997). The most commonly used definition employs the Smith canonical form of the system (Rosenbrock) matrix and determines these zeros (which will be called in the sequel the Smith zeros) as the roots of diagonal (invariant) polynomials of the Smith form (Rosenbrock, 1970, 1973). The output-zeroing problem for continuous-time systems in relationship with the Smith zeros was studied, under certain simplifying assumptions concerning the systems considered, in (MacFarlane and Karcanias, 1976), where the notions of state-zero and input-zero directions were introduced, and was interpreted geometrically in (Isidori, 1995, pp. 164, 296). A more detailed analysis indicates that for characterizing the output-zeroing problem the notion of Smith zeros is too narrow (Tokarzewski, 2002; Tokarzewski and Sokalski, 2004). However, extending in a natural way the concept of the Smith zeros, the above difficulty can be overcomed. Such an extension is based on the definition of invariant zeros which employs the system matrix and zero directions with nonzero state-zero directions (see Tokarzewski, 2002; Tokarzewski and Sokalski, 2004). Because to each invariant zero we can assign a real initial condition and a real-valued input which produce the zero output, the invariant zeros can be easily interpreted in the context of the output-zeroing problem. Of course, since each Smith zero is also an invariant zero, this interpretation remains valid also for Smith zeros.

Taking into account the above concept of invariant zeros, we can state the following question: find a state-space characterization of the output-zeroing problem (at least in the form of necessary conditions) which determines in a simple manner all the possible real-valued inputs and real initial conditions which produce the identically zero system response. For continuous-time systems the question was discussed in (Tokarzewski, 2002) and for the discrete-time case it was outlined for square decouplable systems in (Tokarzewski, 2000).

2 PRELIMINARY RESULTS

Consider a discrete-time system S(**A**,**B**,**C**,**D**) with m inputs and r outputs

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$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)\end{aligned}, \quad k \in N = \{0,1,2,...\}, \qquad (1)$$

where $\mathbf{x}(k) \in R^n, \mathbf{u}(k) \in R^m, \mathbf{y}(k) \in R^r$ and $\mathbf{A}, \mathbf{B} \neq \mathbf{0}, \mathbf{C} \neq \mathbf{0}, \mathbf{D}$ are real matrices of appropriate dimensions. By U we denote the set of *admissible inputs* which consists of all sequences $\mathbf{u}(.) : N \rightarrow R^m$.

The point of departure for our discussion is the following formulation of the *output-zeroing problem* (Isidori, 1995): find all pairs $(\mathbf{x}^o, \mathbf{u}_o(k))$, consisting of an *initial state* $\mathbf{x}^o \in R^n$ and an admissible input $\mathbf{u}_o(k)$, such that the corresponding output $\mathbf{y}(k)$ of (1) is identically zero for all $k \in N$. Any nontrivial pair (i.e., such that $\mathbf{x}^o \neq \mathbf{0}$ or $\mathbf{u}_o(k) \neq \mathbf{0}$) of this kind is called an *output-zeroing input*. Note that in each output-zeroing input $(\mathbf{x}^o, \mathbf{u}_o(k))$, $\mathbf{u}_o(k)$ should be understood simply as an open-loop control signal which, when applied to (1) exactly at $\mathbf{x}(0) = \mathbf{x}^o$, yields $\mathbf{y}(k) = \mathbf{0}$ for all $k \in N$.

Moreover, we consider the following definition of invariant zeros (Tokarzewski, 2000): a complex number λ is an *invariant zero* of (1) if and only if (iff) there exist vectors $\mathbf{0} \neq \mathbf{x}^o \in C^n$ (*state-zero direction*) and $\mathbf{g} \in C^m$ (*input-zero direction*) such that

$$\mathbf{P}(\lambda)\begin{bmatrix}\mathbf{x}^o \\ \mathbf{g}\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0}\end{bmatrix}, \qquad (1a)$$

where $\mathbf{P}(z) = \begin{bmatrix} z\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix}$ denotes the system matrix. *Transmission zeros* of (1) are defined as invariant zeros of its minimal (i.e., reachable and observable) subsystem.

The set of all invariant zeros of (1) is denoted by Z^I. System (1) is called *degenerate* iff Z^I is infinite. Otherwise, the system is said to be *nondegenerate*. The set of all Smith zeros of (1) we denote by Z^S. Recall (Tokarzewski and Sokalski, 2004) that $Z^S \subseteq Z^I$; moreover, (1) is nondegenerate iff $Z^I = Z^S$, and (1) is degenerate iff $Z^I = C$. Recall also that in case of nondegeneracy the Smith and invariant zeros are exactly the same objects (including multiplicities).

The same symbol $\mathbf{x}^o$ is used to denote the state-zero direction in the definition of invariant zeros and the initial state in the definition of output-zeroing inputs. The state-zero direction $\mathbf{x}^o$ must be a nonzero vector (real or complex). Otherwise, the definition of invariant zeros becomes senseless (for any system (1) each complex number may serve as an invariant zero). In other words, in the definition of invariant zeros the condition $\mathbf{x}^o \neq \mathbf{0}$ can not be omitted.

According to the formulation of the output-zeroing problem, the initial state $\mathbf{x}^o$ must be a real vector (but not necessarily nonzero). If the state-zero direction $\mathbf{x}^o$ is a complex vector, then it gives two initial states $\mathrm{Re}\,\mathbf{x}^o$ and $\mathrm{Im}\,\mathbf{x}^o$ (and, of course, at least one of these initial states must be a nonzero vector).

Recall (Tokarzewski, 2000, Remark 1) that if $\lambda = |\lambda| e^{j\varphi}$ is an invariant zero of (1), i.e., a triple $\lambda, \mathbf{x}^o \neq \mathbf{0}, \mathbf{g}$ satisfies (1a), then this triple generates two output-zeroing inputs. Namely, the pair $(\mathrm{Re}\,\mathbf{x}^o, \mathbf{u}_o(k))$, where

$$\mathbf{u}_o(k) = |\lambda|^k (\mathrm{Re}\,\mathbf{g}\,\cos k\varphi - \mathrm{Im}\,\mathbf{g}\,\sin k\varphi),\ k \in N \quad (1b)$$

is an output-zeroing input (and produces the solution of the state equation of (1) of the form $\mathbf{x}(k) = |\lambda|^k (\mathrm{Re}\,\mathbf{x}^o \cos k\varphi - \mathrm{Im}\,\mathbf{x}^o \sin k\varphi)$).

Similarly, the pair $(\mathrm{Im}\,\mathbf{x}^o, \mathbf{u}_o(k))$, where

$$\mathbf{u}_o(k) = |\lambda|^k (\mathrm{Re}\,\mathbf{g}\,\sin k\varphi + \mathrm{Im}\,\mathbf{g}\,\cos k\varphi),\ k \in N \quad (1c)$$

is an output-zeroing input (and produces the solution of the state equation of (1) of the form $\mathbf{x}(k) = |\lambda|^k (\mathrm{Re}\,\mathbf{x}^o \sin k\varphi + \mathrm{Im}\,\mathbf{x}^o \cos k\varphi)$).

We denote by $\mathbf{M}^+$ the Moore-Penrose pseudo-inverse of matrix $\mathbf{M}$ (Ben-Israel and Greville, 2002). Recall (Gantmacher, 1988) that for a given rxm real $\mathbf{M}$ of rank p, a factorization $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$ with an rxp $\mathbf{M}_1$ and a pxm $\mathbf{M}_2$ is called the *skeleton factorization* of $\mathbf{M}$. Then $\mathbf{M}^+$ is uniquely determined as $\mathbf{M}^+ = \mathbf{M}_2^+\mathbf{M}_1^+$, where $\mathbf{M}_1^+ = (\mathbf{M}_1^T\mathbf{M}_1)^{-1}\mathbf{M}_1^T$ and $\mathbf{M}_2^+ = \mathbf{M}_2^T(\mathbf{M}_2\mathbf{M}_2^T)^{-1}$. Moreover, $\mathbf{M} = \mathbf{M}\mathbf{M}^+\mathbf{M}$ and $\mathbf{M}^+ = \mathbf{M}^+\mathbf{M}\mathbf{M}^+$.

Consider the equation $\mathbf{M}\mathbf{z}(k) = \mathbf{b}(k)$, $k \in N$, where $\mathbf{M}$ is a rxm real and constant matrix of rank p and $\mathbf{b}(.) : N \rightarrow R^r$ is a fixed sequence, and suppose that this equation is solvable in the class of all sequences $\mathbf{z}(.): N \to R^m$ (i.e., there is at least one solution). Then any solution can be expressed as $\mathbf{z}(\mathrm{k}) = \mathbf{z}_o^*(\mathrm{k}) + \mathbf{z}_h(\mathrm{k})$, where $\mathbf{z}_o^*(\mathrm{k}) = \mathbf{M}^+\mathbf{b}(\mathrm{k})$ and $\mathbf{z}_h(\mathrm{k})$ is a solution of the equation $\mathbf{Mz}(\mathrm{k}) = \mathbf{0}$.

3 MAIN RESULTS

3.1 Proper Systems $(\mathbf{D} \neq \mathbf{0})$

A general characterization of output-zeroing inputs and the corresponding solutions is given in the following result.

Proposition 1 Let $(\mathbf{x}^o, \mathbf{u}_o(\mathrm{k}))$ be an output-zeroing input for a proper system (1) and let $\mathbf{x}_o(\mathrm{k})$ denote the corresponding solution. Then $\mathbf{x}^o \in \mathrm{Ker}(\mathbf{I}_r - \mathbf{DD}^+)\mathbf{C}$ and $\mathbf{u}_o(\mathrm{k})$, $\mathrm{k} \in N$, has the form

$$\begin{aligned}\mathbf{u}_o(\mathrm{k}) &= -\mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o + \\ &- \mathbf{D}^+\mathbf{C}[\sum_{l=0}^{\mathrm{k}-1}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}-1-l}\mathbf{Bu}_h(l)] + \mathbf{u}_h(\mathrm{k})\end{aligned} \tag{2}$$

for some sequence $\mathbf{u}_h(.) \in U$ satisfying $\mathbf{Du}_h(\mathrm{k}) = \mathbf{0}$ for all $\mathrm{k} \in N$, and $\mathbf{x}_o(\mathrm{k})$, $\mathrm{k} \in N$, has the form

$$\begin{aligned}\mathbf{x}_o(\mathrm{k}) &= (\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o \\ &+ \sum_{l=0}^{\mathrm{k}-1}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}-1-l}\mathbf{Bu}_h(l)\end{aligned} . \tag{3}$$

Moreover, $\mathbf{x}_o(\mathrm{k}) \in \mathrm{Ker}(\mathbf{I}_r - \mathbf{DD}^+)\mathbf{C}$ for all $\mathrm{k} \in N$.

Remark 1 Proposition 1 does not tell us whether the output-zeroing inputs exist. However, if the set of invariant zeros is nonempty, for each such zero there exists an output-zeroing input (see (1b) and (1c)) which in turn may be characterized as in Proposition 1.

Proposition 2 Let $(\mathbf{x}^o, \mathbf{u}_o(\mathrm{k}))$ be an output-zeroing input for a proper system (1) and let $\mathbf{x}_o(\mathrm{k})$ denote the corresponding solution. Then
(i) if $\mathbf{B}(\mathbf{I}_m - \mathbf{D}^+\mathbf{D}) = \mathbf{0}$, then $\mathbf{x}_o(\mathrm{k}) = (\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o$.
Moreover, the pair $(\mathbf{x}^o, \mathbf{u}_o^*(\mathrm{k}))$, where $\mathbf{u}_o^*(\mathrm{k}) = -\mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o$, is also output-zeroing and yields $\mathbf{x}_o(\mathrm{k}) = (\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o$.
(ii) if $\mathbf{D}$ has full column rank, then $\mathbf{u}_o(\mathrm{k}) = -\mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o$ and $\mathbf{x}_o(\mathrm{k}) = (\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o$.

Remark 2 The assumption $\mathbf{B}(\mathbf{I}_m - \mathbf{D}^+\mathbf{D}) = \mathbf{0}$ does not imply in general that $\mathbf{u}_o^*(\mathrm{k}) = \mathbf{u}_o(\mathrm{k})$ for all $\mathrm{k} \in N$. It implies, however, that $\mathbf{u}_o(\mathrm{k})$ and $\mathbf{u}_o^*(\mathrm{k})$ applied at the initial state $\mathbf{x}^o$ affect the state equation of (1) in the same way. This follows from the relation $\mathbf{Bu}_o^*(\mathrm{k}) - \mathbf{Bu}_o(\mathrm{k}) = \mathbf{0}$.

When $\mathbf{D}$ has full row rank, the necessary condition given by Proposition 1 becomes also sufficient.

Proposition 3 In (1) let $\mathbf{D}$ have full row rank. Then $(\mathbf{x}^o, \mathbf{u}_o(\mathrm{k}))$ is an output-zeroing input iff $\mathbf{u}_o(\mathrm{k})$ has the form (2), where $\mathbf{x}^o \in R^n$ and $\mathbf{u}_h(.)$ is an element of U satisfying $\mathbf{Du}_h(\mathrm{k}) = \mathbf{0}$ for all $\mathrm{k} \in N$. Moreover, the solution corresponding to $(\mathbf{x}^o, \mathbf{u}_o(\mathrm{k}))$ has the form (3).

A more detailed characterization of the output-zeroing problem than that obtained in Proposition 2 (ii) is given by the following result.

Proposition 4 In (1) let $\mathbf{D}$ have full column rank. Then $(\mathbf{x}^o, \mathbf{u}_o(\mathrm{k}))$ is an output-zeroing iff

$$\mathbf{x}^o \in \mathrm{S}_{\mathbf{D}}^{cl} := \bigcap_{l=0}^{n-1}\mathrm{Ker}\{(\mathbf{I}_r - \mathbf{DD}^+)\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^l\} \tag{4}$$

and

$$\mathbf{u}_o(\mathrm{k}) = -\mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o . \tag{5}$$

Moreover, the corresponding solution equals

$$\mathbf{x}_o(\mathrm{k}) = (\mathbf{A} - \mathbf{BD}^+\mathbf{C})^{\mathrm{k}}\mathbf{x}^o \tag{6}$$

and is contained in the subspace $\mathrm{S}_{\mathbf{D}}^{cl}$ (4), i.e., $\mathbf{x}_o(\mathrm{k}) \in \mathrm{S}_{\mathbf{D}}^{cl}$ for all $\mathrm{k} \in N$.

Remark 3 Any proper system (1) can be transformed, by introducing an appropriate pre-compensator, into a proper system in which the first nonzero Markov parameter has full column rank. In fact, suppose that $\operatorname{rank}\mathbf{D} = p < m$. Let $\mathbf{D} = \mathbf{D}_1\mathbf{D}_2$ be a skeleton factorization of **D**. Introducing the pre-compensator $\mathbf{D}_2^T$ to (1), we get a sytem with the first nonzero Markov parameter $\mathbf{D}\mathbf{D}_2^T$ of full column rank. Finally, by introducing into a reachable system (1) the precompensator $\mathbf{D}_2^T$, reachability may be lost.

For a proper system (1) we denote by $\mathcal{V}^*(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ the *maximal output-nulling controlled invariant subspace* (Aling and Schumacher, 1984). Recall also that if $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input for (1) and $\mathbf{x}_o(k)$ is the corresponding solution, $\mathbf{x}_o(k) \in \mathcal{V}^*(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ for all $k \in N$ (see e.g. (Tokarzewski, 2002, p. 195)).

Lemma 1 (Tokarzewski, 2002, p.188) Consider a proper system (1). Then

$$Z^I = \emptyset \iff \mathcal{V}^*(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \{\mathbf{0}\}, \tag{7}$$

i.e., the system has no invariant zeros iff its maximal output-nulling controlled invariant subspace is trivial.

Proposition 5 In a proper system (1) let $Z^I = \emptyset$. Then a pair $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input for (1) iff $\mathbf{x}^o = \mathbf{0}$ and $\mathbf{u}_o(k) \in \operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{D}$ for all $k \in N$. Moreover, the corresponding to such a pair solution of the state equation in (1) equals $\mathbf{x}_o(k) = \mathbf{0}$ for all $k \in N$.

Remark 4 It is important to note that Lemma 1 and Proposition 5 are not valid if we replace the assumption $Z^I = \emptyset$ with $Z^S = \emptyset$ (see Example 2).

3.2 Strictly Proper Systems ($\mathbf{D} = \mathbf{0}$)

If $\mathbf{D} = \mathbf{0}$, then the first nonzero Markov parameter of (1) is denoted by $\mathbf{C}\mathbf{A}^{\nu}\mathbf{B}$, where $0 \le \nu \le n-1$ (i.e., $\mathbf{C}\mathbf{B} = \ldots = \mathbf{C}\mathbf{A}^{\nu-1}\mathbf{B} = \mathbf{0}$ and $\mathbf{C}\mathbf{A}^{\nu}\mathbf{B} \ne \mathbf{0}$). Define the matrix (comp. Tokarzewski and Sokalski, 2004)

$$\mathbf{K}_\nu := \mathbf{I} - \mathbf{B}(\mathbf{C}\mathbf{A}^{\nu}\mathbf{B})^{+}\mathbf{C}\mathbf{A}^{\nu}. \tag{8}$$

As a necessary condition for a pair $(\mathbf{x}^o, \mathbf{u}_o(k))$ to be an output-zeroing input we have the following.

Proposition 6 Let $(\mathbf{x}^o, \mathbf{u}_o(k))$ be an output-zeroing input for a strictly proper system (1) and let $\mathbf{x}_o(k)$ denote the corresponding solution. Then $\mathbf{x}^o \in S_\nu := \bigcap_{l=0}^{\nu} \operatorname{Ker}\mathbf{C}\mathbf{A}^l$ and $\mathbf{u}_o(k)$ has the form

$$\begin{aligned}\mathbf{u}_o(k) = &-(\mathbf{C}\mathbf{A}^{\nu}\mathbf{B})^{+}\mathbf{C}\mathbf{A}^{\nu+1}(\mathbf{K}_\nu\mathbf{A})^k\mathbf{x}^o + \\ &-(\mathbf{C}\mathbf{A}^{\nu}\mathbf{B})^{+}\mathbf{C}\mathbf{A}^{\nu+1}[\sum_{l=0}^{k-1}(\mathbf{K}_\nu\mathbf{A})^{k-1-l}\mathbf{B}\mathbf{u}_h(l)], \\ &+\mathbf{u}_h(k)\end{aligned} \tag{9}$$

for some $\mathbf{u}_h(.) \in U$ satisfying $\mathbf{C}\mathbf{A}^{\nu}\mathbf{B}\mathbf{u}_h(k) = \mathbf{0}$ for all $k \in N$, and $\mathbf{x}_o(k)$ has the form

$$\mathbf{x}_o(k) = (\mathbf{K}_\nu\mathbf{A})^k\mathbf{x}^o + \sum_{l=0}^{k-1}(\mathbf{K}_\nu\mathbf{A})^{k-1-l}\mathbf{B}\mathbf{u}_h(l). \tag{10}$$

Moreover, $\mathbf{x}_o(k)$ is entirely contained in the subspace S_ν, i.e., $\mathbf{x}_o(k) \in S_\nu$ for all $k \in N$.

Remark 5 Note that under the assumptions of Proposition 6 the input (9) (at any $\mathbf{u}_h(.) \in U$ satisfying $\mathbf{C}\mathbf{A}^{\nu}\mathbf{B}\mathbf{u}_h(k) = \mathbf{0}$ for all $k \in N$) applied to the system at an arbitrary initial state $\mathbf{x}(0) \in R^n$ yields the solution of the state equation of the form $\mathbf{x}(k) = \mathbf{A}^k(\mathbf{x}(0) - \mathbf{x}^o) + \mathbf{x}_o(k)$, where $\mathbf{x}_o(k)$ is as in (10), and the system output equals $\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k(\mathbf{x}(0) - \mathbf{x}^o)$. In particular, if **A** is stable, then $\mathbf{x}(k) \to \mathbf{x}_o(k)$ and $\mathbf{y}(k) \to \mathbf{0}$ as $k \to \infty$.

Remark 6 Proposition 6 does not tell us whether the output-zeroing inputs exist. However, if the set of invariant zeros is nonempty, for each such zero there exists an output-zeroing input (see (1b) and (1c)) which in turn may be characterized as in Proposition 6.

Proposition 7 Let $(\mathbf{x}^o, \mathbf{u}_o(k))$ be an output-zeroing input for a strictly proper system (1) and let $\mathbf{x}_o(k)$ denote the corresponding solution. Then

(i) if $\mathbf{K}_\nu\mathbf{B} = \mathbf{0}$, then $\mathbf{x}_o(k) = (\mathbf{K}_\nu\mathbf{A})^k\mathbf{x}^o$.

Moreover, at $\mathbf{K}_\nu \mathbf{B} = \mathbf{0}$ the pair $(\mathbf{x}^o, \mathbf{u}_o^*(k))$, where $\mathbf{u}_o^*(k) := -(\mathbf{CA}^\nu \mathbf{B})^+ \mathbf{CA}^{\nu+1} (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o$, is also output-zeroing and yields the solution $\mathbf{x}_o(k) = (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o$.

(ii) if $\mathbf{CA}^\nu \mathbf{B}$ has full column rank, then $\mathbf{u}_o(k) = -(\mathbf{CA}^\nu \mathbf{B})^+ \mathbf{CA}^{\nu+1} (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o$ and $\mathbf{x}_o(k) = (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o$.

Remark 7 The assumption $\mathbf{K}_\nu \mathbf{B} = \mathbf{0}$ does not imply in general the equality $\mathbf{u}_o^*(k) = \mathbf{u}_o(k)$ for all $k \in N$, although it implies $\mathbf{x}(k) \equiv \mathbf{x}_o(k)$. The reason behind this becomes clear if we consider the relations

$\mathbf{Bu}_o^*(k) - \mathbf{Bu}_o(k) = (\mathbf{K}_\nu - \mathbf{I})\mathbf{Ax}_o(k) - \mathbf{Bu}_o(k) =$
$\mathbf{K}_\nu \mathbf{Ax}_o(k) - \mathbf{x}_o(k+1) = -\mathbf{K}_\nu \mathbf{Bu}_o(k)$.

Thus, at $\mathbf{K}_\nu \mathbf{B} = \mathbf{0}$, although in general $\mathbf{u}_o^*(k) \neq \mathbf{u}_o(k)$, both these inputs applied at the initial state $\mathbf{x}(0) = \mathbf{x}^o$ affect the state equation of (1) in exactly the same way.

Proposition 8 In a strictly proper system (1) let $\mathbf{CA}^\nu \mathbf{B}$ have full row rank. Then $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input iff $\mathbf{x}^o \in S_\nu = \bigcap_{l=0}^{\nu} \mathrm{Ker}\, \mathbf{CA}^l$ and $\mathbf{u}_o(k)$ is as in (9) with $\mathbf{u}_h(.) \in U$ satisfying $\mathbf{u}_h(k) \in \mathrm{Ker}\, \mathbf{CA}^\nu \mathbf{B}$ for all $k \in N$. Moreover, the corresponding solution $\mathbf{x}_o(k)$ has the form (10) and is entirely contained in the subspace S_ν, i.e., $\mathbf{x}_o(k) \in S_\nu$ for all $k \in N$.

Proposition 9 In a strictly proper system (1) let $\mathbf{CA}^\nu \mathbf{B}$ have full column rank. Then a pair $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input iff

$$\mathbf{x}^o \in S_\nu^{cl} := \bigcap_{l=0}^{n-1} \mathrm{Ker}\, \mathbf{C}(\mathbf{K}_\nu \mathbf{A})^l \tag{11}$$

and

$$\mathbf{u}_o(k) = -(\mathbf{CA}^\nu \mathbf{B})^+ \mathbf{CA}^{\nu+1} (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o,\ k \in N \tag{12}$$

Moreover, the solution of the state equation corresponding to $(\mathbf{x}^o, \mathbf{u}_o(k))$ has the form

$$\mathbf{x}_o(k) = (\mathbf{K}_\nu \mathbf{A})^k \mathbf{x}^o, \quad k \in N, \tag{13}$$

and is entirely contained in the subspace S_ν^{cl}, i.e., $\mathbf{x}_o(k) \in S_\nu^{cl}$ for all $k \in N$.

Remark 8 Any strictly proper system (1) with nonvanishing transfer-function matrix can be transformed, by introducing an appropriate pre-compensator, into a strictly proper system in which the first nonzero Markov parameter has full column rank. In fact, suppose that $\mathrm{rank}\, \mathbf{CA}^\nu \mathbf{B} = p < m$. Let $\mathbf{CA}^\nu \mathbf{B} = \mathbf{H}_1 \mathbf{H}_2$ be a skeleton factorization. Introducing to (1) the precompensator $\mathbf{H}_2^T$, we get a system with the first nonzero Markov parameter $\mathbf{CA}^\nu \mathbf{BH}_2^T$ of full column rank. By introducing to a reachable system (1) the precompensator $\mathbf{H}_2^T$, reachability may be lost.

For a strictly proper system (1) we denote by $\mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C})$ the maximal output-nulling controlled invariant subspace (Basile and Marro, 1992; Wonham, 1979; Sontag, 1990). Recall also that if $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input for (1) and $\mathbf{x}_o(k)$ is the corresponding solution, then $\mathbf{x}_o(k) \in \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C})$ for all $k \in N$

Lemma 2 (Tokarzewski, 2002, p.168) Consider a strictly proper system (1). Then

$$Z_I = \varnothing \iff \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \{\mathbf{0}\}, \tag{14}$$

i.e., the system has no invariant zeros iff its maximal output-nulling controlled invariant subspace is trivial.

Proposition 10 In a strictly proper system (1) let $Z_I = \varnothing$. Then a pair $(\mathbf{x}^o, \mathbf{u}_o(k))$ is an output-zeroing input for (1) iff $\mathbf{x}^o = \mathbf{0}$ and $\mathbf{u}_o(k) \in \mathrm{Ker}\, \mathbf{B}$ for all $k \in N$. Moreover, the corresponding to such a pair solution of the state equation in (1) equals $\mathbf{x}_o(k) = \mathbf{0}$ for all $k \in N$.

Remark 9 It is important to note that Lemma 2 and Proposition 10 are not valid if we replace the assumption $Z_I = \varnothing$ with $Z_S = \varnothing$ (see Example 1).

4 EXAMPLES

Example 1 In (1) let

$$\mathbf{A} = \begin{bmatrix} -1/3 & 0 & -1 \\ 0 & -2/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, .$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The system has no Smith zeros; on the other hand (see Tokarzewski and Sokalski, 2004, Proposition 9), it is degenerate (i.e., $Z^I = C$). Since **CB** has full row rank, all output-zeroing inputs are as in Proposition 8. Note that $\mathcal{V}^*(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathrm{Ker}\,\mathbf{C}$, i.e., the maximal output-nulling controlled invariant subspace is nontrivial (comp. Remark 9).

Example 2 In (1) let

$$\mathbf{A} = \begin{bmatrix} -2 & -1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, .$$

$$\mathbf{C} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The system has no Smith zeros; on the other hand (see Tokarzewski, 2002, p.188), it is degenerate and $\mathcal{V}^*(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \{\mathbf{x} \in R^3 : x_2 + x_3 = 0\}$ is non-trivial (comp. Remark 4).

5 CONCLUDING REMARKS

In this paper we presented necessary conditions for output-zeroing inputs and the corresponding solutions (Propositions 1 and 6) for a general class of linear discrete-time systems described by the state-space model (1). It is shown that if the first nonzero Markov parameter has full row rank, the necessary conditions become also sufficient (Propositions 3 and 8). Necessary and sufficient conditions for output-zeroing inputs for systems with the first nonzero Markov parameter of full column rank are given in Propositions 4 and 9. Finally, necessary and sufficient conditions for output-zeroing inputs under the assumption that the set of invariant zeros is empty are presented in Propositions 5 and 10.

A more detailed characterization of the output-zeroing problem can be obtained by using singular value decomposition of the first nonzero Markov parameter.

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