DISCRETE–TIME FREE AND FIXED END-POINT OPTIMAL CONTROL PROBLEM

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Abstract: A comparison between the fixed and free end-point discrete time linear quadratic optimal problem is performed. Symmetrical algorithms for both problems are proposed. These algorithms can be easier implemented by comparison with classical procedures. Simulation results are presented.

1 INTRODUCTION

The paper considers the discrete time optimal problems with finite final time, which refer to a quadratic criterion and to a discrete completely controllable linear time invariant system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control vector, $k \in \mathbb{Z}$, A and B are matrices of appropriate dimensions.
Depending on the final state $x(k_f)$, one can formulate the following problems:
P1 (with fixed end-point): Find the feedback control u(x(k)) which transfers the system (1) from the initial state $x(k_0)$ in the imposed final state $x(k_f)=0$ and minimizes the criterion

$$J_1 = \frac{1}{2}\sum_{k=k_0}^{k_f} x^T(k)Q_1x(k) + u^T(k)P_1u(k) \tag{2}$$

(T denotes the transposition).
P2 (with free end–point): Find the optimal feedback control u(x(k)) which transfers the system (1) from the initial state $x(k_0)$ in the free final state $x(k_f)$ and minimizes the criterion

$$J_2 = \frac{1}{2}x^T(k_f)Sx(k_f) + \frac{1}{2}\sum_{k=k_0}^{k_f} x^T(k)Q_1x(k) + u^T(k)P_1u(k) \tag{3}$$

We also mention in addition the problem **P3** with infinite final time, which refers to the criterion

$$J_3 = \frac{1}{2}\sum_{k=0}^{\infty} x^T(k)Q_3x(k) + u^T(k)P_3u(k) \tag{4}$$

For a more relevant comparison we shall consider

$$Q_1 = Q_2 = Q_3 = Q\ ,\ P_1 = P_2 = P_3 = P \tag{5}$$

The matrices of the above criteria are symmetrical and

$$S \geq 0,\ Q \geq 0,\ P > 0 \tag{6}$$

The solution for the above formulated problems are well known (Anderson, Moore, 1990), (Kuo, 1992), but there are some difficulties in implementation of the algorithms. The solution to the **P1** problem is usually presented as an open loop control u(k) because the feedback control u(x(k)) has a complicated form. The **P2** problem is the most frequently meet linear quadratic problem with finite final time. The matrix of the feedback controller is time variant and is designed based on a solution to the Riccati difference matriceal equation. This solution has to be computed in real time and this fact can generate some difficulties in implementation, augmented by the fact that the equation must be solved in inverse time, starting from a final condition.

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The paper uses some previous results of the authors (Botan, Onea, 1999), (Botan, Ostafi, Onea, 2003), and presents a simpler for implementation solution for the formulated problems. Moreover, a symmetrical approach for both problems is established.

2 USUAL APPROACHES

From the Hamilton necessary conditions, the optimal control is obtained as

$$u(k) = -P^{-1}B^{T}\lambda(k+1) \tag{7}$$

and

$$Qx(k) + A^{T}\lambda(k+1) = \lambda(k), \tag{8}$$

where $\lambda(k) \in \mathbb{R}^{n}$ is the co-state vector.

Substituting $\lambda(k+1)$ from (8) in (7) and then in (1), the equations (1) and (8) can be expressed as

$$\gamma(k+1) = G\gamma(k), \quad \gamma(k) = \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \in \mathbb{R}^{2n}, \tag{9}$$

where

$$G = \begin{bmatrix} A + NA^{-T}Q & -NA^{-T} \\ -A^{-T}Q & -A^{-T} \end{bmatrix} \tag{10}$$

with $N = BP^{-1}B^{T}$ and $A^{-T} = (A^{T})^{-1}$.

The solution to the system (9) is

$$\gamma(k) = \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \Gamma(k,k_0)\gamma(k_0) = \begin{bmatrix} \Gamma_{11}(k,k_0) & \Gamma_{12}(k,k_0) \\ \Gamma_{21}(k,k_0) & \Gamma_{22}(k,k_0) \end{bmatrix} \begin{bmatrix} x(k_0) \\ \lambda(k_0) \end{bmatrix} \tag{11}$$

where $\Gamma(.)$ is the transition matrix for G.

The next steps are different for the **P1** and **P2** problems, depending on terminal conditions:
- in the **P1** problem $x(k_0)$ and

$$x(k_f) = 0 \tag{12}$$

are imposed ($\lambda(k_0)$ and $\lambda(k_f)$ are free);
- in the **P2** problem $x(k_0)$ and

$$\lambda(k_f) = Sx(k_f) \tag{13}$$

(from the transversallity condition) are imposed ($x(k_f)$ and $\lambda(k_0)$ are free).

Thus, for the **P1** problem, from (11) and (12), it results

$$\lambda(k_0) = Lx(k_0), \quad L = \Gamma_{12}^{-1}(k_f,k_0)\Gamma_{11}(k_f,k_0) \tag{14}$$

It was proved (Botan, Ostafi, Onea, 2003) that $\Gamma_{12}^{-1}(k_f,k_0)$ is a non-singular matrix if the system (1) is completely controllable. Note also that all inverse matrices which appear in the following equations are non singular.

Therefore, $\gamma(k_0)$ is known and the solution (11) can be obtained. Then it is possible to express u(k) in terms of $x(k_0)$, starting from (7). This expression offers the open loop control u(k) and it is the usual solution presented in the literature. It is possible to obtain the feedback control u(x(k)) if $x(k_0)$ is expressed in terms of x(k). The formula is complicated and contains the inverse of a time variant matrix and this fact introduces great difficulties in the real time implementation.

A similar procedure can be used for the **P2** problem, but in this case it is preferred another way, namely imposing $\lambda(k) = \tilde{R}(k)x(k)$, where $\tilde{R}(k)$ is obtained as a solution to a Riccati difference matriceal equation. The difficulties which arise in this case were mentioned above.

3 MAIN RESULTS

A significant simplification is obtained if we perform a change of variables:

$$\gamma(k) = U\rho(k) \tag{15}$$

with

$$U = \begin{bmatrix} I & 0 \\ R & I \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} I & 0 \\ -R & I \end{bmatrix} \tag{16}$$

where I is the nxn identity matrix and R is a symmetrical nxn matrix. According to (15) and (16), the new system is

$$\rho(k+1) = H\rho(k), \tag{17}$$

where

$$H = U^{-1}GU = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad (18)$$

and

$$\rho(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}, \quad v(k) = \lambda(k) - Rx(k) \quad (19)$$

Using (10), (16) and (18), it is obtained by straightforward computing

$$\begin{aligned} H_{21} &= -RG_{11} - RG_{12}R + G_{21} + G_{22}R \\ &= (I+RN)A^{-T}[R - Q - A^T(I+RN)^{-1}RA] \end{aligned} \quad (20)$$

If we impose the condition

$$R = Q + A^T(I+RN)^{-1}RA\ , \quad (21)$$

then

$$H_{21} = 0 \quad (22)$$

(Note that (21) is the Riccati matriceal equation for the **P3** problem). The others matriceal blocks of H can be similarly computed. Using in addition (21), yields

$$H_{11} = G_{11} + G_{12}R = [I - N(I+NR)^{-1}R]A\ ,$$

or

$$H_{11} = (I+NR)^{-1}A\ , \quad H_{12} = G_{12} = -NA^{-T}$$

$$H_{22} = -RG_{12} + G_{22} = (I+RN)A^{-T} = H_{11}^{-T} \quad (23)$$

The solution to the equation (17) is

$$\rho(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} = \Omega(k,k_0)\rho(k_0)\ , \quad (24)$$

where (for k_0=0)

$$\Omega(k,k_0) = H^k = \begin{bmatrix} \Omega_{11}(k,k_0) & \Omega_{12}(k,k_0) \\ 0 & \Omega_{22}(k,k_0) \end{bmatrix} \quad (25)$$

is the transition matrix for H.

Using (16) and (18), yields

$$\begin{aligned} &\Omega_{11}(k) = H_{11}^k,\ \Omega_{22}(k) = H_{22}^k \\ &\Omega_{12}(k) = H_{12k} = \sum_{i=0}^{k-1} H_{11}^i H_{12} H_{22}^{k-i-1} \end{aligned} \quad (26)$$

From (17), (22), (24) and (26) it results (for k_0=0)

$$\begin{aligned} &v(k+1) = H_{22}v(k) \quad \text{and} \\ &v(k) = \Omega_{22}(k)v(k_0) = H_{22}^k v(k_0) \end{aligned} \quad (27)$$

The optimal control results from (7) and (19)

$$u(k) = -P^{-1}B^T[Rx(k+1) + v(k+1)]$$

Using (17) and (27), we can write

$$u(k) = u_f(k) + u_s(k)\ , \quad (28)$$

where $u_f(k)$ is a feedback component

$$u_f(k) = -P^{-1}B^T RH_{11}x(k) \quad (29)$$

and $u_s(k)$ is a supplementary component, given by

$$u_s(k) = -P^{-1}B^T(RH_{12} + H_{22})v(k)\ , \quad (30)$$

with v(k) given by (27).

Remark 1: For the problem with infinite final time, the vector u(k) contains only the feedback component $u_f(k)$ given by (29).

In order to establish the supplementary component with (30), we have to express $v(k_0)$ in terms of $x(k_0)$, which is the unique known terminal condition.

These operations are different for the two problems.

For **P1** problem:

From (15) and (20) one obtain

$$v(k_0) = (L - R)x(k_0)\ . \quad (31)$$

Using $\Gamma = U\Omega U^{-1}$, we obtain

$$\begin{aligned} &\Gamma_{11}(k,k_0) = \Omega_{11}(k,k_0) - \Omega_{12}(k,k_0)R\ , \\ &\Gamma_{12}(k,k_0) = \Omega_{12}(k,k_0) \end{aligned}$$

and then, it results from (14)

$$L = \Omega_{12}^{-1}(k_f, k_0)\Omega_{11}(k_f, k_0) + R \qquad (32)$$

and one obtain from (31)

$$\begin{aligned} v(k_0) &= -\Omega_{12}^{-1}(k_f, k_0)\Omega_{11}(k_f, k_0)x(k_0) \\ &= H_{11k_f}^{-1} H_{11}^{k_f} x(k_0) \end{aligned} \qquad (33)$$

For **P2** problem:
Using (13) and (19), yields

$$v(k_f) = (S - R)x(k_f) \,. \qquad (34)$$

From (24) and (25)

$$v(k_f) = \Omega_{22}(k_f, k_0)v(k_0) \,,$$

so that

$$v(k_0) = \Omega_{22}^{-1}(k_f, k_0)(S - R)x(k_f) \,. \qquad (35)$$

We can write from (24)

$$x(k_f) = \Omega_{11}(k_f, k_0)x(k_0) + \Omega_{12}(k_f, k_0)v(k_0)$$

and using (35), we obtain

$$x(k_f) = Mx(k_0) \,, \qquad (36)$$

where M is a constant nxn matrix

$$M = [I - \Omega_{12}(k_f, k_0)\Omega_{22}^{-1}(k_f, k_0)]^{-1}\Omega_{11}(k_f, k_0) \qquad (37)$$

Finally, from (35) we can write

$$v(k_0) = \Omega_{22}^{-1}(k_f, k_0)(S - R)Mx(k_0) \qquad (38)$$

Remark 2: Unlike the usual methods which solve the **P1** and **P2** problem by different ways, a symmetrical approach was proposed for the two problems. A similar equation (28) for the optimal control u(k) was obtained for the problems **P1** and **P2**. In both cases, u(k) contains a feedback component $u_f(k)$ (29) and a supplementary one $u_s(k)$ (30). Note that the feedback component is the same in **P1**, **P2** and **P3** problems. The component $u_s(k)$ depends on the vector v(k) given by (27). The difference between the two problems consists in the expression of the initial value $v(k_0)$: (33) for the **P1** problem and (38) for the **P2** problem.

Remark 3: Some of the above established equations are rather complicated, but the most part of the computation is performed off-line, in the stage of controller design. It is important that the real time computing implies only to establish the components $u_f(k)$ and $u_s(k)$ given by (29) and (30), respectively. Therefore, the real time computing volume does not exceed very much the usual state feedback control. Moreover, the supplementary component can be recurrently computed. Indeed, the vector v(k) which appears in (30) can be recurrently computed, as it is indicated in (27), with the initialisation $v(k_0)$ given by (33) or (38) for the **P1** and **P2** problems, respectively.

4 SIMULATION RESULTS

Some simulation tests were performed for both **P1** and **P2** problems. The following discrete completely controllable linear time invariant system was considered (the example is applicable to a servo drive system):

$$x(k+1) = \begin{bmatrix} 1 & 0.0002 & 0 \\ 0 & 1 & 0.04 \\ 0 & -0.007 & 0.962 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0002 \\ 0.0123 \end{bmatrix} u(k)$$

The matrices in the criteria (2) and (3) are:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 3.1 \end{bmatrix}, \; S = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; P=p=1$$

The Figure 1 and Figure 2 show the behaviour of the optimal system in the case of the LQ problem with fixed end-point and with free end-point, respectively. In the both simulations $t_0 = 0$ s (k_0=0), t_f = 1s (k_f=500), the sampling period τ=0.002 s, $x_0=[-2\ 0\ 0]^T$.
Generally, the optimal control refers to a specified time interval. If we are interested to maintain the desired state after the final time $t_f = k_f\tau$, the control law u(k) must be changed for $k>k_f$. For the mentioned example, the control law was changed as

$$u(k) = -\alpha x_1(k) - \beta x_2(k), \; k > k_f \,, \; \alpha>0, \; \beta>0$$

where $x_1(k)$ and $x_2(k)$ are the two first state variables (corresponding to the position and to the speed, if we refer to a servo system) If it is necessary, a

discrete low pass filter can be introduced in order to avoid an abrupt change of u(k) at the moment k_f.
The behaviours in the case of the change of the optimal control law at the moment t_f are presented in the figures 3 and 4, for **P1** and **P2** problems, respectively (the change of the control law for the **P2** problem was performed for $k>0.8k_f$).

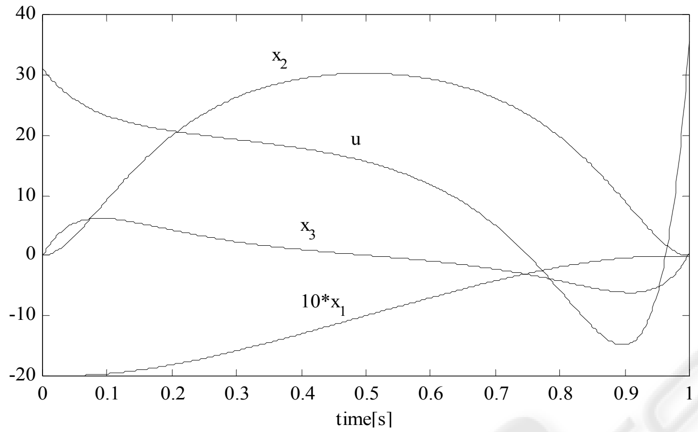

Figure 1: The behaviour of the optimal system in the case of the LQ problem with fixed end-point

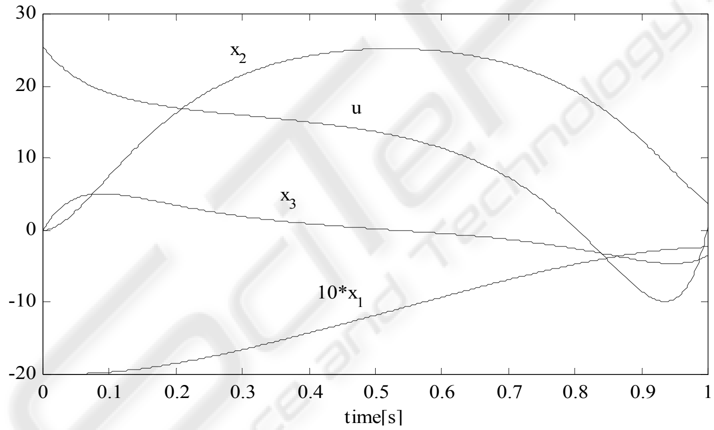

Figure 2: The behaviour of the optimal system in the case of the LQ problem with free end-point

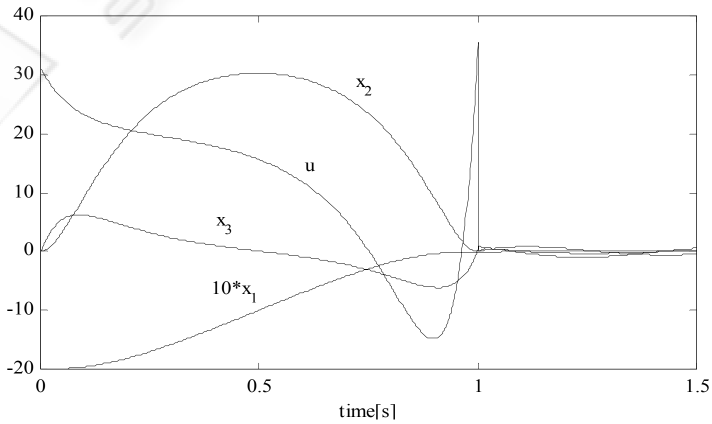

Figure 3: The behaviour of the optimal system for $t<t_f$ and for $t>t_f$ (t_f=1s) in the case of **P1** problem

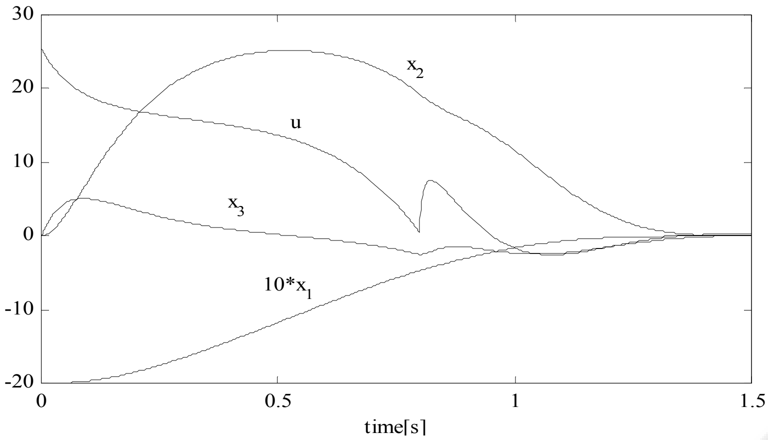

Figure 4: The behaviour of the optimal system for $t<t_f$ and for $t>t_f$ (t_f=1s) in the case of **P2** problem

In both simulated cases (for $k<k_f$ and $k>k_f$), it was calculated $\sum u^2(k)$ and one can remark that this value is bigger in the case of the **P1** problem (with 35%, approximately). This result is expected because in this case the system is forced to reach the imposed final state $x(k_f)=0$.

As it was mentioned, the proposed algorithms can be easier implemented as the classical procedure. Using the MATLAB functions TIC and TOC, the computing time was established. In the case of mentioned example, for all operations performed in a sampling period, it was obtained about 0.06 ms for both **P1** and **P2** problems in the case of the proposed algorithm. In the same conditions, the computing time was 2.6 and 4.6 times grater for the **P1** and **P2** problem, respectively, if a classical approach was used.

5 CONCLUSIONS

A comparison between LQ optimal problems with fixed end-point and free end-point is performed.

By comparison with classical procedures, thealgorithms proposed in the paper for the both problems have advantages and lead to a significant decrease of the computing time.

For the both problems, the proposed approach leads to a similar solution: the optimal control contains similar feedback and supplementary components; the difference is between the last components, which involve different initialisation for a vector.

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