A STOCHASTIC MODEL OF PASSENGER TRANSPORT

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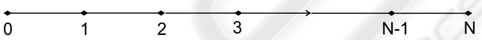
Abstract: We develop a stochastic model in which a recursive formula is derived for computing the mean value of income from sales of bus tickets. The model takes into consideration basic variable costs and the flow of passengers is given by a Poisson process. The recursive formula for the mean income is in the form of a linear discrete system with a random inhomogeneous part.

1 DESCRIPTION OF INITIAL BOARDING OF THE BUS

We assume that the bus has q seats and that the number of passengers does not exceed q. Suppose that the route consists of N segments and that the bus stop only at the beginning of the route and at the stations located at the end of each segment. Passengers board the bus or get off from it only at the stations.

Let us denote by ξ_{2n-1} $(n = 1, 2, \ldots, N)$ the number of passengers in the bus along the $n - th$ segment.



In Fig. 1 we indicate the stops along the route on a line.

Let ξ_{2n} $(n = 1, 2, \ldots, N)$ denote the number of passengers remaining in the bus after some passengers have left at the $n - th$ stop, but not counting the new arrivals. Hence at the n — th stop $(\xi_{2n+1} - \xi_{2n})$ new passengers board the bus. The random variables ξ_n can assume $(q + 1)$ different values $\theta_0 = 0$, $\theta_1 = 1, \ldots, \theta_q = q$ with probabilities

$$p_k(n) = P\{\xi_n = \theta_k\} \qquad (k = 0, 1, \ldots, q).$$

We introduce the vector of probabilities

$$P(n) = [p_0(n), p_1(n), \ldots, p_q(n)]^T,$$

$$(n = 0, 1, 2, \ldots, 2N), \ \dim P(n) = q + 1.$$

We assume that a stationary Poisson process describes the flow of passengers. If the bus waits at the initial stop for a time t_1 and λ denotes the intensity of the passenger flux then the number m passengers boarding the bus during this time is given by the weel — known formula (Ventcel, 1972)

$$P_m = \frac{a^m}{m!} e^{-a}, \ a := \lambda t_1 \quad (m = 0, 1, 2, \ldots).$$

Assume that the bus arrives at the initial stop with ξ_0 passengers. The probabilities of the different numbers of passengers are given by the initial vector $P(0) = [p_0(0), p_1(0), \ldots, p_q(0)]^T$.

Let us introduce the stochastic matrix Π_0 (2) which we will call the boarding matrix.
In the matrix Π_0, to simplify the notation we have

$$\gamma_k := \sum_{s=0}^{k} \frac{a^s}{s!} e^{-a} \quad (k = 0, 1, \ldots, q - 1).$$

Then the equality

$$P(1) = \Pi_0 P(0) \tag{1}$$

defines the vector $P(1) = [p_0(1), p_1(1), \ldots, p_q(1)]$ of probabilities of ξ_1.

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$$\Pi_0 = \begin{bmatrix} e^{-a} & 0 & 0 & \dots & 0 & 0 \\ ae^{-a} & e^{-a} & 0 & \dots & 0 & 0 \\ \frac{a^2}{2!}e^{-a} & ae^{-a} & e^{-a} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{a^{q-1}}{(q-1)!}e^{-a} & \frac{a^{q-2}}{(q-2)!}e^{-a} & \frac{a^{q-3}}{(q-3)!}e^{-a} & \dots & e^{-a} & 0 \\ 1-\gamma_{q-1} & 1-\gamma_{q-2} & 1-\gamma_{q-3} & \dots & 1-\gamma_0 & 1 \end{bmatrix} \tag{2}$$

$$\Pi_1(n) = \begin{bmatrix} 1 & \alpha_1(n) & \alpha_2(n) & \dots & \alpha_{q-1}(n) & \alpha_q(n) \\ 0 & 1-\alpha_1(n) & \alpha_1(n) & \dots & \alpha_{q-2}(n) & \alpha_{q-1}(n) \\ 0 & 0 & 1-\alpha_1(n)-\alpha_2(n) & \dots & \alpha_{q-3}(n) & \alpha_{q-2}(n) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1-\sum_{s=1}^{q-1}\alpha_s(n) & \alpha_1(n) \\ 0 & 0 & 0 & \dots & 0 & 1-\sum_{s=1}^{q}\alpha_s(n) \end{bmatrix} \tag{3}$$

$$\Pi_2(n) = \begin{bmatrix} 1-\sum_{s=1}^{q}\beta_s(n) & 0 & 0 & \dots & 0 & 0 \\ \beta_1(n) & 1-\sum_{s=1}^{q-1}\beta_s(n) & 0 & \dots & 0 & 0 \\ \beta_2(n) & \beta_1(n) & 1-\sum_{s=1}^{q-2}\beta_s(n) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_{q-1}(n) & \beta_{q-2}(n) & \beta_{q-3}(n) & \dots & 1-\beta_1(n) & 0 \\ \beta_q(n) & \beta_{q-1}(n) & \beta_{q-2}(n) & \dots & \beta_1(n) & 1 \end{bmatrix} \tag{4}$$

$$\Pi(n) = \begin{bmatrix} \pi_{00}(n) & \pi_{01}(n) & \pi_{02}(n) & \dots & \pi_{0q}(n) \\ \pi_{10}(n) & \pi_{11}(n) & \pi_{12}(n) & \dots & \pi_{1q}(n) \\ \pi_{20}(n) & \pi_{21}(n) & \pi_{22}(n) & \dots & \pi_{2q}(n) \\ \dots & \dots & \dots & \dots & \dots \\ \pi_{q0}(n) & \pi_{q1}(n) & \pi_{q2}(n) & \dots & \pi_{qq}(n) \end{bmatrix} \tag{5}$$

2 GETTING OFF AND BOARDING AT THE STOP

There are ξ_{2n-1} passengers in the bus arriving at the $n-th$ stop. Some of these passengers may get off. The remaining member is denoted by ξ_{2n}. Next some new passengers board the bus and the total number of passengers in the bus after boarding is denoted by ξ_{2n+1}. The time during which the bus waits be denoted by t_{2n+1}. The probabilities for passengers leaving the bus are related as

$$P(2n) = \Pi_1(n)P(2n-1),\ (n = 1, \ldots, N)$$

where the stochastic matrix $\Pi_1(n)$, for leaving the bus has the form (3)
Here $\alpha_k(n)$denotes the probability of k passengers getting off at the $n-th$ stop. Next we consider new passengers boarding the bus at the $n-th$ stop. The probability may be written by the vector equation

$$P(2n+1) = \Pi_2(n)P(2n), \quad (n = 1, \ldots, N) \quad (6)$$

where the stochastic matrix $\Pi_2(n)$ of boarding has the form (4)
Here $\beta_k(n)$ denotes the probability of k passengers boarding the bus at the $n-th$ stop.
It is clear that $\alpha_k(n), \beta_k(n)$ depend on the number of the stop, since at some stops more passengers gett off or board than at other stops. The coefficients $\alpha_k(n), \beta_k(n), \lambda$ may be set by experiments

3 CALCULATION OF THE INCOME FROM RUNNING THE BUS

The waiting time of the bus at the $n-th$ stop will be denoted by t_{2n-1} and the time for the run along the $n-th$ segment of the route — by t_{2n} $(n = 1, \ldots, N)$.

We will take into account the expenses for paying the driver. If the driver works for a time t, he is paid

$$p_1 = bt$$

where b is some coefficient.
If the bus runs for a time t, we assume that the cost of the fuel consumed and other operating costs is given by

$$p_2 = ct,$$

where c is some coefficient.
A passenger buy a ticket with the cost a.
The price of a ticket for a ride is assumed to be independent of length of the ride.
If we denote by x_{2n-1} $(n = 1, \ldots, N)$ the income derived from passenger transport at the beginning of the $n-th$ segment of the route and by x_{2n} — the income remaining at the end of the $n-th$ segment of route, then we obtain the system of difference equations

$$\begin{array}{l} x_1 = x_0 - bt_1 + a(\xi_1 - \xi_0),\ x_0 = 0 \\ x_2 = x_1 - (b+c)t_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{2n-1} = x_{2n-2} - bt_{2n-1} + a(\xi_{2n-1} - \xi_{2n-2}) \\ x_{2n} = x_{2n-1} - (b+c)t_{2n},\ (n = 2, \ldots, N) \end{array} \quad (7)$$

with the random inhomogeneous part depending on the Markov process ξ_n.

4 OBTAINING THE MOMENT EQUATIONS

Consider the system of difference equations (7) written in general form by letting

$$x_{n+1} = x_n + g(n, \xi_{n+1}, \xi_n) \quad (n = 0, 1, 2, \ldots) \quad (8)$$

where ξ_n is a Markov chain taking values $\theta_0 = 0$, $\theta_1 = 1, \ldots, \theta_q = q$.
Let

$$P(n+1) = \Pi(n)P(n), \quad \dim P(n) = q+1, \quad (9)$$

where the stochastic matrix $\Pi(n)$ has the form (5)
The density distribution of the system (x_n, ξ_n) may be described by the generalised funktion (K.G. Valeev, 1996)

$$f(n, x, \xi) = \sum_{k=0}^{q} f_k(n, x)\delta(\xi - \theta_k). \quad (10)$$

Funktions $f_k(n, x)$ $(k = 0, \ldots, q)$ are called the particular density disrtibutions. They may be defined by the formula

$$P\{x_n < y, \xi_n = \theta_k\} = \int_{-\infty}^{y} f_k(n, x)dx \quad (11)$$

$(k = 0, 1, \ldots, q)$.
We now obtain equations connecting particular density distributions (K. Janglajew, 2003)

$$P\{x_{n+1} < y, \xi_{n+1} = \theta_k\} = \int_{-\infty}^{y} f_k(n+1, x)dx =$$

$$= \sum_{s=0}^{q} P\{x_{n+1} < y, \xi_{n+1} = \theta_k, \xi_n = \theta_s\} =$$

$$= \sum_{s=0}^{q} P\{\xi_{n+1} = \theta_k | x_{k+1} < y, \xi_n = \theta_s\} \times$$

$$\times P\{x_n + g(n, \theta_k, \theta_s) < y, \xi_n = \theta_s\} =$$

$$= \sum_{s=0}^{q} \pi_{ks}(n) \int_{-\infty}^{y - g(n, \theta_k, \theta_s)} f_s(n, x)dx.$$

By differentiating the following equation:

$$\int_{-\infty}^{y} f_k(n+1,x)dx = \\ = \sum_{s=0}^{q} \pi_{ks}(n) \int_{-\infty}^{y-g(n,\theta_k,\theta_s)} f_s(n,x)dx$$

with respect to y and replacing y by x we get the system of equations

$$f_k(n+1,x) = \\ = \sum_{s=0}^{q} \pi_{ks}(n) f_s(n, x - g(n,\theta_k,\theta_s)) \quad (12)$$

$(k = 0, 1, \ldots, q)$.
Let us introduce the initial moments of a random variable x_n

$$m(n) = \int_{-\infty}^{\infty} x f(n,x)dx,$$

$$f(n,x) = \sum_{k=0}^{q} f_k(n,x) \quad (13)$$

$$d(n) = \int_{-\infty}^{\infty} x^2 f(n,x)dx$$

and particular moments

$$m_k(n) = \int_{-\infty}^{\infty} x f_k(n,x)dx \ (k = 0, 1, \ldots, q),$$

$$m(n) = \sum_{k=0}^{q} m_k(n)$$

$$d_k(n) = \int_{-\infty}^{\infty} x^2 f_k(n,x)dx \ (k = 0, 1, \ldots, q),$$

$$d(n) = \sum_{k=0}^{q} d_k(n). \quad (14)$$

Multiplying system (12) by x and integrating over interval $(-\infty, \infty)$ we obtain the system

$$m_k(n+1) = \sum_{s=0}^{q} \pi_{ks}(n)(m_s(n) + \\ + g(n,\theta_k,\theta_s)p_s(n)) \quad (15)$$

$(k = 1, \ldots, q; n = 0, 1, \ldots, 2N)$. Similarly, we get the system

$$d_k(n+1) = \\ = \sum_{s+0}^{q} \pi_{ks}(n)(d_s(n) + 2m_s(n)g(n,\theta_k,\theta_s) + \\ + g^2(n,\theta_k,\theta_s)p_s(n)) \quad (16)$$

$(k = 1, \ldots, q; n = 0, 1, \ldots, 2N)$. From system (15) may be found the mean value $m(2N)$ of the income variable x_{2N}.

5 CALCULATION OF THE MEAN VALUE OF THE INCOME

Consider the system (7) of difference equations with random coefficient. The system of functional equations (12) for our model may be written in the form

$$f_k(1,x) = \sum_{s=0}^{k} \frac{a^s}{s!} e^{-a} f_{k-s}(0, x + bt_1 - a(k-s))$$

$(k = 0, 1, \ldots, q)$

$$f_k(2,x) = \\ = \Big(1 - \sum_{j=1}^{k} \alpha_j(n)\Big) f_k(1, x + (b+c)t_2) + \\ + \sum_{s=k+1}^{q} \alpha_{s-k}(n) f_s((1, x + (b+c)t_2))$$

$(k = 0, 1, \ldots, q)$.
Analogously, we obtain

$$f_k(2n-1,x) = \\ = \Big(1 - \sum_{j=1}^{q-k} \beta_s(n)\Big) f_k(2n-2, x + bt_{2n-1} - \\ - a(k-s)) + \\ + \sum_{s=1}^{k} \beta_s(n) f_{k-s}(2n-2, x + bt_{2n-1} - \\ - a(k-s)),$$

$$f_k(2n,x) = \\ = \Big(1 - \sum_{j=1}^{k} \alpha_j(n)\Big) f_k(2n-1, x + (b+c)t_{2n}) + \\ + \sum_{s=k+1}^{q} \alpha_{s-k}(n) f_s(2n-1, x + (b+s)t_{2n})$$

$(k = 0, 1, \ldots, q; \ n = 2, \ldots, N)$
By using (14) we get

$$m(n) = \int_{-\infty}^{\infty} x f(n,x)dx,$$

$$\text{where } f(n,x) = \sum_{k=0}^{q} f_k(n,x),$$

$$m(n) = \sum_{k=0}^{q} m_k(n).$$

Here the values $m_k(n)$ $(k = 0, 1, \ldots, q)$ are defined by the system of difference equations

$$m_k(1) = \sum_{s=0}^{k} \frac{a^s}{s!} e^{-a} (m_{k-s}(0) + \\ + (a(k-s) - bt_1) p_{k-s}(0));$$

$$m_k(2) = \Big(1 - \sum_{j=1}^{k} \alpha_j(n)\Big)(m_k(1) -$$
$$- (1+c)t_2 p_k(1)) +$$
$$+ \sum_{s=k+1}^{q} \alpha_{s-k}(n)(m_s(1) - (b+c)t_2 p_s(1)),$$

$$m_k(2n-1) = \Big(1 - \sum_{j=1}^{q-k} \beta_s(n)\Big)(m_k(2n-2) +$$
$$+ (a(k-s) - bt_{2n-1})p_k(2n-2) +$$
$$+ \sum_{s=1}^{k} \beta_s(n)(m_{k-s}(2n-2) +$$
$$+ (a(k-s) - bt_{2n-1})p_s(2n-2));$$

$$m_k(2n) = \Big(1 - \sum_{j=1}^{k} \alpha_j(n)(f_k(2n-1) -$$
$$- (b+c)t_{2n} p_k(2n-1)) +$$
$$+ \sum_{s=k+1}^{q} \alpha_{s-k}(n)(m_s(2n-1) -$$
$$- (b+c)t_{2n} p_s(2n-1)) \qquad (17)$$

$(n = 2, \ldots, N; k = 0, 1, \ldots, q)$.
Using formulae (17) we may calculate the mean value $m(2N)$ for the income x_{2M}. Similarly, it is possible to calculate the second moment of the variable x_{2N} by formulae of the form (16).

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