MOMENT-LINEAR STOCHASTIC SYSTEMS

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Abstract: We introduce a class of quasi-linear models for stochastic dynamics, called *moment-linear stochastic systems* (MLSS). We formulate MLSS and analyze their dynamics, as well as discussing common stochastic models that can be represented as MLSS. Further studies, including development of optimal estimators and controllers, are summarized. We discuss the reformulation of a common stochastic hybrid system—the Markovian jump-linear system (MJLS)—as an MLSS, and show that the MLSS formulation can be used to develop some new analyses for MJLS. Finally, we briefly discuss the use of MLSS in modeling certain stochastic network dynamics. Our studies suggest that MLSS hold promise in providing a framework for modeling interesting stochastic dynamics in a tractable manner.

1 INTRODUCTION

As critical networked infrastructures such as air traffic systems, electric power systems, and communication networks have become more interdependent, the need for models for large-scale and hybrid network dynamics has grown. While the dramatic improvement in computer processing speeds in recent years has sometimes facilitated predictive simulation of these networks' dynamics, the development of models that allow not only prediction of dynamics but also network control and design remains a challenge in several application areas (see, e.g., (Bose, 2003)). In this article, we introduce and give the basic analysis for a class of models called *moment-linear stochastic systems* (MLSS) that can represent some interesting stochastic and hybrid system/network dynamics, and yet are sufficiently structured to allow computationally-attractive analyses of dynamics, state estimation, and control. Our studies suggest that MLSS hold promise as a modeling tool for a variety of stochastic and hybrid dynamics, especially because they provide a framework for several common stochastic and/or hybrid models in the literature, and because they can capture some network dynamics in a tractable manner.

Our research is partially concerned with hybrid (mixed continuous- and discrete-state) dynamics. Stochastic hybrid models whose dynamics are constrained to Markovian switching between a finite number of linear time-invariant models have been studied in detail, under several names (e.g., Markovian jump-linear systems (MJLS) and linear jump-Markov systems) (Loparo et al., 1991; Mazor et al., 1998). Techniques for analyzing the dynamics of MJLS, and for developing estimators and controllers, are well-known (e.g., (Fang and Loparo, 2002; Costa, 1994; Mazor et al., 1998)). Of particular interest to us, a linear minimum-mean-square error (LMMSE) estimator for the continuous state of an MJLS was developed by (Costa, 1994), and quadratic controllers have been developed by, e.g., (Chizeck and Ji, 1988). We will show that similar estimation and control analyses can be developed for MLSS, and hence can be applied to a wider range of stochastic dynamics.

We are also interested in modeling stochastic network interactions. There is wide literature on stochastic network models that is largely beyond the scope of this article. Of particular interest to us, several models from the literature on queueing and stochastic automata can be viewed as stochastic hybrid networks (see (Kelly, 1979; Rothman and Zaleski, 1997; Durrett, 1981) for a few examples). By and large, the aims of the analyses for these models differ from our aims, in that transient dynamics are not characterized, and estimators and controllers are not developed. One

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class of stochastic automata of particular interest to us are Hybrid Bayesian Networks (e.g., (Heskes and Zoeter, 2003)). These are graphical models (i.e., models in which stochastic interactions are specified using edges on a graph) with hybrid (i.e., mixed continuous-state and discrete-state) nodal variables. Dynamic analysis, inference (estimation), and parameter learning have been considered for such networks, but computationally feasible methods are approximate.

We observe that stochastic systems models with certain linear or quasi-linear structures (e.g., linear systems driven by additive white Gaussian noise, MJLS) are often widely tractable: statistics of transient dynamics can be analyzed, and LMMSE estimators and optimal quadratic controllers can be constructed. Several stochastic network models with quasi-linear interaction structures have also been proposed—examples include the *voter model* or *invasion process* (Durrett, 1981), and our *influence model* (Asavathiratham, 2000). For these network models, the quasi-linear structure permits partial characterization of state occupancy probabilities using low-order recursions. In this article, we view these various linear and quasi-linear models as examples of *moment-linear models*—i.e., models in which successive moments of the state at particular times can be inferred from equal and lower moments at previous times using linear recursions. This common representation for quasi-linear system and network dynamics motivates our formulation of MLSS, which are similarly tractable to the examples listed above. The MLSS formulation is further useful, in that it suggests some new analyses for particular examples and elucidates some other types of stochastic interactions that can be tractably modeled.

The remainder of the article is organized as follows: in Section 2, we describe the formulation and basic analysis—namely, the recursive computation of state statistics—of MLSS. We also list common models, and types of stochastic interactions, that can be captured using MLSS. Section 3 contains a summary of further results on MLSS. In Section 4, we reformulate the MJLS as an MLSS, and apply the MLSS analyses to this model. We also list other common hybrid models that can be modeled as MLSS. Section 5 summarizes our studies on using MLSS to model network dynamics. In this context, a discrete-time flow network model is developed in some detail.

2 MLSS: FORMULATION AND BASIC ANALYSIS

An MLSS is a discrete-time Markov model in which the conditional distribution for the next state given the current state is specially constrained at each time-step. These conditional distributions are constrained so that successive moments and cross-moments of state variables at each time-step can be found as linear functions of equal and lower moments and cross-moments of state variables at the previous time-step, and hence can be found recursively.

Formally, consider a discrete-time Markov process with an m-component real state vector. The state (i.e., state vector) of the process at time k is denoted by $\mathbf{s}[k]$. We write $\{\mathbf{s}[k]\}$ to represent the state sequence $\mathbf{s}[0], \mathbf{s}[1], \ldots$ and $s_i[k]$ to denote the ith element of the state vector at time k. We stress that we do not in general enforce any structure on the state variables, other than that they be real; the state vector may be continuous-valued, discrete-valued, or hybrid.

For this Markov process, we consider the conditional expectation $E(\mathbf{s}[k+1]^{\otimes r} \,|\, \mathbf{s}[k])$, for $r = 1, 2, \ldots$, where the notation $\mathbf{s}[k+1]^{\otimes r}$ refers to the Kronecker product of the vector $\mathbf{s}[k+1]$ with itself r times and is termed the rth-*order state vector* at time k. This expectation vector contains all rth moments and cross-moments of the state variables $s_1[k+1], \ldots, s_n[k+1]$ given $\mathbf{s}[k]$, and so we call the vector the *conditional* rth *(vector) moment* for $\mathbf{s}[k+1]$ given $\mathbf{s}[k]$. We say that the process $\{\mathbf{s}[k]\}$ is rth-*moment linear* at time k if the conditional rth moment for $\mathbf{s}[k+1]$ given $\mathbf{s}[k]$ can be written as follows:

$$E(\mathbf{s}[k+1]^{\otimes r} \,|\, \mathbf{s}[k]) = H_{r,0}[k] + \sum_{i=1}^{r} H_{r,i}[k]\mathbf{s}[k]^{\otimes i}, \tag{1}$$

for some set of matrices $H_{r,0}[k], \ldots, H_{r,r}[k]$.

The Markov process $\{\mathbf{s}[k]\}$ is called a *moment-linear stochastic system* (MLSS) of degree $\widehat{r}$ if it is rth-moment linear for all $r \leq \widehat{r}$, and for all times k. If a Markov model is moment linear for all r and k, we simply call the model an MLSS. We call the constraint (1) the rth-*moment linearity condition* at time k, and call the matrices $H_{r,0}[k], \ldots, H_{r,r}[k]$ the rth-*moment recursion matrices* at time k. These recursion matrices feature prominently in our analysis of the temporal evolution of MLSS.

MLSS are amenable to analysis, in that we can find statistics of the state $\mathbf{s}[k]$ (i.e., moments and cross-moments of state variables) using linear recursions. In particular, for an MLSS of degree $\widehat{r}$, $E(\mathbf{s}[k+1]^{\otimes r})$ (called the rth *moment* of $\mathbf{s}[k+1]$) can be found in terms of the first r moments of $\mathbf{s}[k]$ for any $r \leq \widehat{r}$. To find these rth moments, we use the law of iterated expectations and then invoke the rth-moment linearity condition:

$$E(\mathbf{s}[k+1]^{\otimes r}) = E(E(\mathbf{s}[k+1]^{\otimes r}\,|\,\mathbf{s}[k])) \quad (2)$$
$$= E(H_{r,0}[k] + \sum_{i=1}^{r} H_{r,i}[k]\mathbf{s}[k]^{\otimes i})$$
$$= H_{r,0}[k] + \sum_{i=1}^{r} H_{r,i}[k]E(\mathbf{s}[k]^{\otimes i}).$$

We call (2) the rth-*moment recursion* at time k. Considering equations of the form (2), we see that the first r moments of $\mathbf{s}[k+1]$ can be found as a linear function of the first r moments of $\mathbf{s}[k]$. Thus, by applying the moment recursions iteratively, the rth moment of $\mathbf{s}[k]$ can be written in terms of the first r moments of the initial state $\mathbf{s}[0]$.

The recursions developed in equations of the form (2) can be rewritten in a more concise form, by stacking rth and lower moment vectors into a single vector. In particular, defining $\mathbf{s}'_{(r)}[k] = \left[\ \mathbf{s}'[k]^{\otimes r} \quad \ldots \quad \mathbf{s}'[k]^{\otimes 1} \quad 1\ \right]$, we find that

$$E(\mathbf{s}_{(r)}[k+1]) = \tilde{H}_{(r)}[k]E(\mathbf{s}_{(r)}[k]), \quad (3)$$

where

$$\tilde{H}_{(r)}[k] = \quad (4)$$
$$\begin{bmatrix} H_{r,r}[k] & H_{r,r-1}[k] & \ldots & H_{r,1}[k] & H_{r,0}[k] \\ 0 & H_{r-1,r-1}[k] & \ldots & H_{r-1,1}[k] & H_{r-1,0}[k] \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & H_{1,1}[k] & H_{1,0}[k] \\ 0 & \ldots & 0 & 0 & 1 \end{bmatrix} .$$

Interactions and Examples Captured by MLSS MLSS provide a convenient framework for representing several models that are prevalent in the literature. These MLSS reformulations are valuable in that they expose similarity in the moment structure of several types of stochastic interactions, and in that they permit new analyses (e.g., development of linear state estimators, or characterization of higher moments) in the context of these examples. Common models that can be represented as MLSS are listed:

- A linear system driven by independent, identically distributed noise samples with known statistics is an MLSS.
- A finite-state Markov chain can be formulated as an MLSS, by defining the MLSS state to be a 0–1 *indicator vector* of the state of the Markov chain.
- An MJLS is a hybrid model that can be reformulated as an MLSS. This reformulation of an MJLS is described in Section 4.
- A Markov-modulated Poisson process (MMPP) is a stochastic model that has commonly been used to represent sources in communications and manufacturing systems (e.g., (Baiocchi et al., 1991), (Nagarajan et al., 1991), (Ching, 1997)). It consists of a finite-state *underlying Markov chain*, as well as a Poisson arrival process with (stochastic) rate modulated by the underlying chain. An MMPP can be formulated as an MLSS using a state vector the indicates the underlying Markov chain's state and also tracks the number of arrivals over time (see (Roy, 2003) for details). The MLSS reformulation of MMPPs highlights that certain state-parametrized stochastic updates, such as a Poisson generator with mean specified as a linear function of the current state, can be represented using MLSS. More generally, various stochastic updates with state-dependent noise can be represented.
- Certain infinite server queues can be represented as MLSS (see (Roy, 2003) for details).

Observations and Inputs Observations and external inputs are naturally incorporated in MLSS. These are structured so as to preserve the moment-linear structure of the dynamics, and hence to allow development of recursive linear estimators and controllers for MLSS.

At each time k, we observe a real vector $\mathbf{z}[k]$ that is independent of the past history of the system, (i.e., $\mathbf{s}[0], \ldots, \mathbf{s}[k-1]$ and $\mathbf{z}[0], \ldots, \mathbf{z}[k-1]$), given the current state $\mathbf{s}[k]$. The observation $\mathbf{z}[k]$ is assumed to be first- and second-moment linear given $\mathbf{s}[k]$. That is, $\mathbf{z}[k]$ is generated from $\mathbf{s}[k]$ in such a way that the first moment (mean) for $\mathbf{z}[k]$ given $\mathbf{s}[k]$ can be written as an affine function of $\mathbf{s}[k]$:

$$E(\mathbf{z}[k]\,|\,\mathbf{s}[k]) = C_{1,1}\mathbf{s}[k] + C_{1,0} \quad (5)$$

for some $C_{1,1}$ and $C_{1,0}$, and the second moment for $\mathbf{z}[k]$ given $\mathbf{s}[k]$ can be written as an affine function of $\mathbf{s}[k]$ and $\mathbf{s}[k]^{\otimes 2}$:

$$E(\mathbf{z}[k]^{\otimes 2}\,|\,\mathbf{s}[k]) = C_{2,2}\mathbf{s}[k]^{\otimes 2} + C_{2,1}\mathbf{s}[k] + C_{2,0} \quad (6)$$

for some $C_{2,2}$, $C_{2,1}$, and $C_{2,0}$.

We have restricted observations to a form for which analytical expressions for LMMSE estimators can be found, yet relevant stochastic interactions that are not purely linear can be captured. Observation generation from the state in MLSS admits the same variety of stochastic interactions—e.g., linear interactions, finite-state Markovian transitions, certain state-parameterized stochastic updates—as captured by the MLSS state update. This generality allows us to model, e.g., Hidden Markov Model observations (Rabiner, 1986), random selection among multiple linear observations, and observations of a Poisson arrival process with mean modulated by the state.

The inputs in our model are structured in such a way that the next state is first- and second-moment linear with respect to both the current state and the current input. Specifically, a system is a 2nd-degree

MLSS with state sequence $\{\mathbf{s}[k]\}$ and input $\{\mathbf{u}[k]\}$ if the conditional distribution for the next state is independent of the past history of the system, given the current state and input, and if there exist matrices $H_{1,1}$, $D_{1,1}$, $H_{1,0}$, $H_{2,2}$, $G_{2,2}$, $D_{2,2}$, $H_{2,1}$, $D_{2,1}$, and $H_{2,0}$ such that

$$E(\mathbf{s}[k+1] \,|\, \mathbf{s}[k], \mathbf{u}[k]) = H_{1,1}\mathbf{s}[k] + D_{1,1}\mathbf{u}[k] + H_{1,0}$$

and

$$\begin{aligned} &E(\mathbf{s}[k+1]^{\otimes 2} \,|\, \mathbf{s}[k], u[k]) \\ &= H_{2,2}\mathbf{s}[k]^{\otimes 2} + G_{2,2}(\mathbf{s}[k] \otimes \mathbf{u}[k]) + D_{2,2}\mathbf{u}[k]^{\otimes 2} \\ &+ H_{2,1}\mathbf{s}[k] + D_{2,1}\mathbf{u}[k] + H_{2,0}. \end{aligned} \quad (7)$$

That is, a Markov process is a 2nd-degree MLSS with input $\mathbf{u}[k]$ if the first and second moments and cross-moments of the next state, given the current state and input, are first- and second-degree polynomials, respectively, of current state and input variables.

We have restricted MLSS inputs to a form for which analytical expressions for optimal quadratic controllers can be found, yet several relevant types of input interactions can nevertherless be represented. The dependence of the state on the input has the same generality as the state update.

3 FURTHER RESULTS

In the following paragraphs, we summarize four results on MLSS that are elaborated further in (Roy, 2003).

Cross Moments Cross-moments of state variables across multiple time-steps can be calculated, by recursively rewriting cross-moments across time-steps as linear functions of moments and cross-moments at single times. Our expressions for cross-moments are similar in flavor to the Kronecker product-based expressions for higher-order statistics of linear systems given by, e.g., (Mendel, 1975; Swami and Mendel, 1990), but apply to MLSS.

Asymptotics We develop necessary and sufficient conditions for convergence (approach to a steady-state value) of MLSS moments[1] in (Roy, 2003). Conditions for moment convergence are useful because they can help to characterize asymptotic dynamics: they can in some instances be used to prove convergence of the state itself, or to prove distributional convergence, for example.

Because the moments satisfy linear recursions, conditions for convergence can straightforwardly be expressed in terms of the modes of the moment recursion matrices. However, we note that redundancy in the moment vectors, which is inherent to the MLSS formulation, complicates development of good convergence conditions because it allows introduction of spurious unstable modes that do not actually alter the moments. We therefore develop reduced forms of the moment recursions to construct the necessary and sufficient conditions for moment convergence. Details are in (Roy, 2003).

Estimation We have developed a recursive algorithm for linear minimum mean square error (LMMSE) filtering of MLSS. Because tractability of estimation and control is a primary goal in our formulation of MLSS, it is worthwhile for us to present our estimator, and to connect it to related literature.

Our LMMSE filter for an MLSS is a generalization of the discrete-time Kalman filter (see, e.g., (Catlin, 1989)), in which the state update and observation processes are constrained to be moment-linear rather than purely linear. Equivalently, we can view our filter as applying to a linear system in which certain quasi-linear state-dependence of state and observation noise is permitted. From this viewpoint, our filter is related to the LMMSE filter introduced in (Zehnwirth, 1988), which allows for state-dependent observation variance. Also of interest to us are linear estimation techniques for arrival processes whose underlying rate processes are themselves random, and/or arrival-dependent (e.g., (Snyder, 1972),(Segall et al., 1975)). Segall and Kailath describe a broad class of arrival processes of this type, for which a martingale formulation facilitates nonlinear recursive estimation (Segall et al., 1975). We also can capture some arrival processes with stochastic rates (e.g., MMPPs), and hence develop recursive state estimators for these processes. The arrival processes that are MLSS are a subset of those in (Segall et al., 1975), but for which linear filtering is possible, and hence exact finite-dimensional estimators can be constructed.

The derivation of our LMMSE filter for MLSS, which closely follows the derivation of the discrete-time Kalman filter, can be found in (Roy, 2003). Here, we only present the results of our analysis. We use the following notation: we define $\widehat{\mathbf{s}}_{k|k}$ as the LMMSE estimate for $\mathbf{s}[k]$ given $\mathbf{z}[0], \ldots, \mathbf{z}[k]$, and define $\Sigma_{k|k} \triangleq E\left((\mathbf{s}[k] - \widehat{\mathbf{s}}_{k|k})(\mathbf{s}[k] - \widehat{\mathbf{s}}_{k|k})'\right)$ as the error covariance of this estimate. Also, we define $\widehat{\mathbf{s}}_{k+1|k}$ as the LMMSE estimate for $\mathbf{s}[k+1]$ given $\mathbf{z}[0], \ldots, \mathbf{z}[k]$, and let define $\Sigma_{k+1|k} \triangleq E\left((\mathbf{s}[k+1] - \widehat{\mathbf{s}}_{k+1|k})(\mathbf{s}[k+1] - \widehat{\mathbf{s}}_{k|k})'\right)$ as the error covariance of this estimate.

As with the Kalman filter, the estimates are found recursively, in two steps. First, a *next-state update* is

[1] Our study of moment convergence is limited to the state update of an MLSS; input-output stability has not yet been considered.

used to determine $\widehat{\mathbf{s}}_{k+1|k}$ and $\Sigma_{k|k}$ in terms of $\widehat{\mathbf{s}}_{k|k}$, $\Sigma_{k|k}$, and the *a priori* statistics of $\mathbf{s}[k]$. The next-state update for our filter is

$$\widehat{\mathbf{s}}_{k+1|k} = H_{1,1}\widehat{\mathbf{s}}_{k|k} + H_{1,0} \quad (8)$$

$$\Sigma_{k+1|k} = H_{1,1}\Sigma_{k|k}H'_{1,1} + M_k(E(\mathbf{s}[k], E(\mathbf{s}[k]^{\otimes 2})) \quad (9)$$

In (9), $M_k(E(\mathbf{s}[k], E(\mathbf{s}[k]^{\otimes 2}))$ is the (*a priori*) expectation for the conditional variance of $z[k]$ given $s[k]$; an explicit expression is given in (Roy, 2003).

Second, a *measurement update* is used to determine $\widehat{\mathbf{s}}_{k|k}$ and $\Sigma_{k|k}$ in terms of $\mathbf{z}[k]$, $\widehat{\mathbf{s}}_{k|k-1}$, $\Sigma_{k|k-1}$, and the *a priori* statistics of $\mathbf{s}[k]$. The measurement update for our filter is

$$\widehat{\mathbf{s}}_{k|k} = \widehat{\mathbf{s}}_{k|k-1} + \Sigma_{k|k-1}C'_{1,1}(C_{1,1}\Sigma_{k|k-1}C'_{1,1} + N_k(E(\mathbf{s}[k], E(\mathbf{s}[k]^{\otimes 2})))^{-1}(\mathbf{z}[k] - C_{1,1}\widehat{\mathbf{s}}_{k|k-1} - C_{1,0})$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1}C'_{1,1}(C_{1,1}\Sigma_{k|k-1}C'_{1,1} + N_k\left(E(\mathbf{s}[k]), E(\mathbf{s}[k]^{\otimes 2})\right))^{-1}C_{1,1}\Sigma_{k|k-1}. \quad (10)$$

In (10), $N_k\left(E(\mathbf{s}[k]), E(\mathbf{s}[k]^{\otimes 2})\right)$ is the *a priori* expectation for the conditional variation of $\mathbf{s}[k+1]$ given $\mathbf{s}[k]$; an explicit expression is given in (Roy, 2003).

Control Considering the duality between linear estimation and quadratic control, it is not surprising that optimal dynamic quadratic control can be achieved for MLSS, given full state information. In (Roy, 2003), we use a dynamic programming formulation for the quadratic control problem to find a closed-form recursion for the optimal control. The optimal control at each time is linear with respect to the current state, in analogy to the optimal quadratic control for a linear system. The reader is referred to (Roy, 2003) for presentation and discussion of the optimal controller, as well as a description of relevant literature.

4 EXAMPLE: MJLS

We describe the reformulation of MJLS as MLSS, and then present some analyses of MJLS using the MLSS formulation. For simplicity, we only detail the reformulation of a Markovian jump-linear state-update equation here; reformulation of the entire input-output dynamics is a straightforward extension.

Let's consider the update equation

$$\mathbf{x}[k+1] = A(\mathbf{q}[k])\mathbf{x}[k] + \mathbf{b}_k(\mathbf{q}[k]), \quad (11)$$

where $\{\mathbf{q}[k]\}$ is a 0–1 indicator vector sequence representation for an underlying Markov chain with finite state-space. We denote the transition matrix for the underlying Markov chain by $\Theta = [\theta_{ij}]$. We denote the number of components of $\mathbf{x}[k]$ as n, and the number of statuses in the underlying Markov chain as m.

For convenience, we rewrite (11) in an extended form as

$$\tilde{\mathbf{x}}[k+1] = \tilde{A}(\mathbf{q}[k])\tilde{\mathbf{x}}[k], \quad (12)$$

where $\tilde{\mathbf{x}}[k] = \begin{bmatrix}\mathbf{x}[k]\\ 1\end{bmatrix}$ and $\tilde{A}(\mathbf{q}[k]) = \begin{bmatrix}A(\mathbf{q}[k]) & \mathbf{b}(\mathbf{q}[k])\\ \mathbf{0} & 1\end{bmatrix}$.

To reformulate the jump-linear system as an MLSS, we define a state vector that captures both the continuous state and underlying Markov dynamics of the jump-linear system. In particular, we define the state vector as $\mathbf{s}[k] = \mathbf{q}[k] \otimes \tilde{\mathbf{x}}[k]$, and consider the first conditional vector moment $E(\mathbf{s}[k+1] \,|\, \mathbf{s}[k])$. Since $\mathbf{s}[k]$ uniquely specifies $\mathbf{x}[k]$ and $\mathbf{q}[k]$, we can determine this first conditional vector moment as follows:

$$\begin{aligned} E(\mathbf{s}[k+1] \,|\, \mathbf{s}[k]) &= E(\mathbf{s}[k+1] \,|\, \mathbf{q}[k], \mathbf{x}[k]) \quad (13)\\ &= E(\mathbf{q}[k+1] \otimes \tilde{\mathbf{x}}[k+1] \,|\, \mathbf{x}[k], \mathbf{q}[k])\\ &= E(\mathbf{q}[k+1] \,|\, \mathbf{q}[k]) \otimes \tilde{A}(\mathbf{q}[k])\tilde{\mathbf{x}}[k]\\ &= \Theta'\mathbf{q}[k] \otimes \tilde{A}(\mathbf{q}[k])\tilde{\mathbf{x}}[k] \end{aligned}$$

With a little bit of algebra, we can rewrite Equation 13 as

$$E(\mathbf{s}[k+1] \,|\, \mathbf{s}[k]) = \begin{bmatrix} \theta_{11}\tilde{A}(\mathbf{q}[k] = \mathbf{e}(1)) & \dots & \theta_{m1}\tilde{A}(\mathbf{q}[k] = \mathbf{e}(m))\\ \vdots & \ddots & \vdots\\ \theta_{1m}\tilde{A}(\mathbf{q}[k] = \mathbf{e}(1)) & \dots & \theta_{mm}\tilde{A}(\mathbf{q}[k] = \mathbf{e}(n)) \end{bmatrix} \mathbf{s}[k], \quad (14)$$

where $\mathbf{e}(i)$ is an indicator vector with the ith entry equal to 1.

Equation (14) shows that the first-moment linearity condition holds for $\{\mathbf{s}[k]\}$. To justify that higher-moment linearity conditions hold, let's consider entries of the rth-conditional moment vector $E(\mathbf{s}[k+1]^{\otimes r} \,|\, \mathbf{s}[k]) = E((\mathbf{q}[k+1] \otimes \tilde{\mathbf{x}}[k+1])^{\otimes r} \,|\, \mathbf{x}[k], \mathbf{q}[k])$. Because $\mathbf{q}[k]$ is an indicator, the non-zero entries of the rth-conditional moment vector can all be written in the form $E(q_i[k+1]\prod_{i=1}^{n} x_i[k+1]^{\alpha_i} \,|\, \mathbf{x}[k], \mathbf{q}[k])$, where each $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i = \widehat{r} \leq r$. Given that $\mathbf{q}[k] = \mathbf{e}(i)$, $1 \leq i \leq m$, $E(\prod_{i=1}^{n} x_i[k+1]^{\alpha_i} \,|\, \mathbf{x}[k], \mathbf{q}[k])$ is an $\widehat{r}$th degree polynomial of $x_1[k], \dots, x_n[k]$, say $p_i[k]$. Using that $\mathbf{q}[k]$ is an indicator vector, we can rewrite $E(q_i[k+1]\prod_{i=1}^{n} x_i[k+1]^{\alpha_i} \,|\, \mathbf{x}[k], \mathbf{q}[k])$ as $\sum_{i=1}^{m} q_i[k]p_i[k]$. Hence, we see that each entry in the rth-conditional moment vector is linear with respect to the entries in $\mathbf{s}[k]^{\otimes r} = (\mathbf{q}[k+1] \otimes \tilde{\mathbf{x}}[k+1])^{\otimes r}$, and so the state vector is rth-moment linear. Some bookkeeping is required to express the higher-moment linearity conditions in vector form; these expressions are omitted.

We believe that the MLSS reformulation of MJLS is valuable, because it places MJLS in the context of a broader class of models, and because it facilitates several analyses of MJLS dynamics. Some analyses of MJLS dynamics that can be achieved using the MLSS formulation are listed below. Our analyses are illustrated using an example MJLS with a two-status underlying Markov chain and scalar continuous-valued state. The underlying Markov chain for this example has transition matrix $\Theta = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$. The scalar continuous state is updated as follows: if the Markov chain is in the first status at time k, then the time-$(k+1)$ continuous state is $x[k+1] = -0.9x[k]+0.5$; if the Markov chain is in the second status, the time-$(k+1)$ continuous state is $x[k+1] = x[k]+1$.

The MLSS formulation allows us to compute statistics (moments and cross-moments) for both the continuous-valued state and underlying Markov status, as well as conditional statistics for the continuous-valued state given the underlying Markov status (at one or several times). Recursions for the first- and second-moments of the continuous-valued state are well-known (see, e.g., (Costa, 1994)), though our second-moment recursion differs in form from the recursion on the covariance matrix that is common in the literature. We have not seen general computations of higher-order statistics, or of statistics across time-steps: the MLSS formulation provides a concise notation in which to develop recursions for these statistics. The higher-moment recursions are especially valuable because they can provide characterizations for MJLS asymptotics. We can specify conditions for δ-*moment stability* (see, e.g., (Fang et al., 1991)) for all even integer δ in terms of the eigenvalues of the higher moment recursions. We can also characterize the asymptotics of MJLS in which the state itself does not stabilize, by checking for convergence of moments (to non-zero values). We are currently exploring whether the methods of (Meyn and Tweedie, 1994) can be used to prove distributional convergence from moment convergence.

For illustration, first- and second-order statistics of the example MLSS are shown along with a fifty time-step simulation in Fig. 1. Additionally, the steady-state values for the first three moments of the continuous-valued state have been obtained from the moment recursions, and are shown along with the corresponding steady-state distribution. We note that the first three moments provide significant information about the steady-state distribution and, in this example, require much less effort to compute than the full distribution.

The MLSS formulation allows us to develop conditions for moment convergence in MJLS. In (Roy, 2003), we have presented necessary and sufficient conditions for moment convergence of an MJLS with scalar continuous-valued state and two-status underlying Markov chain, in terms of its parameters. Our

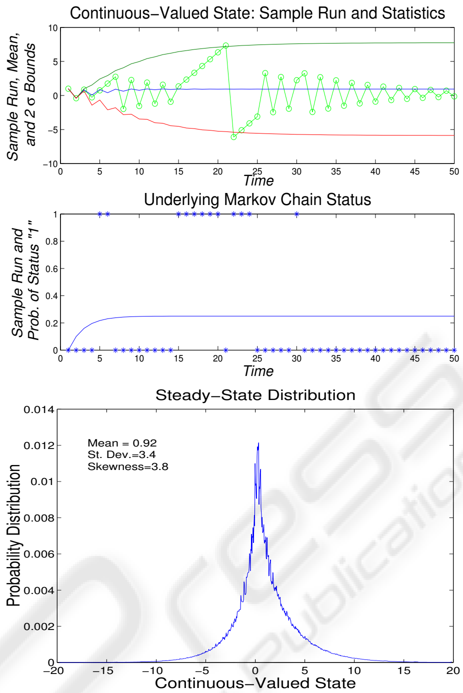

Figure 1: This figure illustrates the MLSS-based analysis of moments for an example MJLS. The top plot in this figure specifies the continuous-valued state during a 50 time-step simulation, along with the computed mean value and two standard deviation intervals for this continuous-valued state. The middle plot indicates the status of the underlying Markov chain during the simulation and also shows the probability that the Markov chain is in status "1". The bottom plot shows the steady-state distribution for the continuous-valued state (found through iteration of the distribution) and lists the moments of this continuous-valued state (found with much less computation using the moment recursions).

example MJLS, with statistics shown in Fig. 1, can be shown to be moment-convergent. Our study of moment convergence complements stability studies for MJLS (e.g., (Fang and Loparo, 2002)), by allowing identification of MJLS whose moments reach a steady-state but that are not stable (in the sense that state itself does not reach a steady-state).

The MLSS reformulation can be used to develop LMMSE estimators for MJLS. LMMSE estimation of the continuous-valued state of an MJLS from noisy observations has been considered by (Costa, 1994). The observation structure (i.e., the generator of the observation from the concurrent state) assumed by (Costa, 1994) can be captured using our MLSS formulation, and in that case our estimator is nearly identical to that of (Costa, 1994); only the squared error that is minimized by the two estimators is slightly different.

Our MLSS formulation allows for estimation for a variety of observation structures. For instance, we can capture observations that are Poisson random variables, with parameter equal to a linear function of the state variables. (Such an observation model may be realistic, for example, if the state process modulates an observed Poisson arrival process.) We can also capture other types of state-dependent noise in the observation process, as well as various discrete and continuous-valued state-independent noise. Further, hybrids of multiple observation structures can be captured in the MLSS formulation.

Another potential advantage of the MLSS formulation is that, because the underlying Markov status is part of the state vector, this underlying status can be estimated. The accuracy of our estimator for the underlying state remains to be assessed. A direction for future study is to compare our estimator for the underlying status with the nonlinear estimators of, e.g., (Sworder et al., 2000; Mazor et al., 1998).

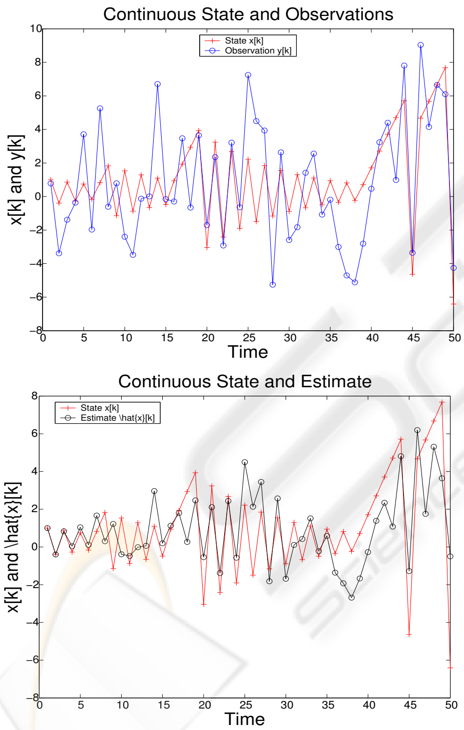

Figure 2: In the left plot, the continuous-valued state and observations during a 50 time-step simulation of the example MJLS are shown. In the right plot, the LMMSE estimate for the continuous-valued state is compared with the actual continuous-valued state. The LMMSE estimate is a better approximation for the continuous-valued state than the observation sequence.

For illustration, we have developed an LMMSE filter for our example MJLS. Here, we observe the (scalar) continuous-valued state upon corruption by additive white Gaussian noise. Fig. 2 illustrates the performance of the LMMSE estimator, for a particular sample run. Our analysis shows that the error covariance of our estimate is about half the measurement error covariance, verifying that our estimate is usually a more accurate estimate for the state than the unfiltered observation sequence.

5 MLSS MODELS FOR NETWORK DYNAMICS: SUMMARY

We believe that MLSS representations for networks are of special interest because analysis of stochastic network dynamics is quite often computationally intensive (e.g., exponential in the number of vertices), while MLSS representations can permit analysis at much lower computational cost. Below, we summarize our studies of MLSS representations for network dynamics. More detailed description of MLSS models for network dynamics can be found in (Roy, 2003).

The following are examples of MLSS models for network dynamics that we have considered.

- The *influence model* was introduced in (Asavathiratham, 2000) as a network of interacting finite-state Markov chains, and is developed and motivated in some detail in (Asavathiratham et al., 2001). We refer the reader to (Basu et al., 2001) for one practical application, as a model for conversation in a group. The influence model consists of **sites** with discrete-valued statuses that evolve in time through stochastic interaction. The influence model's structured stochastic update permits formulation of the model as an MLSS, using 0–1 indicator vector representations for each site's status. The rth moment recursion permits computation of the joint probabilities of the statuses of groups of r sites, and hence the configurations of small groups of sites can be characterized using low-order recursions. The MLSS formulation for the influence model can also be used to develop good state estimators for the model, and to prove certain results on the convergence of the model.
- MLSS can be used to represent single-server queueing networks with linear queue length-dependent arrival and service rates, operating under heavy-traffic conditions. The MLSS formulation allows us to find statistics and cross-statistics of the queue lengths. Of particular interest is the possibility for using the MLSS formulation to develop queue-length estimators for these state-dependent models and, indirectly, for Jackson networks (see, e.g., (Kelly, 1979)).

- In (Roy et al., 2004), we have developed an MLSS model for aircraft counts in regions of the U.S. airspace, and have used the MLSS formulation to compute statistics of aircraft counts in regions. In the context of this MLSS model, we have also developed techniques for parameter estimation from data.

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