SOME FEASIBILITY ISSUES RELATED TO CONSTRAINED GENERALIZED PREDICTIVE CONTROL

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Abstract This paper analyzes the feasibility of the generalized predictive control law under constraints on the input, output or other auxiliary signals that depend linearly on the system variables. These constraints are formulated as sets of linear equalities or inequalities; the control sequence is therefore elaborated based on a quadratic optimization problem. The feasibility issues are related on one hand to the well posedness feature, and on the other hand to the compatibility with the set-point constraints. The prediction of the feasibility is of great interest from this point of view and necessary feasibility conditions are presented. Two possible approaches are followed, one strictly related to the specific set-point and the second, more general, examines the geo-metrical description of the optimization domain. The main practical advantage is that all the results are based on off-line numerical procedures offering qualitative information prior to the effective implementation.

1 INTRODUCTION

The computer aided design of control laws must overcome important difficulties when some imposed constraints must be satisfied. These constraints may be forced by practical considerations as limitations on the input control signal amplitude or rate. Other constraints may arise from the qualities desired for the control law, a classical example being the output constraints (Maciejowski, 2002). Other hidden constraints, from the end-user point of view, could be forced for example with end-point stability constraints.

All can be expressed as linear equality or inequality constraints that have to be further considered in the control design procedure (Erhlinger, *et al*., 1996). This set describes in fact a polyhedral domain for which a dual representation in terms of generators is available (Wilde, 1993). Analyzing the geometry and the evolutions of this polyhedron due to the dynamic evolution of the controlled system variables could highlight the characterization of the control algorithm.

This paper considers the model predictive control (MPC) in the presence of such operational constraints that alienate the performance of the control sequence provided by the unconstrained optimum. The effects could be severe, as for example unstable systems regulated by a constrained controller cannot be stabilized for all initial conditions. An exhaustive analysis of the system of constraints may reveal useful properties such as the expression of the “switching surfaces” for the linear control laws and the corresponding affine formulations (Bemporad, *et al*., 2002), (Seron, *et al*., 2003). This paper deals with another important aspect related to the constraints analysis, the feasibility of the optimization problem to be solved (Kouvaritakis, *et al*., 2000). This is a sensitive one as long as, in the case of an infeasibility message coming from the optimization solver, the entire control law is invalidated and the control performances are damaged in an irreversible way. Consequently, an analysis of the infeasibility is crucial for the validation of the predictive control law (Olaru and Dumur, 2003), (Scokaert and Clarke, 1994b). This is equivalent with an off-line prediction of infeasibility. It must be mentioned that even for the analytical close form description the feasibility domain represents an important problem.

The main contribution of this paper is to provide results towards feasibility and to stress their implications in the case of general types of set-points. Theoretical aspects related to some classes of necessary feasibility conditions are covered and an algorithm is built in order to check these conditions, based on off-line information. In practice, although

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it cannot be analytically proved that they are sufficient, these conditions offer a good test for the on-line feasibility. The advantage of this approach is that it covers the CGPC feasibility directly linked with the structure of the set-points to be followed. The paper concludes with a numerical example proving the usefulness of the specified procedures.

The paper is organized as follows: Section 2 describes the generalized predictive control formulation. Section 3 examines the constrained case and provides some basic results about feasibility. Section 4 is dedicated to the geometrical description of the constraints and the main results towards necessary conditions to be satisfied for on-line feasibility. Section 5 presents a simulation on a second order non minimum phase system. Finally, Section 6 gives some concluding remarks.

2 GENERALIZED PREDICTIVE CONTROL

Generalized predictive control (GPC) is part of the long-range predictive control (LRPC) or model predictive control (MPC) family (Rossiter, 2003). All these controllers are based on the fact that the process evolution can be predicted over a horizon taking into account the history of the control inputs, plant outputs and the potential future control sequence. The quantification of suitability for the predicted response is measured by a cost function that considers the fitness with respect to the desired characteristics. GPC is characterized by two major characteristics. It uses first a CARIMA plant model

$$A(q^{-1})y(t) = B(q^{-1})u(t-d) + C(q^{-1})\xi(t)/\Delta(q^{-1}) \quad (1)$$

where u, y are the system input and output respectively, $\xi(t)$ represents a centered Gaussian white noise, d the system time delay, A and B are polynomials in q^{-1} (the backward shift operator) of degree n_a and n_b, and $\Delta(q^{-1}) = 1 - q^{-1}$.

Then the cost function to be minimized is quadratic in the tracking error and control effort over a receding horizon

$$J = \sum_{j=N_1}^{N_2}\left[\hat{y}(t+j) - w(t+j)\right]^2 + \lambda\sum_{j=1}^{N_u}\left[\Delta u(t+j-1)\right]^2 \quad (2)$$

where $\hat{y}(t+j)$ is the output prediction, N_1, N_2 are the minimum and maximum costing horizon, N_u the control horizon, λ a control weighting factor and w the setpoint.

Based on the model mentioned earlier and following the ideas of GPC (Clarke, *et al.*, 1987) an optimal j-step ahead predictor can be constructed

$$\hat{y}(t+j) = \underbrace{F_j(q^{-1})y(t) + H_j(q^{-1})\Delta u(t-1)}_{l = \text{free response}} + \underbrace{G_j(q^{-1})\Delta u(t+j-1)}_{\text{forced response}} \quad (3)$$

where the F_j, G_j, H_j polynomials are solutions of the Diophantine equations

$$\begin{aligned} &\Delta(q^{-1})A(q^{-1})J_j(q^{-1}) + q^{-j}F_j(q^{-1}) = 1 \\ &G_j(q^{-1}) + q^{-j}H_j(q^{-1}) = B(q^{-1})J_j(q^{-1}) \end{aligned} \quad (4)$$

The index (2) is rewritten for optimization purpose

$$\begin{aligned} J &= (\mathbf{G}\,\mathbf{k}_u + \mathbf{l} - \mathbf{w})^{\mathrm{T}}(\mathbf{G}\,\mathbf{k}_u + \mathbf{l} - \mathbf{w}) + \lambda\,\mathbf{k}_u^{\mathrm{T}}\mathbf{k}_u = \\ &= 0.5\,\mathbf{k}_u^{\mathrm{T}}\mathbf{Q}\,\mathbf{k}_u + \mathbf{f}\,\mathbf{k}_u + J_0 \end{aligned} \quad (5)$$

with the vector form of (3)

$$\hat{\mathbf{y}} = \mathbf{G}\,\mathbf{k}_u + \mathbf{l} = \mathbf{G}\,\mathbf{k}_u + \mathbf{if}\,\mathbf{y}_{past}(t) + \mathbf{ih}\,\Delta\mathbf{u}_{past}(t)$$

with:

$$\mathbf{k}_u = \begin{bmatrix} \Delta u(t) \\ \vdots \\ \Delta u(t+N_u-1) \end{bmatrix}; \mathbf{u}_{past}(t) = \begin{bmatrix} u(t-1) \\ \vdots \\ u(t-n_b) \end{bmatrix}$$

$$\mathbf{y}_{past}(t) = \begin{bmatrix} y(t) \\ \vdots \\ y(t-n_a) \end{bmatrix}; \hat{\mathbf{y}} = \begin{bmatrix} \hat{y}(t+N_1) \\ \vdots \\ \hat{y}(t+N_2) \end{bmatrix}; \mathbf{w} = \begin{bmatrix} w(t+N_1) \\ \vdots \\ w(t+N_2) \end{bmatrix}$$

$$\mathbf{ih} = \begin{bmatrix} H_{N_1}(1)\cdots H_{N_1}(n_b-1) \\ \vdots \qquad \vdots \\ H_{N_2}(1)\cdots H_{N_2}(n_b-1) \end{bmatrix}; \mathbf{if} = \begin{bmatrix} F_{N_1}(1)\cdots F_{N_1}(n_a) \\ \vdots \qquad \vdots \\ F_{N_2}(1)\cdots F_{N_2}(n_a) \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} g_{N_1} & g_{N_1-1} \cdots & 0 \\ \vdots & \ddots \quad \vdots & \vdots \\ g_{N_2-1} & \cdots \quad \vdots & \vdots \\ g_{N_2} & \cdots & \cdots g_{N_2-N_u+1} \end{bmatrix}$$

In the unconstrained case, the optimum of J derived through analytical minimization is given by the relation $\mathbf{k}_u = -\mathbf{Q}^{-1}\mathbf{f}$. By applying the first control action $\mathbf{k}_u(1)$ of this optimal sequence and restarting the procedure, a control law with improved performances under a "two degrees of freedom" polynomial RST form is obtained (Boucher and Dumur, 1998). Such a formulation takes advantage of all the properties related to a closed loop control law as at each sampling instant it uses the new measured values of the plant output.

3 CONSTRAINED GPC

All these properties have to be reanalyzed when constraints are taken into consideration (Camacho, 1993). The design procedures most often have to consider specific types of constraints originated by amplitude limits in the control signal, slew rate limits of the actuator, limits on the output signal or equality constraints at the end of the prediction horizon for stability purposes.

3.1 Constraints formulation

Generally the formal mathematical description is

$$\begin{cases} -\Delta u_{\min} \le u(t+k) - u(t+k-1) \le \Delta u_{\max} \\ -u_{\min} \le u(t+k) \le u_{\max}, \quad 0 \le k \le N_u - 1 \\ -y_{\min} \le \hat{y}(t+k) \le y_{\max}, \quad N_1 \le k \le N_2 \\ \hat{y}(t+N_2+k) = w(t+N_2), \quad k = 1...m \end{cases} \quad (6)$$

These constraints on the control action and outputs can be restated in a form depending only on control updates. Further, this description could be translated in a matrix form like in (Ehrlinger, *et al.*, 1996)

$$\begin{cases} -\mathbf{M}(N_u,1)\,\Delta u_{\min} \le \mathbf{I}\,\mathbf{k}_u \le \mathbf{M}(N_u,1)\,\Delta u_{\max} \\ -\mathbf{M}(N_u,1)\,\underline{u} \le \mathbf{L}\,\mathbf{k}_u \le \mathbf{M}(N_u,1)\,\overline{u}, \\ \qquad \underline{u} = -u_{\min} - u(t-1),\ \overline{u} = u_{\max} - u(t-1) \\ -\mathbf{M}(N,1)\,y_{\min} \le \mathbf{G}\,\mathbf{k}_u + \mathbf{l} \le \mathbf{M}(N,1)\,y_{\max} \\ \mathbf{G}_c\mathbf{k}_u + \mathbf{l}_c = \mathbf{M}(m,1)\,w(t+N_2) \end{cases} \quad (7)$$

where $N = N_2 - N_1 + 1$, $\mathbf{M}(q,r)$ is a matrix of dimension $q\mathrm{x}r$ whose entries are one on the first column and zero for the others, $\mathbf{L}$ is a $N_u \mathrm{x} N_u$ lower triangular matrix whose entries are one. $\mathbf{G}_c$ and $\mathbf{l}_c$ describe the dynamics and the free response of the constrained system, both found as in (3), (5).

3.2 Feasibility

When minimizing the index J in (2) with respect to the constraints, the methods presented in the relaxed case cannot be applied since they do not provide a solution when the global optimum violates the constraints. In this case, practical GPC implementation is dealing with nonlinearities in the control law due to the entrance in the frontier hyperspace of the polyhedral domain defined by the set of constraints. These nonlinearities affect the controller expression that is usually found by solving on-line the quadratic program and applying then the receding horizon principle. However, the control law is affected irreversibly when on-line optimization returns infeasibility messages as in this case no pertinent control action can be applied.

Recalling the definition of the two types of infeasibility (Olaru and Dumur, 2003), type I is easy to analyze by a simple inspection of the optimization domain. It is not the same for the type II infeasibility as long as it depends on the set-point, which may conflict with the system dynamics and the other inequality constraints. Notice that there always exists a set-point which causes the infeasibility of the optimization for a system with a given set of constraints.

3.3 Necessary conditions

The following result concerns the degrees of freedom and the dynamics of the predicted output. However, it does not give any insight for the constrained domain point of view. To do that, a geometrical approach must be examined, which will be considered in Section 4. The optimization problem to be solved at instant t will be noted here $P(t, N, N_u, m, w(N+m))$. The argument for the solution to this problem will be noted $K(t, N, N_u, m, w(N+m))$.

The following proposition introduces a necessary condition for feasibility of a GPC law.

Proposition 1: If a GPC law is feasible at each instant $t > 0$, then the existence of all the following sequences is assured

$$K(0, N+k, N_u + k, m, w(N+m+k)), \forall k \ge 0 \quad (8)$$

Proof: GPC feasibility is equivalent with the feasibility of $P(t, N, N_u, m, w(N+m))$ for any t and as result with the existence of the optimal solution $K(t, N, N_u, m, w(N+m))$.

For $t = 0$, $P(0, N, N_u, m, w(N+m))$ is feasible and thus $K(0, N, N_u, m, w(N+m))$ exists which is (8) for $k = 0$. Assume the existence of the first $k-1$ $K(0, N+i, N_u + i, m, w(N+m+i))$, $i = 0 \cdots k-1$, solutions. Based on the optimization problem until time k, the following sequence can be built

$$\aleph = \{K(0, N, N_u, m, w(N+m))(1), \ldots \\ \ldots, K(k-1, N, N_u, m, w(N+m))(1)\}$$

which is the sequence of the first k GPC control actions. On the other hand at instant k, from the hypothesis the CGPC law defines a feasible optimization problem $P(k, N, N_u, m, w(N+m))$ with solution $K(k, N, N_u, m, w(N+m))$, adding it to the existing $\aleph$, a $N_u + k$ control sequence is obtained

$$\overline{\aleph} = \{\aleph, K(k, N, N_u, m, w(N+m))\} \quad (9)$$

Each element of this vector satisfies the operational constraints of $P(0, N+k, N_u + k, m, w(N+k+m))$ as long as its set of constraints is included in the union of all subsets upon which the elements of $\aleph$ were constructed. Now, as the final part of the sequence $\aleph$, $K(k, N, N_u, m, w(N+m))$ satisfies the end-point constraints for $P(k, N, N_u, m, w(N+m))$ being the same as for $P(0, N+k, N_u+k, m, w(N+k+m))$. $\aleph$ is thus a feasible solution to this problem and there exists $K(0, N+k, N_u+k, m, w(N+m+k))$. The result is completely proved by induction. ■

To illustrate how the result of *Prop. 1* can be exploited, consider a simple second order system

$$(1-q^{-1}+0.25q^{-2})y(t)=u(t) \qquad (10)$$

An unconstrained GPC law with horizons $N_1=1$, $N_2=4$, $N_u=2$ controls without problems the system for a pulse train set-point of magnitudes 0.3 and 1 (Figures 1a, b). Similar performances are available even if constraints on the input update $-0.1 \le \Delta u \le 0.1$ are introduced. Difficulties arise when endpoint constraints are added to the previous ones ($m=1$ in the case of Figures 1c, d). The second ascending front of the set-point requires an important control effort to satisfy the endpoint constraints and the control law in this constrained case becomes infeasible. The infeasibility only appears at the 12th iteration while from the necessary condition it was obvious that as long as $P(0, N+12, N_u+12, m, w(N+13))$ is infeasible, the CGPC law is infeasible. Figures 1e, f shows that the control law with only endpoint constraint is feasible.

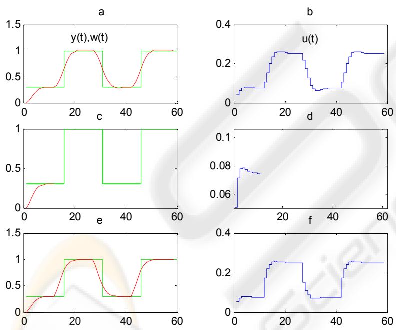

Figure 1: a, b) Output, setpoint and input for GPC c, d) output, setpoint and input for CGPC e, f) output, setpoint and input for CGPC with endpoint constraints only.

Generally, if for a set of GPC parameters there exists a k for which $P(0, N+k, N_u+k, m, w(N+k+m))$ is infeasible, then there exists a t such that $P(t, N, N_u, m, w(N+m))$ is infeasible. In practice, all (N, N_u, m) combinations with this property must be avoided. Thought these are useful principles, a finite time procedure checking the necessary conditions is not achievable in the general case. However, the necessary condition can be checked for some specific k values, which in the case of regular set-points might cover all the possible cases (see (Olaru and Dumur 2003) for a step set-point example).

Furthermore, *Prop. 1* considers necessary, but not sufficient, feasibility conditions. With the same previous system (10) with endpoint constraints, if the set-point is a pulse train of magnitudes 0.35 and -0.35, all the optimization problems (8) are feasible but the GPC law is infeasible.

4 GEOMETRICAL ANALYSIS

More complex results towards sufficient feasibility conditions based on invariant sets exist, which are too conservative. Consequently, an alternative approach in order to achieve some tractable necessary conditions considers a dual representation of the polyhedral domain coming from the constraints.

4.1 Constrained domain evolution

Trying to describe the feasibility domain for a system under all types of constraints, a compact form is deduced from (7)

$$\underbrace{\begin{bmatrix} \mathbf{F} \\ -\mathbf{F} \end{bmatrix}}_{\tilde{\mathbf{F}}} \boldsymbol{\theta}(t) \le \underbrace{\begin{bmatrix} \boldsymbol{\Gamma}_{max} \\ \boldsymbol{\Gamma}_{min} \end{bmatrix}}_{\boldsymbol{\Gamma}}; \ \boldsymbol{\Gamma} > \mathbf{0} \qquad (11)$$

with

$$\boldsymbol{\Gamma}_{min} = \begin{bmatrix} \mathbf{M}(N_u,1)\,\Delta u_{min} \\ \mathbf{M}(N_u,1)\,u_{min} \\ \mathbf{M}(N,1)\,y_{min} \\ \mathbf{M}(m,1)\,\varepsilon_{min} \end{bmatrix} \quad \boldsymbol{\Gamma}_{max} = \begin{bmatrix} \mathbf{M}(N_u,1)\,\Delta u_{max} \\ \mathbf{M}(N_u,1)\,u_{max} \\ \mathbf{M}(N,1)\,y_{max} \\ \mathbf{M}(m,1)\,\varepsilon_{max} \end{bmatrix}$$

$$\mathbf{F} = \left[\begin{array}{ccc|cc} 0 & 0 & \mathbf{I}_{N_u} & 0 & 0 \\ 0 & \mathbf{M}(N_u,n_b) & \mathbf{L} & 0 & 0 \\ \mathbf{if} & \Delta\,\mathbf{ih} & \mathbf{G} & 0 & 0 \\ \hline \mathbf{if}_c & \Delta\,\mathbf{ih}_c & \mathbf{G}_c & 0 - \mathbf{M}(m,n_w) & \end{array}\right]$$

$$\boldsymbol{\theta}(t) = \left[\mathbf{y}_{past}(t)\,\mathbf{u}_{past}(t)\,\mathbf{k}_u(t) \middle| \mathbf{w}(t)\,\mathbf{w}_c(t)\right]^T$$

where the epsilon machine will represent the bounds for equality constraints, n_w is the required number of past known values that are necessary to properly evaluate the future setpoint evolution.

A possible way of modeling (11) considers the dual representation of the inequalities in (7)

$$\begin{aligned} P = conv.hull\{x_1, \ldots, x_v\} + \\ + cone\{y_1, \ldots, y_r\} + lin.spaceP \end{aligned} \qquad (12)$$

where *conv.hullX* denotes the set of all convex combinations of points in *X*, *coneY* denotes nonnegative combinations of unidirectional rays and *lin.spaceP* represents a linear combination of bi-directional rays. It can be rewritten as

$$P=\sum_{i=1}^{v}\lambda_i x_i+\sum_{i=1}^{r}\gamma_i y_i+\sum_{i=1}^{l}\mu_i z_i$$
$$0\le\lambda_i\le 1,\ \sum_{i=1}^{v}\lambda_i=1,\gamma_i\ge 0\ ,\forall\mu_i \tag{13}$$

The complete procedure for finding the dual representation evaluates the system of constraints through Chernikova algorithm (Le Verge, 1992) implemented in libraries like POLYLIB.

Usually the polyhedral domain related with practical CGPC laws are in fact polytopes. These domains in a compact form can be analysed by their evolution, providing the dynamics of the constrained variables vector. This is the purpose of the next part.

Let us before examine the explicit linear controller structure. It comes from (5)

$$J=\mathbf{k}_u{}^{\mathrm{T}}\left(\mathbf{G}^{\mathrm{T}}\mathbf{G}+\lambda\mathbf{I}\right)\mathbf{k}_u+2\mathbf{k}_u{}^{\mathrm{T}}\mathbf{G}(\mathbf{l}-\mathbf{w})=$$
$$=\mathbf{k}_u{}^{\mathrm{T}}\Big(\underbrace{\mathbf{G}^{\mathrm{T}}\mathbf{G}+\lambda\mathbf{I}}_{\mathbf{H}}\Big)\mathbf{k}_u+2\mathbf{k}_u{}^{\mathrm{T}}\mathbf{G}\,\mathbf{E}\,\boldsymbol{\theta}^*(t) \tag{14}$$

where $\mathbf{E}$ is a matrix which allows the description of the vector $\mathbf{l}-\mathbf{w}=\mathbf{E}\boldsymbol{\theta}^*(t)$ when

$$\boldsymbol{\theta}^*(t+1)=\begin{bmatrix}\mathbf{y}_{past}(t+1)\\ \mathbf{u}_{past}(t+1)\\ \mathbf{w}(t+1)\\ \mathbf{w}_c(t+1)\end{bmatrix}=\boldsymbol{\Phi}^*\boldsymbol{\theta}(t)=$$
$$=\begin{bmatrix}\mathbf{D}_1(\mathbf{if})&\mathbf{D}_2(\mathbf{ih})&\mathbf{D}_3(\mathbf{G})&0&0\\0&\mathbf{I}_{dev}&\mathbf{D}_4&0&0\\0&0&0&\mathbf{M}_1&0\\0&0&0&\mathbf{M}_3&\mathbf{M}_4\end{bmatrix}\begin{bmatrix}\mathbf{y}_{past}(t)\\ \mathbf{u}_{past}(t)\\ \mathbf{k}_u(t)\\ \mathbf{w}(t)\\ \mathbf{w}_c(t)\end{bmatrix} \tag{15}$$

One can find the description for the vector $\mathbf{k}_u$ by minimizing this index *J* under the constraints

$$\overline{\mathbf{A}}\,\mathbf{k}_u\le\overline{\mathbf{b}}-\mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t-1) \tag{16}$$

where $\mathbf{K}$ is a matrix allowing the description of the affine part of the inequalities as a linear dependence on the context parameters $\boldsymbol{\theta}^*(t)$. The close form of the optimal control sequence for the CGPC is

$$\mathbf{k}_u{}^*=(\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{A}}_0\mathbf{H}^{-1}-\mathbf{H}^{-1})\mathbf{G}\mathbf{E}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t-1)$$
$$+\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\left(\overline{\mathbf{b}}-\mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t-1)\right)=$$
$$=\Big[\Big(\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{A}}_0\mathbf{H}^{-1}-\mathbf{H}^{-1}\Big)\mathbf{G}\mathbf{E}\boldsymbol{\Phi}^*$$
$$-\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\mathbf{K}\boldsymbol{\Phi}^*\Big]\boldsymbol{\theta}(t-1)+$$
$$+\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{b}}$$

with $\overline{\mathbf{A}}_0$ the matrix constructed by the subset of lines in $\mathbf{A}$ for whom the inequality constraints are saturated.

As a conclusion, the elaborated control law is affine in the parameter vector $\boldsymbol{\theta}(t-1)$. However, the difficulties arise from the fact that the matrix $\overline{\mathbf{A}}_0=\overline{\mathbf{A}}_0(\boldsymbol{\theta}(t-1))$ is not allowing an explicit dependence on the vector of parameters.

Remark: A parameterized polyhedron like the one in (16)

$$P=\left\{\mathbf{k}_u\,\middle|\,\overline{\mathbf{A}}\mathbf{k}_u\le\overline{\mathbf{b}}-\mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t-1)\right\}$$

has a dual representation where only the vertices are affected by the parameters

$$P=\left\{\begin{array}{l}\mathbf{k}_u\,\middle|\,\mathbf{k}_u=\sum_{i=1}^{v}\lambda_i x_i(\boldsymbol{\theta}^*)+\sum_{i=1}^{r}\gamma_i y_i+\sum_{i=1}^{l}\mu_i z_i\\ \lambda_i\ge 0,\ \sum_{i=1}^{v}\lambda_i=1,\gamma_i\ge 0\ ,\forall\mu_i\end{array}\right\}$$

4.2 Necessary conditions by means of extremal point feasibility

Considering the polyhedral domain as described earlier, with the dual representation by the vertices, it can be interesting to look at the evolution of these vertices at each sampling time.

Proposition 2: The optimal control sequence corresponding to all extremal combinations of context parameters must lead to points inside the projection of the initial polyhedral domain for a feasible CGPC law.

Proof sketch: As explained earlier, the constraints on the CGPC law define a polyhedral domain, given in the case of a polytope by

$$D=\left\{\mathbf{k}_u\,\middle|\,\mathbf{k}_u=\sum_{i=1}^{v}\lambda_i k_{u\,i}(\boldsymbol{\theta}^*);\lambda_i\ge 0,\ \sum_{i=1}^{v}\lambda_i=1\right\} \tag{17}$$

By considering the involved system variables as parameters, this parameterized polyhedron can be extended to a fixed one of higher dimension. Further, a corresponding representation as a generators combination may be found. In the case of a polytope, this will be

$$P = \left\{ \boldsymbol{\theta} \middle| \; \boldsymbol{\theta} = \sum_{i=1}^{v} \lambda_i \boldsymbol{\theta}_i ; \lambda_i \geq 0, \; \sum_{i=1}^{v} \lambda_i = 1 \right\} \qquad (18)$$

The existence of these vertices does not guaranty the fact that the CGPC law will have the opportunity to reach each of them. Multiple vertices may correspond to the same context parameters. Thus, a useful manipulation may be the orthogonal projection of this domain on the subspace of the context parameters (as in Figure 2).

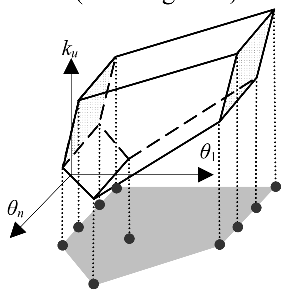

Figure 2: The polyhedral domain and its projection on the context parameters subspace.

This operation can be done explicitly by multiplying each vertex θ_i by a matrix $\lfloor e_j \rfloor$ where j are the indices of the context parameters in the vector $\boldsymbol{\theta}$. The resulting set is P^*, the convex combination of the points

$$P^* = \left\{ \boldsymbol{\theta}^* \middle| \; \boldsymbol{\theta}^* = \lfloor e_j \rfloor \theta_i \right\} \qquad (19)$$

Once the projection available, a redundancy check must be operated in order to obtain the minimal set of generators.

The resulting domain P^* can provide by its vertices the extremal points for the context parameters that can further be used for figuring the whole domain. Solving the parameterized quadratic problem related to CGPC, one can retrieve a hyper surface inside the original polyhedral domain D .

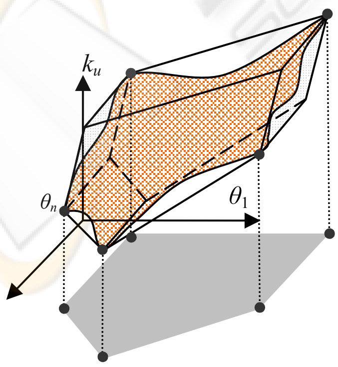

Figure 3: The optimal solution of CGPC for each possible context parameters combination.

The elaboration of this shape enables to solve all the analysis problems as it defines the whole behavior of the CGPC law. However, this is not a trivial task although systematic results exist at least for the MPC with state space models. The investigated case is slightly different as long as it incorporates also a model of the reference, even if the optimization problem is still a part of the quadratic multi-parametric programs.

As far as the evolution of the context parameters domain is concerned, the image of the points on the CGPC shape must be found by the linear transformation (15). If this domain is denoted as P_+^*, the necessary and sufficient conditions for feasibility are resumed by the relation

$$P^* \supset P_+^* \qquad (20)$$

Due to limitations in the knowledge on the topology of the CGPC shape, this will resume on necessary conditions based on the extremal points. These necessary conditions may be expressed as in Figure 4 by a set of inequalities

$$\boldsymbol{\theta}^*(t+1) \in P_+^* \qquad (21)$$

which resumes the proposition. ■

In practice this condition seems to be quite general and covers with sufficiency all the special cases verified by the authors. An analytical proof of the fact that the extremal points of the CGPC shape corresponding with the extended polyhedron vertices will have as image the vertices of the domain P_+^* can not yet be obtained.

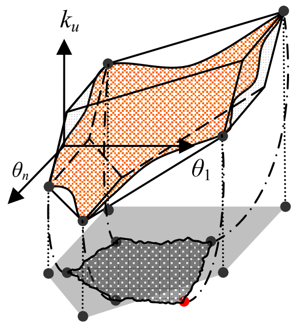

Figure 4: The evolution of the extremal points of context parameters domain.

For a complete analysis of the CGPC law, all the points inside the polyhedral domain P_+^* have to be checked in order to confirm the feasibility. This is not an obvious task as long as the optimal control sequence is affine in the context parameters, and the affine part even if linear in the parameter vector is changing the linear dependence in concordance with the active set of constraints. It is clear that the number of active constraints is maximal for the vertices and is subsequently decreasing for the points on the frontier where subsets of these sets of constraints are active.

Following the same line as the proof, an algorithm based on tools of polyhedral computations and quadratic optimization can be designed in order to validate these necessary conditions. Such an algorithm can be resumed by the following steps.

Algorithm 1

1. Compute the vertices of the polyhedral set by dual representation of the constraints
2. Project the polytope on the parameters subspace
3. Remove the redundant points
4. Compute the close form of the control law in all the vertices of the constrained domain. (Compulsatory as it is not always equal with the value in the original polyhedron)
5. For each such law, construct the evolution matrix and compute the corresponding next step parameters $\boldsymbol{\theta}(t+1)$
6. Check if for each such point $\boldsymbol{\theta}(t+1)$, its projection is inside the projected polyhedron found at step 2. If it is not the case, that means that there exists at least one point in the constrained domain which, if reached, will lead to infeasibility. The necessary conditions are thus not accomplished.

5 EXAMPLE

A simple constrained generalized predictive control is examined in order to illustrate the analysis technique procedure proposed in the previous section. Consider in the following a second order linear system as the one reported in (Olaru and Dumur, 2003), with non-minimum phase characteristics

$$(1-q^{-1}+0.25q^{-2})y(t)= \\ (-0.25-0.25q^{-1}+0.75q^{-2})u(t) \qquad (22)$$

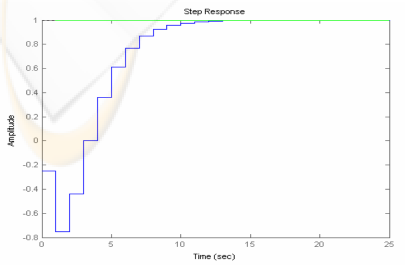

Figure 5: Open loop step response.

The step response of this system is given in Figure 5. For CGPC law with $N_1=1$, $N_2=4$, $N_u=2$, the system proves to have an infeasible behavior for step setpoints and constraints on the output of magnitude $-1\le y\le 1$, based on snow-ball attitude (Scokaert and Clarke, 1994a) (Figure 6).

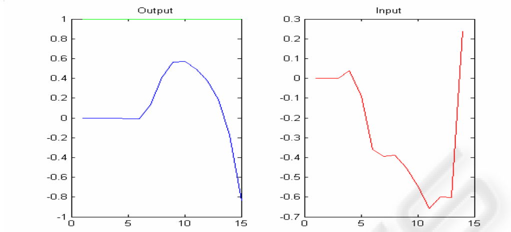

Figure 6: CGPC closed loop behavior.

Proceeding as explained in ***Algorithm 1***, the constrained domain can be described as

$$D=\left\{\mathbf{k}_u \middle| \mathbf{1}\underline{y}\le \mathbf{G}\,\mathbf{k}_u+\mathbf{l}\le \mathbf{1}\overline{y}\right\} \qquad (23)$$

where $\mathbf{l}$ is like in (3) and

$$\mathbf{G}=\begin{bmatrix}-0.25 & 0\\ -0.75 & -0.25\\ -0.40 & -0.80\\ 0 & -0.40\end{bmatrix} \qquad (24)$$

The elaboration of the extended polyhedron requires the definition of

$$P=\{-\boldsymbol{\Gamma}_{\min}\le \mathbf{F}\ \boldsymbol{\theta}(t)\le \boldsymbol{\Gamma}_{\max}\quad \boldsymbol{\Gamma}_{\min},\boldsymbol{\Gamma}_{\max}\ge \mathbf{0}\} \ (25)$$

with: $\boldsymbol{\Gamma}_{\min}{}^{T}=\mathbf{1}^{T}\underline{y}=\mathbf{1}^{T}$; $\boldsymbol{\Gamma}_{\max}{}^{T}=\mathbf{1}^{T}\overline{y}=\mathbf{1}^{T}$

$$\mathbf{F}=\begin{bmatrix}2 & -1.25 & 0.25 & -0.25 & 0.75 & -0.25 & 0\\ 2.75 & -2.25 & 0.5 & 0.25 & 1.5 & -0.75 & -0.25\\ 3.2 & -3 & 0.7 & 0.8 & 2 & -0.4 & -0.8\\ 3.6 & -3.4 & -0.8 & -1.2 & -2.4 & 0 & 0.4\end{bmatrix}$$

$$\boldsymbol{\theta}(t)=\left[\mathbf{y}_{past}(t)\ \ \Delta\mathbf{u}_{past}(t)\ \ \mathbf{k}_u(t)\right]^{T}$$

As the context parameters include the past outputs, three implicit constraints have been added as an upper part of $\mathbf{F}$ in order to avoid the analysis of non-reachable regions. The result is a square matrix of constraints describing a polytope with a dual representation containing 128 vertices

θ_1= [148 -148 -148 -59 -467 -158 -315]/148
θ_2= [-148 148 -148 -459 689 -138 241]/148
θ_3= [148 148 148 -309 15 354 -97]/148
…
θ_{128}= [-148 -148 148 -357 207 90 23]/148

Figure 7: Convex hull for D computed by POLYLIB.

Now the projection on the subspace of the first five variables leads to a domain P^* that can be reduced by removing redundant pairs to the convex hull of 64 vertices like in Figure 8.

θ_1= [-148 -148 148 335 -193]/148
θ_2= [148 148 -148 677 -303]/148
θ_3=[148 -148 -148 793 -767]/148
…
θ_{64}= [-148 148 148 -793 767]/148

Figure 8: Convex hull for P^* computed by POLYLIB.

Now the corresponding quadratic problems have to be solved in order to find the optimal control law in each such extreme context. The next step aims at computing the image of the resulting extended vectors $\boldsymbol{\theta}_{1..64}$ by the linear transformation

$$\boldsymbol{\theta}^*(t+1) = \begin{bmatrix} \mathbf{y}_{past}(t+1) \\ \Delta\mathbf{u}_{past}(t+1) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1.25 & 0.25 & -0.25 & 0.75 & -0.25 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{past}(t) \\ \Delta\mathbf{u}_{past}(t) \\ \mathbf{k}_u(t) \end{bmatrix}$$

Checking their membership inside D ends the algorithm. In the studied case, there are 32 vertices which are positioned outside the feasible context polyhedron P^*. This means that there are at least 32 combinations of past inputs and outputs for which there is no feasible control sequence able to retain the system inside the constraints

$$-1 \le y \le 1$$

Thus as the necessary conditions are not fulfilled, the overall CGPC law is infeasible.

6 CONCLUSION

This paper presented some possible approaches for the off-line analysis of the feasibility in the case of a constrained generalized predictive control strategy. The advantages of these kinds of analysis consist in the set-point dependent procedure that may prove to be useful in the decisions of tuning predictive control law parameters.

However, a gap between the necessary and sufficient conditions for off-line feasibility of CGPC exists as long as the dependence of affine linear control law corresponding to the saturated constraints as functions of the context parameters can not be explicitly computed.

REFERENCES

Bemporad, A., Morari, M., Dua, V. and Pistikopoulos, E., 2002. The explicit linear quadratic regulator for constrained systems, *Automatica, Volume No 38, pp.3-20.*

Camacho, E.F., 1993. Constrained Generalized Predictive Control, *IEEE Transactions on Automatic Control, Volume No 38-2, pp.327-331.*

Clarke, D.W., Mohtadi C., and Tuffs, P.S., 1987. Generalised Predictive Control: Part I: The Basic Algorithm, Part II: Extensions and Interpretation, *Automatica, Volume No 23-2, pp.137-160.*

Dumur D. and Boucher, P., 1998. A Review Introduction to Linear GPC and Applications, *Journal A, Volume No 39-4, pp.21-35.*

Ehrlinger, A., Boucher, P. and Dumur, D., 1996. Unified Approach of Equality and Inequality Constraints in G.P.C.. *5th IEEE Conference on Control Applications, pp.893-899, Dearborn, September.*

Kouvaritakis, B., Cannon, M. and Rossiter, J.A., 2000. Stability, feasibility, optimality and the degrees of freedom in constrained predictive control, in *Nonlinear Model Predictive Control, F. Allgower and A. Zheng (eds.), Progress in Systems and Control Theory Series, Volume No 26, pp.403-417, Birkhauser Verlag, Bâle.*

Le Verge, H., 1992. A note on Chernikova's Algorithm, *Technical Report 635, IRISA-Rennes, France.*

Maciejowski, J., 2002. Predictive Control with Constraints, *Prentice Hall.*

Olaru, S., and Dumur, D., 2003. Feasibility analysis of constrained predictive control, *14th Conference on Control Systems and Computer Sciences, pp.164-169, Bucharest, July 2003.*

Rossiter, J.A., 2003. Model based predictive control – a practical approach, *CRC Press.*

Scokaert, P. and Clarke, D.W., 1994a. Stability and feasibility in constrained predictive control, In: *Advances in model-based predictive control, pp.217-230. Oxford University Press.*

Scokaert, P. and Clarke, D.W., 1994b. Stabilising properties of constrained predictive control, *IEE Proceedings Control Theory Applications, Volume No 141-5, pp.295-304, September.*

Seron, M.M., Goodwin, G.C. and De Dona, J.A., 2003. Characterisation of Receding Horizon Control for Constrained Linear Systems, *Asian Journal of Control, Volume No 5-2, pp. 271-286.*

Wilde, D.K., 1993. A library for doing polyhedral operations, *Technical report 785, IRISA-Rennes, France.*