PARAMETRIC ESTIMATION OF SINUSOIDS IN NOISE

A comparison between parametric approaches and the definition of a regularized Smyth algorithm

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Abstract: A comparison between well-established parametric algorithms and the more recent Smyth algorithm for estimation of sinusoidal signals in white noise is presented. The comparison is performed through a pseudo-Monte Carlo analysis on simulated data. The results obtained show that Smyth algorithm has a slightly better performance at large Signal-to Noise Ratios. However, when the SNR drops down, the performance of the Smyth algorithm dramatically decreases. A better performance with respect to both ESPRIT and Smyth algorithms at low SNR can be obtained by a regularized filtering procedure on the data.

1 INTRODUCTION

The detection of sinusoidal signals embedded in noise is a classic problem, of wide interest in numerous applications, from sonar to radar and tracking problems. The available algorithms can be classified as parametric and non parametric methods. The non parametric methods are based on classical periodogram analysis (Stoica and Moses, 1997). Parametric methods include Auto Regressive (AR) – based algorithms and high resolution eigenanalysis approaches.

Popular eigenanalysis algorithms include MUSIC (Schmidt, 1983), ESPRIT (Roy and Kailath, 1989) and the classic Pisarenko algorithm (Pisarenko, 1973). More recently, Smyth has proposed a novel eigenanalysis algorithm by combining a constrained Pisarenko approach with an iterative least square estimator (Smyth, 2000).

The purpose of this paper is to compare the above mentioned algorithms at different Signal to Noise Ratios (SNR), in order to evaluate pros and cons of the various approaches. The comparison is done through a pseudo Monte Carlo method on simulated data. The results obtained show that the Smyth algorithm has a slightly better performance with respect to the other methods at high SNR. At low SNR, with specific values depending on the number of different sinusoids to be estimated, MUSIC and ESPRIT seems to yield the best performance.

The comparison reported is in itself interesting and valuable, since to the Authors knowledge no comparison of the traditional methods with the Smyth approach has appeared in the literature. In addition to that, the above results have also led to the definition of a novel estimation algorithm. The novel approach is essentially an iterated application of the Smyth algorithm to a succession of regularized data set. The regularization is obtained as a weighted sum of the predicted, noise-free data from the previous step solution with the noise-corrupted measurements. The data weighting depends on a regularization parameter which is changed from step to step in order to reach, in a finite number of steps, the situation in which the algorithm uses only the data measurements. The initialization of this filtering procedure is done with the predicted, noise free solution obtained from ESPRIT. By using the simulative approach which has been employed for the algorithms comparison, it is shown that there does exist an optimal value of the regularization parameter such that for this value the proposed algorithm has performance better than both ESPRIT and Smyth algorithms.

The paper is organized as follows. In the next section, a formal statement of the problem is given and the Smyth algorithm is briefly reviewed. In section 3 the simulative trials and the results obtained by comparing the performance of ESPRIT, MUSIC, Pisarenko and Smyth algorithms are reported. In section 4 the novel regularization algorithm is presented, and the existence of an

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optimal continuation parameter is enlightened. In section 5, some comments on the computational burden of the various approaches are given, and future efforts in order to determine algorithmically the optimal parameter are briefly described. Finally, some conclusions are given.

2 THE SMYTH PROBLEM AND THE SMYTH ALGORITHM

It is supposed to have available the measurements:

$$y(k) = \sum_{j}^{n} s_j(k) + w(k) \ , \qquad k = 1,...,K \qquad (1)$$

where $s_j(k), j = 1,...,n$ are the samples of n sinusoidal signals, each at the frequency $\omega_j, \omega_j \neq \omega_i$ if $i \neq j$, and $w(k)$ is a finite power white process uncorrelated with the sinusoidal signals. Given the measurement of K samples of the output signal $y(k)$, and given the knowledge of the number n of frequencies present in the measurement, the problem is to obtain an estimate of the frequencies $\omega_j, j = 1,...,n$.

The following estimation procedure has been introduced by Smyth (Smyth, 2000). The procedure is composed of three steps. In the first one, a constrained Pisarenko algorithm is applied. The estimate thus obtained is used as starting point of the second step, in which the Osborne-Bresler-Makovsky (OBM) estimation algorithm is applied (Bresler and Makovsky, 1986). Finally, the results of the second step are used to initialize a least square iterative estimation (Osborne, 1975). The idea behind the Smyth approach is that the three algorithms, Pisarenko, OBM and Osborne, are applied in order of increased sensitivity to the initialization. So, the least sensitive to the initial condition is applied as first, while the others are used to progressively refine the solution.

Formally, let Y be the auto correlation matrix of the measurement signal $y(k)$; let $\mathbf{c}$ be the eigenvector associated to the smallest eigenvalue of Y, and let:

$$c(z) = c_0 + c_1 z + ... c_{2n} z^{2n}$$

be the annihilator polynomial whose coefficients are given by the components of $\mathbf{c}$. By observing that $c(z)$ and $c(z^{-1})$ must have the same roots, Smyth has introduced a constrained Pisarenko estimate for the eigenvector $\mathbf{c}$ in the following iterative form: at each iteration h it must be solved the constrained least square problem:

$$\min_{\mathbf{c}^T D \mathbf{c} = 1} \mathbf{c}^{(h+1)^T} Y(c^{(h)}) \mathbf{c}^{(h+1)} \qquad (2)$$

where D is a matrix imposing the root constraint. Equation (2) is successively solved by Pisarenko, OBM and Osborne least-square algorithms, using as starting point the solution obtained in the previous step.

3 SIMULATIVE EVALUATION OF THE SMYTH ALGORITHM

In this section a pseudo Monte Carlo simulative study is presented, reporting the performance obtained by the Smyth algorithm as compared with the standard Pisarenko, MUSIC and ESPRIT algorithms. Results from a non-parametric, FFT-based, estimation algorithm are also reported. Three cases are considered. In the first, the signal is assumed to be composed by a single sinusoid, in the second by two and in the third by four sinusoids, all added up with random phases. Gaussian white noise, is added to the signal, with varying Signal-to-Noise Ratio (SNR), from –20 up to 40 dB, at 2.5 dB steps. All simulations have been carried out in Matlab environment, version 6.5. In each trial, performances are obtained as follows. For each algorithm, and for each SNR the signal $y(k)$ is generated as the sum of the predefined number of tones plus noise. Then all the algorithms are applied to $y(k)$. Such a procedure is repeated N = 100 times. The estimated Mean Square Error from each algorithm is computed as follows:

$$MSE(\omega_j) = \frac{1}{N} \sum_{i=1}^{N} \left(\omega_j - \hat{\omega}^{(i)}{}_j \right)^2 \qquad (3)$$

$$MSE = \sum_{j=1}^{n} MSE(\omega_j) \qquad (4)$$

where $\hat{\omega}^{(i)}{}_j$ is the estimate of the *j*-th frequency in the *i-th* run.

In the figures reported in the following, the value in dB of 1/MSE is plotted as a function of the SNR, so that for each SNR, the higher the y-scale value, the better is the performance for each algorithm. Numerically, a normalized sampling time of 1 second has been used, with a window of K=300 samples. The chosen frequencies have been taken as ω_1 = 0.025 Hz, ω_2 = 0.062 Hz, ω_3 = 0.1021 Hz, ω_4 = 0.2848 Hz.

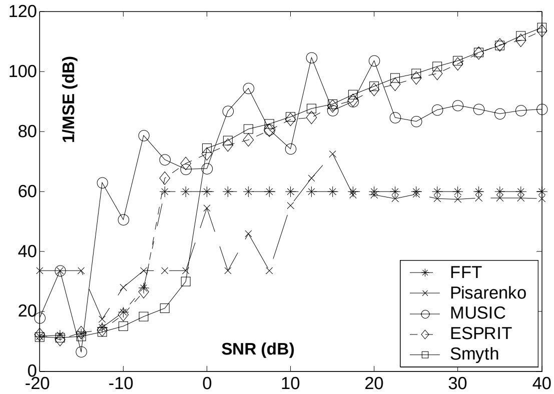

Figure 1: Performance results from pseudo Monte Carlo analysis for the estimation of a single frequency. dash-dot line: FFT; dashed, with crosses: Pisarenko; continuous line, with circles: MUSIC; dotted line, with diamonds: ESPRIT; continuous line with squares: Smyth.

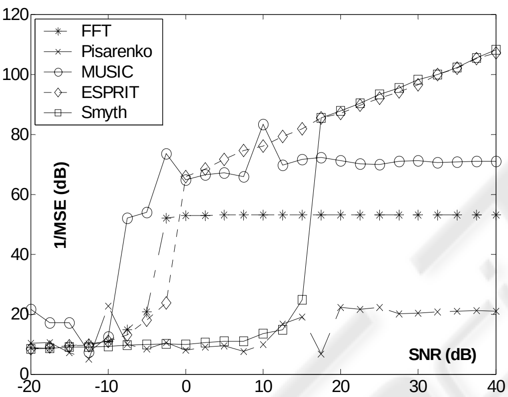

Figure 2: Performance results from pseudo Monte Carlo analysis for the simultaneous estimation of two frequencies; dash-dot line: FFT; dashed, with crosses: Pisarenko; continuous line, with circles: MUSIC; dotted line, with diamonds: ESPRIT; continuous line with squares: Smyth.

In figures 1, 2 and 3 the results obtained in the estimation of one, two and four sinusoids respectively are reported.

The results reported show that, at high SNR, the Smyth approach can lead to some slight improvement (few dB at the best) with respect to ESPRIT, which has the best performance among the standard methods. At low SNR, the performance of Smyth algorithm drops down quite dramatically; moreover, at low SNR MUSIC becomes competitive with ESPRIT, succeeding in obtaining the best performance in some cases. The behaviour of MUSIC as a function of the SNR, however, is less predictable with respect to that of the other methods, and of ESPRIT in particular, showing weak correlation among the algorithm performance at varying SNR.

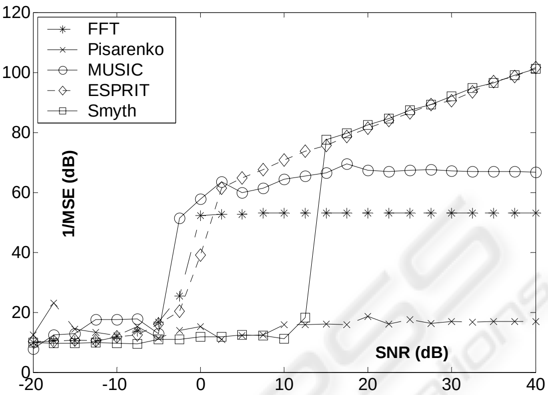

Figure 3: Performance results from pseudo Monte Carlo analysis for the simultaneous estimation of four frequencies. dash-dot line: FFT; dashed, with crosses: Pisarenko; continuous line, with circles: MUSIC; dotted line, with diamonds: ESPRIT; continuous line with squares: Smyth.

4 SMYTH ESTIMATION WITH REGULARIZED DATA

The results presented in the previous section have led to the exploration of a novel procedure in which the Smyth algorithm is applied to a filtered, regularized version of the data. The starting motivation was to increase the performance of the Smyth algorithm by increasing the SNR by a process-guided data filtering. As a result, as it will be shown in the following, a consistent improving in the performance can be obtained. The proposed data regularization procedure is the following: the filtered data are obtained as a weighted sum of the predicted, noise free data obtained from an estimated solution (by whatever method) and the noise-corrupted measurements. In order to generate the noise free predicted data $\hat{y}(k)$, one has to start from the estimated frequencies; keeping fixed these frequencies it is possible, through standard Fourier analysis, to estimate also the phase and amplitude of these frequency components. This estimate leads to the estimate $\hat{y}(k)$ of the time series (1) without the noise components. The regularized data set is then obtained as:

$$y_a(k) = \alpha y(k) + (1-\alpha)\hat{y}(k) \quad (5)$$

where $0 \le \alpha \le 1$ is the regularization parameter, trading off between measurement fidelity and confidence in the estimate. When $\alpha = 0$ the filtered data are equal to the prediction (complete confidence in the estimate), when $\alpha = 1$ no filtering is applied to the measurement.

In applying the Smyth algorithm to the filtered data of equation (5) one has to select the regularization parameter. In order to investigate the algorithm performance as a function of the regularized parameter, the following procedure has been applied: the regularization parameter α is made varying from 0 to 1 in $\overline{h}+1$ steps:

$$\alpha_0 = 0, \quad \alpha_h = \alpha_{h-1} + \frac{1}{\overline{h}}, h = 1,\ldots,\overline{h}, \quad \alpha_{\overline{h}} = 1$$

The noise free predicted data $\hat{y}_{\alpha_{h-1}}(k)$ from the estimate obtained with parameter α_{h-1} are used to generate the filtered data with parameter α_h, accordingly to the equation:

$$y_{a_h}(k) = \alpha_h y(k) + (1-\alpha_h)\hat{y}_{a_{h-1}}(k) \qquad (6)$$

The Smyth algorithm is applied to this data; then its results are used to compute a new data prediction, and the procedure is iterated. The Smyth algorithm is also intialized with the solution obtained at the previous step. Substantially, the algorithm is successively applied to convex combinations of noise-free data estimates obtained from the previous step estimation and measurements The procedure is initialized at step 0 with the ESPRIT algorithm, whose data prediction is employed to compute the regularized data with $\alpha = 0$. The same procedure described in the previous section has then been applied to evaluate the performance of the Smyth algorithm as a function of the regularization parameter. The results presented report the value in dB of 1/MSE (equation (4)) against the value of α from 0 to 1, at 0.1 steps. The maximum corresponds to the optimal regularization parameter. The two limit cases corresponds to the independent application of ESPRIT ($\alpha = 0$) and Smyth ($\alpha = 1$) algorithms. A sample of the results obtained are reported in Figures 4, 5, 6: the figures refer to the estimation of two sinusoids, at SNR of –5, -10, -20 dB respectively. The results reported show the existence of an optimal value of the regularization parameter; the Smyth algorithm, applied with the initialization process described and with the optimal regularization parameter, has a performance at low SNR which is better than that of both the "pure Smyth" algorithm and ESPRIT. The average performance gain with respect to ESPRIT of the regularization approach ranges from the 8 dB of the –10 dB SNR case (Figure 5) to the 3 dB of the –20 dB SNR case (Figure 6). Note that the performance gain with respect to the "pure Smyth" algorithm can be much larger (up to 45 dB, Figure 4). This general behaviour is systematic in all the simulations test performed, with sometimes even larger performance gains.

It is natural to ask what is the optimal regularization parameter at high SNR. A typical behaviour is shown in Figure 7, obtained in the case of estimation of a single tone with SNR of 30 dB: the optimal value is 1, indicating that the best performance is given by the unregularized Smyth algorithm.

It needs to be observed, though, that the performance curve as a function of the regularization parameter as determined through the pseudo Monte Carlo approach does not exhibit any specific behaviour that can be further exploited. In particular, Figures 4-6 has been purposely chosen to illustrate the several different behaviours observed. Figure 4 illustrates the situation in which the experimental curve exhibits a unique maximum, with monotone behaviour of the performance curve before and after the extremal point; Figure 5 shows the presence of multiple maxima and minima; Figure 6 presents a case in which, though there does exist a unique maximum, there is also a minimum within the regularization interval, and the performance function does not have a monotone behaviour before the reaching of the maximum. Moreover, for some values of the regularization parameter, the performance is worse with respect to both ESPRIT and unregularized Smyth. This diversity in the performance curve behaviour may represent a serious obstacle to an efficient implementation of a computational scheme aimed at exploiting the better performance of the regularized approach. Some guidelines and critical evaluations toward this goal are reported in the next section.

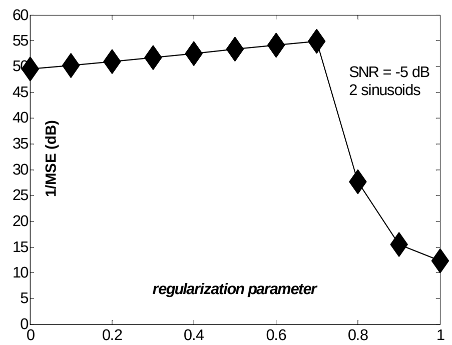

Figure 4: Estimation results obtained from the Smyth algorithm and the regularization procedure as a function of the regularization parameter; SNR: -5 dB.

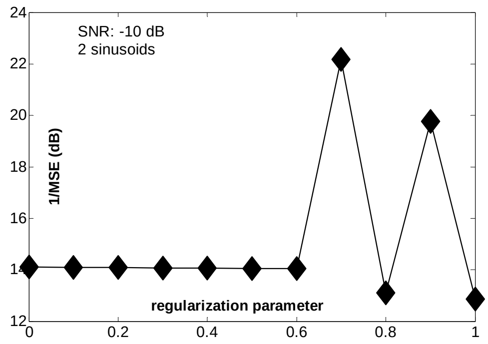

Figure 5: Estimation results obtained from the Smyth algorithm and the regularization procedure as a function of the regularization parameter: SNR: -10 dB.

5 REGULARIZED ALGORITHM: COMPUTATIONAL ASPECTS

The results reported in section 4 seems promising: they establish the existence of a regularized Smyth procedure through which a performance gain of several dB with respect to ESPRIT can be obtained by the proper selection of the regularization parameter. This may lead to a substantial improvement of the performance curves (Figures 1-3) at low SNR, where the drop in performance had been observed. On the other hand, there are some non trivial computational aspects to be discusses before the regularization approach can be successfully implemented in a real life situation.

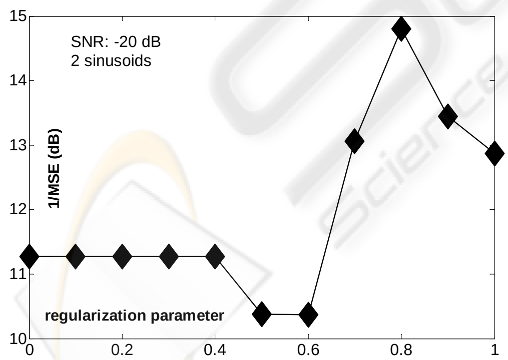

Figure 6: Estimation results obtained from the Smyth algorithm and the regularization procedure as a function of the regularization parameter: SNR: -20 dB

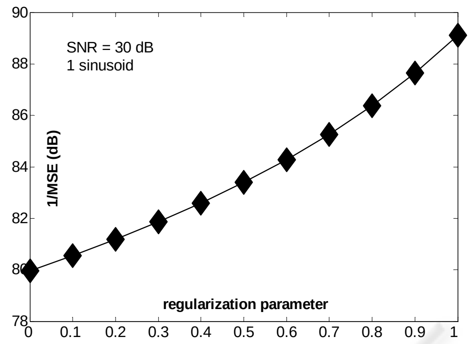

Figure 7: Estimation results obtained from the Smyth algorithm and the regularization procedure as a function of the regularization parameter: SNR: 30 dB.

In particular, the performance curves of Figures 4-6 have been determined with the knowledge of the true solution, which of course is never available in practice. A practical implementation may consist in repeatedly applying the algorithm varying the regularization parameter over the whole admissible range, and looking for the best performance by comparison of the predicted time series with the measured time domain data. How this is best accomplished (by direct residual minimization, by whiteness test on the residual time series, etc.) it is not clear at the moment. Some preliminary tests with the residual mean square error show both light and shadows: in Figure 8 the time domain MSE is plotted as a function of the regularization parameter (as before, the plot refers to the quantity 1/MSE in dB). Figure 8 correspond to the case reported in Figure 6: -20 dB SNR and 2 unknown sinusoids. The performance curve measured on the residual in time domain has the maximum in correspondence of the same regularization parameter as determined in the frequency domain with the knowledge of the true solution; however, the two performance curves do not have the same behaviour, so that a systematic exploitation of the time domain curve may also lead to inaccuracies or undetected mistakes. Additional considerations are related to the computational burden of the Smyth approach, either in its "pure" or regularized version, with respect to the attainable performance gain. In Table 1 the mean computation time of ESPRIT and of the Smyth algorithm, taken over the 100 simulations described in Section 3, are reported. The computation time refers to our Matlab implementation running on a Pentium 4 PC, 2.4 GHz clock, under Windows XP operating system.

Table 1: comparison between ESPRIT and Smyth algorithm computation times

	Mean computation time (ms)		
	1 freq.	2 freq.	4 freq.
ESPRIT	9.9	10.2	8.5
SMYTH	41.6	77.4	105.4

The regularized Smyth algorithm consists of the successive application of several Smyth algorithms, with in addition the computational effort of computing the new data set through Equation (6). So there may as well be a factor of 100 in the computational speed in favour of ESPRIT.

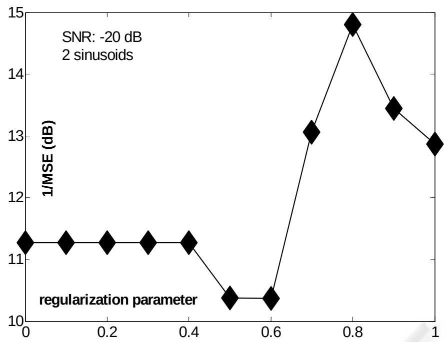

Figure 8: Performance results computed from the time domain residual as a function of the regularization parameter: SNR: -20 dB

6 CONCLUSIONS

The paper has concentrated on two main topics: a pseudo Monte Carlo evaluation of standard parametric algorithms with respect to the Smyth algorithm; an investigation of an innovative regularization technique to improve the performance of the Smyth algorithm at low SNR. The evaluation study has shown that the Smyth algorithm can lead to a slightly better performance with respect to ESPRIT (the best among the standard parametric algorithms tested) at high SNR. The Smyth performance drops down dramatically at low SNR, with a SNR threshold that is higher the greater the number of sinusoids to be estimated. It has been shown that, with the regularization approach introduced, the Smyth algorithm is capable to improve consistently its performance at low SNR, with respect to its own performance, and also with respect to ESPRIT. In particular, performance gains up to 10 dB have been obtained. The results reported are an experimental verification of the existence of an optimal regularization parameter. Steps toward the definition of an algorithm for the search of the optimal parameter have been reported, as well as the computational issues to be taken into account.

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