ESTIMATION ALGORITHM FROM RANDOMLY DELAYED OBSERVATIONS WITH WHITE PLUS COLOURED NOISES

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Abstract: A recursive algorithm for the least-squares linear one-stage prediction and filtering problems of discrete-time signals using randomly delayed measurements perturbed by additive white plus coloured noises are presented. It is assumed that the autocovariance function of the signal and the coloured noise are expressed in a semi-degenerate kernel form and the delay is modelled by a sequence of independent Bernoulli random variables, which indicate if the measurements arrive in time or are delayed by one sampling time. The estimators are obtained by an innovation approach and do not use the state-space model of the signal, but only the covariance information about the signal and the observation noises and the delay probabilities.

1 INTRODUCTION

There are many situations, such as the ones relative to telecommunication scope, in which it is possible that the measurements available to estimate the state of a system may not arrive in time, but delayed by a any sampling time. Although sometimes these delays have been treated as measurement errors or as deterministic functions of the time, these assumptions are not always accurate and, in these cases, the best way to model the delay is to interpret it as a stochastic process, including its statistical properties in the system model.

Many recent works have used stochastic time-delay models to treat estimation problems. For example, the state estimation in a model with randomly varying sensor delays has been described as a estimation problem in systems with stochastic parameters (Yaz and Ray, 1998). Also, the state estimation has been treated in the case where a finite-state Markov chain is applied to model the random delay in the observations (Evans and Krishnamurthy, 1999).

The above studies consider that the state-space generating the signal is known but, in many situations, it is not available and estimation algorithms using another kind of information, such as covariance one, must be used. In (Nakamori et. al, 2004b), the least-squares linear filtering and fixed-point smoothing problems from measurements with stochastic delays, perturbed by white noise, is treated by using covariance information.

In this paper, we treat the least-squares linear prediction and filtering problems of signals using randomly delayed measurements which are perturbed by additive white plus coloured noises. The delay is modelled by a binary white noise, whose values, zero or one, indicate if the measurements arrive in time or are delayed by one sampling period.

This study also generalizes the work (Nakamori et. al, 2004a), which consider uncertain observations affected by additive white plus coloured noises without delay in time.

The estimators are obtained without requiring the state-space model generating the signal, but just using the covariance functions of the signal and the noises, assuming a semi-degenerate kernel form for the signal and coloured noise autocovariance functions, and the delay probabilities. Finally, the effectiveness of the proposed algorithms is shown in a computer simulation example.

2 PROBLEM FORMULATION

We consider the estimation problem of a $n \times 1$ signal z_k from delayed observations described by

$$\begin{aligned} \widetilde{y}_k &= z_k + v_k + w_k, \quad k \geq 0, \\ y_k &= (1-\gamma_k)\widetilde{y}_k + \gamma_k \widetilde{y}_{k-1}, \quad k \geq 1. \end{aligned}$$

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The following hypotheses are assumed:
H1. The signal process $\{z_k;\ k\geq 0\}$ has zero mean and its autocovariance function is expressed as

$$K_z(k,s) = E[z_k z_s^T] = \begin{cases} A_k B_s^T, & 0 \leq s \leq k \\ B_k A_s^T, & 0 \leq k \leq s \end{cases}$$

where A and B are known $n \times M$ matrix functions.
H2. The noise process $\{v_k;\ k \geq 0\}$ is a zero-mean white sequence with $E[v_k v_s^T] = R_k \delta_K(k-s)$, being δ_K the Kronecker delta function.
H3. The process $\{w_k;\ k\geq 0\}$ is a zero-mean coloured noise with autocovariance function expressed as

$$K_w(k,s) = E[w_k w_s^T] = \begin{cases} \alpha_k \beta_s^T, & 0 \leq s \leq k \\ \beta_k \alpha_s^T, & 0 \leq k \leq s \end{cases}$$

where α and β are known $n \times N$ matrix functions.
H4. The noise $\{\gamma_k;\ k \geq 0\}$ is a sequence of independent Bernoulli variables with $P[\gamma_k = 1] = p_k$ (probability of a delay in the measurement y_k).
H5. $\{z_k;\ k \geq 0\}$, $\{v_k;\ k \geq 0\}$, $\{w_k;\ k \geq 0\}$ and $\{\gamma_k;\ k \geq 0\}$ are mutually independent.

In this paper, we consider the least-squares (LS) linear estimation problem of the signal, z_k, based on the randomly delayed observations up to time j, $\{y_1, \ldots, y_j\}$; more specifically, our aim is to obtain the one-stage predictor ($j = k-1$) and the filter ($j = k$). For this purpose, we will use an innovation approach; if $\widehat{y}_{k,k-1}$ denotes the LS linear estimator of y_k based on the observations $\{y_1, \ldots, y_{k-1}\}$ and $\nu_k = y_k - \widehat{y}_{k,k-1}$ represents the innovation, the estimator of the signal is given by

$$\widehat{z}_{k,j} = \sum_{i=1}^{j} s_{k,i} \Pi_i^{-1} \nu_i, \tag{1}$$

being $s_{k,i} = E[z_k \nu_i^T]$ and $\Pi_i = E[\nu_i \nu_i^T]$. So, as the estimator is been in terms of the innovation process, we must begin by determining it. For it, since $\widehat{y}_{1,0} = 0$, we only need to determine

$$\widehat{y}_{k,k-1} = \sum_{i=1}^{k-1} E[y_k \nu_i^T] \Pi_i^{-1} \nu_i, \quad k \geq 2. \tag{2}$$

By denoting $\overline{s}_{k,i} = E[w_k \nu_i^T]$, $\mathcal{S}_{k,i} = s_{k,i} + \overline{s}_{k,i}$ and taking into account the model hypotheses, we have

$$E[y_k \nu_i^T] = (1-p_k)\mathcal{S}_{k,i} + p_k \mathcal{S}_{k-1,i}, \ \ i \leq k-2,$$

$$E[y_k \nu_{k-1}^T] = (1-p_k)\mathcal{S}_{k,k-1} + p_k \mathcal{S}_{k-1,k-1} + p_k(1-p_{k-1})R_{k-1}.$$

Substituting these last expressions in (2) and denoting $\widehat{w}_{k,j}$ to the LS linear estimator of the noise w_k based on the observations $\{y_1, \ldots, y_j\}$, we conclude that

$$\begin{aligned} \widehat{y}_{k,k-1} = {} & (1-p_k)\widehat{z}_{k,k-1} + p_k \widehat{z}_{k-1,k-1} \\ & +(1-p_k)\widehat{w}_{k,k-1} + p_k \widehat{w}_{k-1,k-1} \\ & +H_k \nu_{k-1}, \end{aligned} \tag{3}$$

where $H_k = p_k(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}$.

Hence, in order to determine ν_k we need to obtain the linear one-stage predictor and the filter of the signal and the coloured noise.

3 ESTIMATION ALGORITHM

The next theorem proposes a estimation algorithm for the one-stage predictor and filter of the signal from randomly delayed measurements.

Theorem 1. If we consider the delayed observation model given in Section 2, the one-stage predictor and filter of the signal z_k are obtained, respectively, as

$$\widehat{z}_{k,k-1} = A_k O_{k-1}, \quad \widehat{z}_{k,k} = A_k O_k, \tag{4}$$

where the vectors O_k are recursively calculated from

$$O_k = O_{k-1} + J_k \Pi_k^{-1} \nu_k, \quad O_0 = 0, \tag{5}$$

and ν_k, the innovation, satisfies

$$\begin{aligned} &\nu_k = y_k - G_{A,k} O_{k-1} - G_{\alpha,k} \overline{O}_{k-1} - H_k \nu_{k-1}, \\ &\nu_0 = 0, \end{aligned} \tag{6}$$

with

$$\overline{O}_k = \overline{O}_{k-1} + \overline{J}_k \Pi_k^{-1} \nu_k, \quad \overline{O}_0 = 0, \tag{7}$$

being

$$\begin{aligned} &J_k = G_{B,k}^T - r_{k-1} G_{A,k}^T - c_{k-1} G_{\alpha,k}^T - J_{k-1} H_k^T, \\ &J_0 = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\overline{J}_k = G_{\beta,k}^T - c_{k-1}^T G_{A,k}^T - d_{k-1} G_{\alpha,k}^T - \overline{J}_{k-1} H_k^T, \\ &\overline{J}_0 = 0 \end{aligned} \tag{9}$$

where, for $Y = A,\ B,\ \alpha$ and β, the matrices $G_{Y,k}$ are given by

$$G_{Y,k} = (1-p_k) Y_k + p_k Y_{k-1} \tag{10}$$

and $H_k = p_k(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}$.
The matrices r, c and d are recursively calculated by

$$r_k = r_{k-1} + J_k \Pi_k^{-1} J_k^T, \quad r_0 = 0, \tag{11}$$

$$c_k = c_{k-1} + J_k \Pi_k^{-1} \overline{J}_k^T, \quad c_0 = 0, \tag{12}$$

$$d_k = d_{k-1} + \overline{J}_k \Pi_k^{-1} \overline{J}_k^T, \quad d_0 = 0, \tag{13}$$

and Π_k, the covariance of the innovation ν_k, verifies

$$\begin{aligned} \Pi_k = {} & (1-p_k)\left[A_k B_k^T + \alpha_k \beta_k^T + R_k\right] \\ & +p_k\left[A_{k-1} B_{k-1}^T + \alpha_{k-1}\beta_{k-1}^T + R_{k-1}\right] \\ & -G_{A,k}\left[r_{k-1} G_{A,k}^T + c_{k-1} G_{\alpha,k}^T + J_{k-1} H_k^T\right] \\ & -G_{\alpha,k}\left[d_{k-1} G_{\alpha,k}^T + c_{k-1}^T G_{A,k}^T + \overline{J}_{k-1} H_k^T\right] \\ & -H_k\left[\Pi_{k-1} H_k^T + J_{k-1}^T G_{A,k}^T + \overline{J}_{k-1} G_{\alpha,k}^T\right] \\ \Pi_0 = {} & 0. \end{aligned} \tag{14}$$

Proof. Taking into account (1), the determining of the filter needs the calculation of the coefficients $s_{k,i} = E[z_k\nu_i^T]$, for $i \leq k$. Using expression (3) for $\widehat{y}_{i,i-1}$ and taking into account that

$$\widehat{w}_{k,j} = \sum_{i=1}^{j}\overline{s}_{k,i}\Pi_i^{-1}\nu_i,$$

the hypotheses on the model leads to

$$\begin{aligned} s_{k,i} &= A_k G_{B,i}^T - (1-p_i)\sum_{j=1}^{i-1}s_{k,j}\Pi_j^{-1}\mathcal{S}_{i,j}^T \\ &\quad -p_i\sum_{j=1}^{i-1}s_{k,j}\Pi_j^{-1}\mathcal{S}_{i-1,j}^T - s_{k,i-1}H_i^T, \ \ 2 \leq i \leq k \\ s_{k,1} &= A_k G_{B,1}^T. \end{aligned}$$

This expression for $s_{k,i}$ guarantees that

$$s_{k,i} = A_k J_i, \qquad 1 \leq i \leq k, \tag{15}$$

where J is a function satisfying

$$\begin{aligned} J_i &= G_{B,i}^T - (1-p_i)\sum_{j=1}^{i-1}J_j\Pi_j^{-1}\mathcal{S}_{i,j}^T \\ &\quad -p_i\sum_{j=1}^{i-1}J_j\Pi_j^{-1}\mathcal{S}_{i-1,j}^T - J_{i-1}H_i^T, \ \ 2 \leq i \leq k \\ J_1 &= G_{B,1}^T. \end{aligned} \tag{16}$$

Hence, if we denote

$$O_k = \sum_{i=1}^{k} J_i\Pi_i^{-1}\nu_i, \quad O_0 = 0, \tag{17}$$

the expression (4) for the predictor and the filter of z_k is deduced. The recursive relation (5) for the vectors O_k is immediate from (17).

From (3), in order to obtain the innovation ν_k, we also need to obtain the predictor and the filter of w_k; using an analogous reasoning to that realized to obtain (4), we have that $\overline{s}_{k,i} = \alpha_k\overline{J}_i$, for $1 \leq i \leq k$, being $\overline{J}$ a function verifying

$$\begin{aligned} \overline{J}_i &= G_{\beta,i}^T - (1-p_i)\sum_{j=1}^{i-1}\overline{J}_j\Pi_j^{-1}\mathcal{S}_{i,j}^T \\ &\quad -p_i\sum_{j=1}^{i-1}\overline{J}_j\Pi_j^{-1}\mathcal{S}_{i-1,j}^T - \overline{J}_{i-1}H_i^T, \ \ 2 \leq i \leq k \\ \overline{J}_1 &= G_{\beta,1}^T. \end{aligned} \tag{18}$$

So, we obtain that

$$\widehat{w}_{k,k-1} = \alpha_k\overline{O}_{k-1}, \quad \widehat{w}_{k,k} = \alpha_k\overline{O}_k, \tag{19}$$

where

$$\overline{O}_k = \sum_{i=1}^{k}\overline{J}_i\Pi_i^{-1}\nu_i, \quad \overline{O}_0 = 0 \tag{20}$$

Substituting (4) and (19) in (3), and considering the expressions (10) for $Y = A$ and α, we have that the innovation is given by (6). From (20), the recursive relation (7) is immediate.

Now, taking into account that $\mathcal{S}_{k,i} = A_k J_i + \alpha_k\overline{J}_i$, for $1 \leq i \leq k$, and by denoting

$$r_k = E\left[O_k O_k^T\right] = \sum_{i=1}^{k} J_i\Pi_i^{-1}J_i^T, \quad r_0 = 0,$$

$$c_k = E\left[O_k\overline{O}_k^T\right] = \sum_{i=1}^{k} J_i\Pi_i^{-1}\overline{J}_i^T, \quad c_0 = 0,$$

$$d_k = E\left[\overline{O}_k\overline{O}_k^T\right] = \sum_{i=1}^{k} \overline{J}_i\Pi_i^{-1}\overline{J}_i^T, \quad d_0 = 0,$$

we easily derive the recursive expressions (8) and (9) for J_k and $\overline{J}_k$, and the formulas (11), (12) and (13) for r_k, c_k and d_k, respectively.

Finally, the expression (14) for the innovation covariance is deduced from expression (6) together with the recursive relations (5) and (7) for the vectors O_{k-1} and $\overline{O}_{k-1}$, respectively, taking into account that O_{k-2} and $\overline{O}_{k-2}$ are orthogonal to ν_{k-1}. □

The performance of the estimates can be measured by the covariance matrices of the estimation errors

$$P_{k,j} = E\left[\{z_k - \widehat{z}_{k,j}\}\{z_k - \widehat{z}_{k,j}\}^T\right].$$

Since the error $z_k - \widehat{z}_{k,j}$ is orthogonal to the estimator $\widehat{z}_{k,j}$, it is immediate to verify that

$$P_{k,j} = K_z(k,k) - E\left[\widehat{z}_{k,j}\widehat{z}_{k,j}^T\right].$$

and taking into account the hypotheses on $K_z(k,k)$ and the expressions for the one-stage predictor and filter given in Theorem 1, we deduce the following formulas for the filtering and one-stage prediction error covariance matrices,

$$P_{k,k} = A_k\left[B_k^T - r_k A_k^T\right],$$

$$P_{k,k-1} = A_k\left[B_k^T - r_{k-1}A_k^T\right].$$

4 COMPUTER EXAMPLE

This section presents a numerical simulation example to estimate a scalar signal $\{z_k;\ k \geq 0\}$ generated by a first-order autoregressive model.

We consider a delayed observation model given by

$$\begin{aligned} \widetilde{y}_k &= z_k + v_k + w_k, \quad k \geq 0 \\ y_k &= (1-\gamma_k)\widetilde{y}_k + \gamma_k\widetilde{y}_{k-1}, \quad k \geq 1 \end{aligned}$$

where the scalar signal $\{z_k;\ k \geq 0\}$ has zero mean and autocovariance function

$$K_z(k,s) = 1.025641 \times (0.95)^{k-s}, \quad 0 \leq s \leq k.$$

The zero-mean white noise $\{v_k;\ k \geq 0\}$ is a Gaussian process with $Var[v_k] = 0.9$, for all k; the process $\{w_k;\ k \geq 0\}$ is a zero-mean coloured noise with autocovariance function

$$K_w(k,s) = 0.1 \times (0.5)^{k-s}, \quad 0 \leq s \leq k,$$

and, finally, $\{\gamma_k;\ k \geq 0\}$ is a sequence of independent Bernoulli random variables with $P[\gamma_k = 1] = p$, for all k; that is, we assume that the probability of a delay in the measurement is constant at any time.

In order to demonstrate the effectiveness of the algorithms proposed in this paper, we have performed a program in MATLAB, which simulates the signal value at each iteration, and provides the prediction and filtering estimates, as well as the corresponding error variances.

Firstly, the prediction and filtering error variances have been calculated for different values of the probability of delay, specifically, for $p = 0.2$ and $p = 0.9$. The results are displayed in Figure 1 which shows, on the one hand, that the error variances corresponding to the filtering estimates are less than the prediction ones and, on the other, that both, the prediction and filtering error variances, are smaller (and, consequently, the performance of the estimators is better) as the probability p decreases.

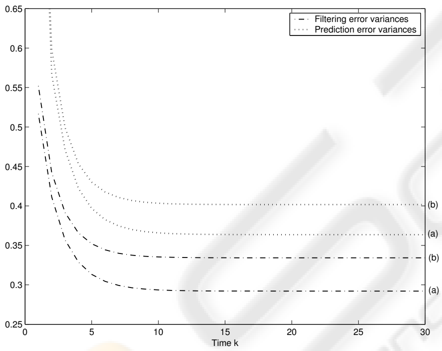

Figure 1: Filtering and prediction error variances for $p = 0.2$ [(a)] and $p = 0.9$ [(b)]

Finally, Figure 2 presents filtering estimates of a simulated signal from delayed measurements for $p = 0.2$ and $p = 0.9$. This figure shows that the filter follows the signal evolution better as the delay probability, p, is smaller, and, therefore, confirms the comments about Figure 1.

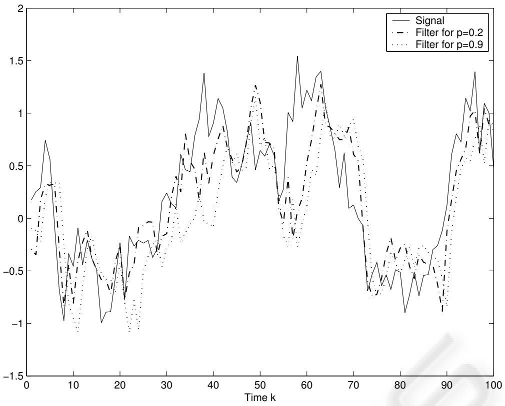

Figure 2:Signal and filtering estimates for $p = 0.2, 0.9$

5 CONCLUSION

In this paper, the linear one-stage predictor and filter are derived from randomly delayed measurements of the signal, for the case of white plus coloured noises. It is assumed that the delay is modelled by a sequence of independent Bernoulli random variables, which indicate if the measurements arrive in time or are delayed by one sampling period. The estimators are obtained by an innovation approach and do not require the knowledge of the state-space model of the signal, but just the second-order moments of the signal and noises, assuming a semi-degenerate kernel form for the autocovariance functions of the signal and the coloured noise, and the delay probabilities. A numerical example shows that the obtained algorithms are computationally feasible.

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