POLYNOMIAL ESTIMATION OF SIGNALS FROM UNCERTAIN OBSERVATIONS USING COVARIANCE INFORMATION

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Abstract: The least-squares νth-order polynomial filtering and fixed-point smoothing problems of uncertainly observed signals are considered. The proposed estimators do not require the knowledge of the state-space model generating the signal, but only the moments (up to the 2νth one) of the signal and the observation noise, as well as the probability that the signal exists in the observations.

1 INTRODUCTION

Systems with uncertain observations are characterized by including an observation multiplicative noise described by a sequence of Bernoulli random variables whose values -one or zero- indicate the presence or absence of signal in the observation, respectively. So, these systems constitute an appropriate model for analyzing those situations in which the observation may not contain the signal to be estimated and, hence, it consists only of noise (for example, situations of fading or reflection of transmitted signals from the ionosphere).

Due to the multiplicative noise component, even if the additive observation noise is gaussian, the least-squares (LS) estimator is not a linear function of the observations and, usually, it is not easily obtainable. This difficulty has motivated the necessity of looking for suboptimal estimators which are easier to obtain; particularly, linear and polynomial estimation problems from uncertain observations have been treated by several authors, as NaNacara and Yaz (1997), Caballero et al. (2003), etc., assuming a full knowledge of the state-space model for the signal process.

Nevertheless, usually the state-space model is not available and the estimation problem must be addressed using another kind of information, such as covariance information about the processes involved. The LS linear estimation problem from uncertain observations using this kind of information has been considered, for example, in Nakamori et al. (2003a) and these results are extended in Nakamori et al. (2003b) by proposing algorithms for the LS quadratic estimators, which improve the linear ones.

In this paper the results in Nakamori et al. (2003a, 2003b) are generalized. More specifically, we address the LS polynomial filtering and fixed-point smoothing problems of arbitrary degree (ν) from uncertain observations perturbed by white noise. Besides the probability that the signal exists in the observations, the proposed estimators only require the knowledge of the moments (up to the 2νth one) of the signal and the observation additive noise.

2 PROBLEM FORMULATION

Let $z(k)$ and $y(k)$ be $n \times 1$ vectors describing the signal and its observation at time k, respectively. Let us suppose that

$$y(k) = U(k)z(k) + v(k). \qquad (1)$$

Our aim is to obtain the least-squares (LS) νth-order polynomial estimator of the signal $z(k)$ based

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on the observations $\{y(1), \ldots, y(L)\}$, being $\nu \geq 1$ arbitrary. Defining the random vectors

$$y^{[2]}(i) = y(i) \otimes y(i)$$
$$y^{[j]}(i) = y^{[j-1]}(i) \otimes y(i), \;\; j > 2$$

($\otimes$ denotes the Kronecker product, Magnus and Neudecker (1988)), and assuming that $E\left[y^{[2\nu]}(i)\right] < \infty$, this estimator is the orthogonal projection of $z(k)$ on the space of n-dimensional linear transformations of $y(1), \ldots, y(L)$ and their Kronecker powers $y^{[2]}(1), \ldots, y^{[\nu]}(1), \ldots, y^{[2]}(L), \ldots y^{[\nu]}(L)$. More specifically, we are interested in obtaining the LS νth-order polynomial filter ($L = k$) and fixed-point smoother ($L > k$) of the signal. For this purpose, we assume the following hypotheses:

(H.1) The signal process $\{z(k);\; k \geq 0\}$ has zero mean and, for $i, j = 1, \ldots, \nu$, the covariance function of the vectors $z^{[i]}(k)$ and $z^{[j]}(k)$, $K_{ij}(k,s) = E[\overline{z^{[i]}}(k)\overline{z^{[j]}}^T(s)]$, can be expressed as

$$K_{ij}(k,s) = \begin{cases} A_{ij}(k)B_{ij}^T(s), & 0 \leq s \leq k \\ B_{ji}(k)A_{ji}^T(s), & 0 \leq k \leq s \end{cases}$$

where A_{ij} and B_{ij} are $n^i \times N_{ij}$ and $n^j \times N_{ij}$ known matrix functions, respectively, and $\overline{x} := x - E[x]$.

(H.2) The noise process $\{v(k);\; k \geq 0\}$ is a zero-mean white sequence with $E[v^{[2\nu]}(k)] < \infty$ and $E[v^{[i]}(k)]$ is known for $i = 1, \ldots, 2\nu$.

(H.3) The multiplicative noise $\{U(k);\; k \geq 0\}$ is a sequence of independent Bernoulli random variables with known $P\left[U(k) = 1\right] = p(k)$.

(H.4) The processes $\{z(k);\; k \geq 0\}$, $\{U(k);\; k \geq 0\}$ and $\{v(k);\; k \geq 0\}$ are mutually independent.

To address the LS νth-order polynomial estimation problem, we define the augmented signal and observation vectors as

$$\mathcal{Z}(k) = \begin{pmatrix} z(k) \\ \vdots \\ z^{[\nu]}(k) \end{pmatrix}, \quad \mathcal{Y}(k) = \begin{pmatrix} y(k) \\ \vdots \\ y^{[\nu]}(k) \end{pmatrix}.$$

Then, the vector constituted by the first n entries of the LS linear estimator of $\mathcal{Z}(k)$ based on $\mathcal{Y}(1), \ldots, \mathcal{Y}(L)$ provides the LS νth-order polynomial estimator of the original signal $z(k)$.

Next we study the properties of $\mathcal{Z}(k)$ and $\mathcal{Y}(k)$ which will be used to obtain the LS linear estimator of $\mathcal{Z}(k)$.

3 AUGMENTED EQUATION

In order to analyze the properties of the vector $\mathcal{Y}(k)$, we start by obtaining an appropriate expression for $y^{[j]}(k),\; j = 2, \ldots, \nu$. By employing the Kronecker product properties and noting that $U(k) = U^2(k) = \cdots = U^\nu(k)$, $y^{[j]}(k)$ can be written as

$$y^{[j]}(k) = U(k)\sum_{l=1}^{j} L_{jl}(k)z^{[l]}(k) + E[v^{[j]}(k)] + g_j(k).$$

where

$$L_{jl}(k) = M_{j-l}^j(n)\left(E[v^{[j-l]}(k)] \otimes I_{n,l}\right), \; l \leq j-1$$
$$L_{jj}(k) = I_{n,j}$$

and

$$g_j(k) = U(k)\sum_{l=1}^{j-1} M_{j-l}^j(n)\left(\overline{v^{[j-l]}}(k) \otimes I_{n,l}\right)z^{[l]}(k) + \overline{v^{[j]}}(k)$$

with

$$M_0^j(n) = M_j^j(n) = I_{n,j},$$
$$M_r^j(n) = (M_r^{j-1}(n) \otimes I_{n,1}) + (M_{r-1}^{j-1}(n) \otimes G_{j-r}), \quad 1 \leq r \leq j-1$$
$$G_l = (I_{n,1} \otimes G_{l-1})(G_1 \otimes I_{n,l-1}), \quad G_1 = K_{n,n}$$

($I_{n,l}$ denotes the $n^l \times n^l$ identity matrix; $K_{n,m}$ is the $nm \times nm$ commutation matrix).

Then, by denoting

$$\mathcal{C}(k) = \begin{pmatrix} L_{11}(k) & 0_{n \times n^2} & \cdots & 0_{n \times n^\nu} \\ L_{21}(k) & L_{22}(k) & \cdots & 0_{n^2 \times n^\nu} \\ \cdots & \cdots & \cdots & \cdots \\ L_{\nu 1}(k) & L_{\nu 2}(k) & \cdots & L_{\nu\nu}(k) \end{pmatrix},$$

$$V_k = \begin{pmatrix} 0_{n \times 1} \\ E\{v^{[2]}(k)\} \\ \vdots \\ E\{v^{[\nu]}(k)\} \end{pmatrix}, \quad \mathcal{G}(k) = \begin{pmatrix} v(k) \\ g_2(k) \\ \vdots \\ g_\nu(k) \end{pmatrix}$$

we obtain that $Y(k) = \mathcal{Y}(k) - E[\mathcal{Y}(k)]$ satisfy the following *augmented observation equation*

$$Y(k) = U(k)\mathcal{C}(k)Z(k) + V(k) \tag{2}$$

where $Z(k) = \mathcal{Z}(k) - E[\mathcal{Z}(k)]$ and $V(k) = \left[U(k) - p(k)\right]\mathcal{C}(k)E[\mathcal{Z}(k)] + \mathcal{G}(k)$.

In the following propositions the statistical properties of the processes involved in equation (2) are established.

Proposition 1. Under hypotheses (H.1)-(H.4), the process $\{Z(k);\; k \geq 0\}$ has zero mean and its autocovariance function, $K_Z(k,s) = E[Z(k)Z^T(s)]$, is expressed as

$$K_Z(k,s) = \begin{cases} \mathcal{A}(k)\mathcal{B}^T(s), & 0 \leq s \leq k \\ \mathcal{B}(k)\mathcal{A}^T(s), & 0 \leq k \leq s \end{cases}$$

with

$$\mathcal{A}(k) = \begin{pmatrix} A_{11}(k) \cdots A_{1\nu}(k) \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots A_{\nu 1}(k) \cdots A_{\nu\nu}(k) \end{pmatrix}$$

$$\mathcal{B}(k) = \begin{pmatrix} B_{11}(k)\cdots & 0 & \cdots B_{\nu 1}(k)\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots B_{1\nu}(k)\cdots & 0 & \cdots B_{\nu\nu}(k) \end{pmatrix}$$

Moreover, the process $\{Z(k);\ k \geq 0\}$ is independent of the multiplicative noise $\{U(k);\ k \geq 0\}$.

Proposition 2. If hypotheses (H.1)-(H.4) are satisfied, the noise $\{V(k);\ k \geq 0\}$ of equation (2) is a sequence of zero-mean, mutually uncorrelated random vectors with covariance matrices

$$R_V(k) = p(k)\,(1-p(k))\,\mathcal{C}(k)E\left[\mathcal{Z}(k)\right]E\left[\mathcal{Z}^T(k)\right] \times \mathcal{C}^T(k) + R_{\mathcal{G}}(k)$$

being

$$E\left[\mathcal{Z}(k)\right] = \begin{pmatrix} 0 \\ vec\left(A_{11}(k)B_{11}^T(k)\right) \\ \vdots \\ vec\left(A_{1\ \nu-1}(k)B_{1\ \nu-1}^T(k)\right) \end{pmatrix}$$

and $R_{\mathcal{G}}(k) = E\left[\mathcal{G}(k)\mathcal{G}^T(k)\right]$ is a matrix whose (r,s)-block is given by

$$\begin{aligned} R_{\mathcal{G}}^{(r,s)}(k) &= E\left\{g_r(k)g_s^T(k)\right\} \\ &= p(k)\left\{\sum_{l=1}^{r-1}\sum_{i=0}^{s-1} M_{r-l}^r(n)P_{l,i}^{r,s}(v(k))(M_{s-i}^s(n))^T \right. \\ &\quad \left. + \sum_{i=1}^{s-1} P_{0,i}^{r,s}(v(k))(M_{s-i}^s(n))^T\right\} + P_{0,0}^{r,s}(v(k)) \end{aligned}$$

with

$$\begin{aligned} P_{l,i}^{r,s}(v(k)) = vec^{-1}\Big[&\left(I_{n,s-i}\otimes K_{n^i,n^{r-l}}\otimes I_{n,l}\right) \\ &\times \left(\left(E\left\{v^{[r+s-l-i]}(k)\right\} - E\left\{v^{[s-i]}(k)\right\}\right.\right. \\ &\left.\left.\otimes E\left\{v^{[r-l]}(k)\right\}\right)\otimes I_{n,l+i}\right)E\left\{z^{[l+i]}(k)\right\}\Big] \end{aligned}$$

Moreover, $\{V(k);\ k \geq 0\}$ is uncorrelated with the processes $\{Z(k);\ k \geq 0\}$ and $\{U(k)Z(k);\ k \geq 0\}$.

4 LINEAR ESTIMATION OF $Z(k)$

As it has been indicated, the LS polynomial estimator of the original signal $z(k)$ is obtained by extraction of the first n entries of the LS linear estimator of $Z(k)$. Our aim is then to establish a recursive algorithm for the linear filtering and fixed-point smoothing estimators, $\widehat{Z}(k,L),\ L \geq k$, of the signal $Z(k)$ based on the observations $Y(1),\ldots,Y(L)$. Taking into account the properties established in propositions 1 and 2, the following recursive algorithm is derived.

Theorem 1. The filtering and fixed-point smoothing algorithm of the augmented signal $Z(k)$ based on the observations $Y(1),\ldots,Y(L),\ L \geq k$, is given by

$$\widehat{Z}(k,L) = \widehat{Z}(k,L-1) + g(k,L)\nu(L), \quad L > k$$

where the innovation, $\nu(L)$, verifies

$$\nu(L) = Y(L) - p(L)\mathcal{C}(L)\mathcal{A}(L)O(L-1), \ \ L \geq 1$$

and the vector $O(L)$ can be calculated from

$$O(L) = O(L-1) + \Delta(L)\Pi^{-1}(L)\nu(L), \ \ O(0) = 0$$

$$\Delta(L) = p(L)\left[\mathcal{B}^T(L) - r(L-1)\mathcal{A}^T(L)\right]\mathcal{C}^T(L)$$

where $r(L) = E[O(L)O^T(L)]$ satisfies

$$r(L) = r(L-1) + \Delta(L)\Pi^{-1}(L)\Delta^T(L), \ \ r(0) = 0.$$

The smoother gain, $g(k,L)$, is expressed as

$$g(k,L) = p(L)\left[\mathcal{B}(k) - E(k,L-1)\right] \times \mathcal{A}^T(L)\mathcal{C}^T(L)\Pi^{-1}(L)$$

where $\Pi(L)$, the covariance matrix of the innovation, is given by

$$\Pi(L) = R_V(L) + p(L)\mathcal{C}(L)\mathcal{A}(L)\left[\mathcal{B}^T(L) - p(L)r(L-1)\mathcal{A}^T(L)\right]\mathcal{C}^T(L)$$

and the matrices $E(k,L)$ are calculated from

$$\begin{aligned} &E(k,L) = E(k,L-1) + g(k,L)\Delta^T(L), \quad L > k \\ &E(k,k) = \mathcal{A}(k)r(k). \end{aligned}$$

The filter, $\widehat{Z}(k,k)$, which provides the initial condition, is given by $\widehat{Z}(k,k) = \mathcal{A}(k)O(k)$.

The smoothing and filtering error covariance matrices, $P(k,L),\ \ L \geq k$, satisfy

$$\begin{aligned} &P(k,L) = P(k,L-1) - g(k,L)\Pi(L)g^T(k,L), \\ &P(k,k) = \mathcal{A}(k)\left[\mathcal{B}^T(k) - r(k)\mathcal{A}^T(k)\right]. \end{aligned}$$

5 SIMULATION RESULTS

Consider a scalar signal $\{z(k);\ k \geq 0\}$ generated by the following first-order autoregressive model

$$z(k+1) = 0.95z(k) + w(k)$$

where $\{w(k);\ k \geq 0\}$ is a zero-mean white Gaussian noise with $Var\left[w(k)\right] = 0.1$, for all k.

For $0 \leq s \leq k$, the autocovariance and cross-covariance functions of this signal and its Kronecker powers are

$$\begin{aligned} &K_{11}(k,s) = 1.0256 \cdot 0.95^{k-s} \\ &K_{13}(k,s) = K_{31}(k,s) = 3.1558 \cdot 0.95^{k-s} \\ &K_{22}(k,s) = 2.1039 \cdot 0.95^{2(k-s)} \\ &K_{33}(k,s) = 9.7102 \cdot 0.95^{k-s} + 6.4735 \cdot 0.95^{3(k-s)} \end{aligned}$$

and, for all s,k,

$$K_{12}(k,s) = K_{21}(k,s) = K_{23}(k,s) = K_{32}(k,s) = 0.$$

In view of these expressions, these functions can be easily factorized according to hypothesis (H.1).

The observation equation is given by

$$y(k) = U(k)z(k) + v(k)$$

where $\{U(k);\ k \geq 0\}$ is a sequence of independent Bernoulli random variables with $P[U(k) = 1] = p$, for all k, and $\{v(k);\ k \geq 0\}$ is a white sequence with

$$E[v(k)] = 0, \qquad E[v^2(k)] = 9.1429,$$
$$E[v^3(k)] = -62.6939, \qquad E[v^4(k)] = 513.4928,$$
$$E[v^5(k)] = -4094.2941, \qquad E[v^6(k)] = 32769.95.$$

In order to show the effectiveness of the algorithm proposed in Theorem 1, we compare the linear, quadratic and cubic estimates for different values of the parameter p, specifically, $p = 0.7$ and $p = 1$ (case in which the signal is always present in the observations). The filtering error variances are displayed in Figure 1 which shows that, for both values of p, the error variances are smaller as the degree of the polynomial function increases, that is, the linear filter is improved by the quadratic one which, in turn, is improved by the cubic one. This figure also shows that, as p increases, the error variances are smaller, which means that the performance of the filters is better as the probability of the signal being missing is smaller.

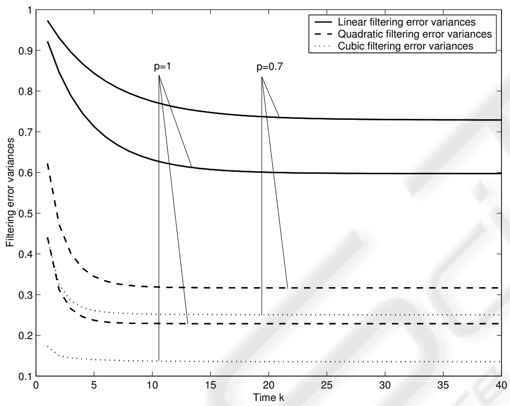

Figure 1: Linear, quadratic and cubic filtering error variances for $p = 0.7$ and $p = 1$.

Figure 2 displays a simulated signal together with the linear, quadratic and cubic filtering estimates for the value $p = 0.7$. The result, as expected, is that the better performance corresponds to the cubic filtering estimate, according to the comments about Figure 1.

6 CONCLUSION

In this paper a recursive algorithm for the LS νth-order polynomial filter and fixed-point smoother from uncertain observations is presented, when the state-space model of the signal is unknown. The available information is only the autocovariance and cross-covariance functions of the signal and its Kronecker powers, as well as the corresponding functions of the additive noise. It is also assumed that the probabilities of existence of the signal in the observed values are available. An augmented observation equation suitably defined allows us to obtain the polynomial estimator of the original signal from the linear estimator of the augmented signal.

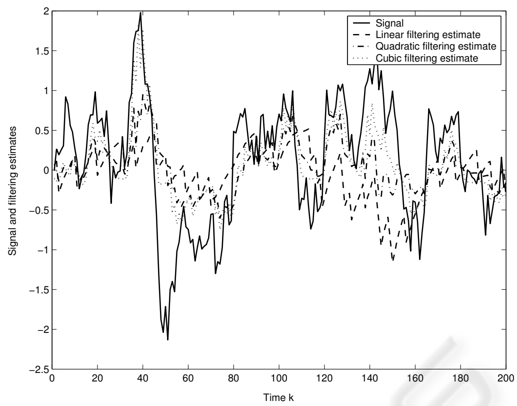

Figure 2: Signal and filtering estimates for $p = 0.7$.

The effectiveness of the quadratic and cubic filters in contrast to the linear one is shown by applying the proposed algorithm to estimate a signal generated by a first-order autoregressive model.

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