# A Stochastic Model of Diffusion in Opinion Dynamics

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Abstract: This paper studies analytically the dynamics of the opinion in multi-agent systems when only the sociological phenomenon known as diffusion is considered. First, the paper recalls a framework for the analytic study of opinion dynamics which has been already applied to describe the effects of a number of sociological phenomena. Then, the framework is specialized to the study of diffusion, according to which the opinion of an agent can be influenced by the social context. Diffusion is introduced in the framework by stating stochastic rules meant to describe at the microscopic level how diffusion contributes to change the opinion of an agent. The obtained model is used to derive collective and asymptotic properties of multi-agent systems when only diffusion is considered, which are verified against specific simulations shown in the last part of the paper. The paper is concluded with a recapitulation of presented results and an outline of future work.

## 1 INTRODUCTION

This paper describes a stochastic model that is used to derive analytic results on collective and long-time asymptotic properties of the opinion in multi-agent systems. Considered multi-agent systems are completely decentralized and without supervised coordination, and they are studied using a generic framework that depends only on the description of the effects of microscopic interactions among agents. Microscopic rules that govern how the opinion of two interacting agents change are used to derive analytically observable macroscopic properties of the opinion in the multi-agent system, under proper assumptions.

In this context, the term *interaction* is used to denote a symmetric message exchange among two agents, and it is assumed that message exchanges are asynchronous. Time is modeled as a sequence of discrete steps, which may not have the same duration, and each step corresponds to a single interaction among two agents. Each agent is free to interact with any other agent, and no restriction is imposed on the topological properties of the multi-agent system. The adopted framework assumes that each agent is associated with a scalar attribute, and since the target application of the framework in this paper regards the study of the dynamics of the opinion, we assume that such an attribute represents the opinion of an agent on a fact, which may vary within a finite interval.

Most of existing agent-based models used to study opinion dynamics are based on simulations (Deffuant et al., 2000; Hegselmann and Krause, 2002; Monica and Bergenti, 2014). Hence, their validity depends on the specific type of multi-agent system that is simulated, and on the actual values assigned to the parameters of simulations. Actually, their validity depends on how representative are simulated scenarios with respect to studied phenomena (Flache et al., 2017). At the opposite, the framework discussed in this paper is analytic, and it provides parametric results which are valid as long as its hypotheses remain valid.

Even if analytic models are typically simpler than models based on simulations, they are preferred when the interest is on how collective and long-time asymptotic behaviours are affected by the parameters of the models, or when no representative cases to be simulated can be found. Simulations are still very important even if analytic models are available because they become a means to independently verify analytic models in interesting cases. Simulations reported in the last part of the paper are performed independently of discussed analytic results, and they are uniquely intended to verify analytic results.

Note that the proposed framework is not limited to the study of opinion dynamics and its approach can be adopted to describe other collective behaviours of multi-agent systems, provided that studied phenomena emerge from decentralized interactions.



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The analytic framework adopted in this paper and described in next section is inspired by a branch of physics known as kinetic theory of gases and its generalizations, as described, e.g., in (Bellouquid and Delitala, 2006). According to this approach, analytic models of microscopic interactions among molecules can be considered to derive macroscopic characteristics of gases concerning, for instance, temperature and pressure. Analytic frameworks that follow the approach of the kinetic theory of gases has been applied successfully in different fields. For instance, the kinetic approach has been successfully applied in economics to describe the evolution of market economy (Cordier et al., 2005) and the distribution of wealth (Slanina, 2004), giving birth to a discipline known as *econophysics* (Chakrabarti et al., 2006). In this paper, we focus on an application of the kinetic approach to describe sociological processes and the dynamics of the opinion in multi-agent systems. The idea is not new, and it is part of an emerging discipline known as *sociophysics* (Galam et al., 1982). Many analogies between the kinetic theory of gases and the study of opinion dynamics in multi-agent systems can be found. First, it is evident that a parallelism between *molecules* in gases and *agents* in multi-agent systems can be drawn. Similarly, *collisions among molecules* in gases can be re-interpreted as *interactions among agents* in multi-agent systems. Obviously, the detailed mathematical modeling of collisions among molecules is based on the laws of physics and it is different from that of interactions among agents, which is typically suggested by sociological studies. Due to such differences, only few results of kinetic theory of gases hold in the study of the dynamics of the opinion.

Many sociological phenomena can be studied analytically using the kinetic approach (Monica and Bergenti, 2017a), like:

- *Compromise*, which is the tendency of agents to move their opinions towards those of agents they interact with (De Groot, 1974);
- *Diffusion*, according to which the opinion of each agent can be influenced by the social context (Bonabeau, 2002);
- *Homophily*, according to which agents interact only with those with similar opinions (Nowak et al., 1990; Mark, 2003);
- *Negative Influence*, according to which agents evaluate their peers, and they only interact with some peers (Mäs and Flache, 2013);
- *Opinion Noise,* according to which a random additive process may lead to arbitrary opinion changes with small probability (Pineda et al., 1982); and
- *Striving for Uniqueness*, which is based on the idea that agents want to distinguish from others and, hence, they decide to change their opinions if too many agents share the same opinion (Mäs et al., 2010).

Kinetic models that study compromise alone have been already proposed in (Monica and Bergenti, 2016a; Monica and Bergenti, 2016b; Monica and Bergenti, 2017b; Monica and Bergenti, 2017c). In addition, in (Monica and Bergenti, 2015a; Monica and Bergenti, 2015b; Monica and Bergenti, 2015c), both compromise and diffusion were analytically modeled and investigated under proper assumptions. In this paper, instead, we focus uniquely on diffusion and we analytically derive results concerning the dynamics of the opinion when only this phenomenon is considered. The major contribution of this paper is to study analytically collective and asymptotic properties of the opinion when only diffusion is relevant, so that the properties of diffusion can be isolated and studied independently of other phenomena. Note that other models of opinion formation based on the kinetic theory of gases were also proposed to account for the possibility of having different types of agents in a multi-agent system (Bergenti and Monica, 2016; Bergenti and Monica, 2017).

This paper is organized as follows. Section 2 summarizes the main ideas of the kinetic approach and it outlines the adopted kinetic framework. Section 3 presents a stochastic model to describe the sociological phenomenon of diffusion. Section 4 shows major analytic results concerning collective and asymptotic properties of a multi-agent system when only diffusion is considered. Section 5 describes simulations that were ran to verify analytic results. Finally, Section 6 concludes the paper and outlines future work.

## 2 A KINETIC FRAMEWORK TO STUDY OPINION DYNAMICS

In this section, we present the main ideas of classic kinetic theory of gases and we focus on the reinterpretation of such ideas to study the dynamics of the opinion in multi-agent systems. Classic kinetic theory of gases assumes that, at each time $t$, each molecule of the considered gas can be associated with relevant parameters, such as its position $\underline{x}$ and velocity $\underline{v}$, which are both modeled as three-dimensional vectors. The characteristics of the considered gas are described in terms of a density function $f(\underline{x}, \underline{v}, t)$, which represents the number of molecules whose position is in $(\underline{x}, \underline{x} + \mathrm{d}\underline{x})$ and whose velocity is in $(\underline{v}, \underline{v} + \mathrm{d}\underline{v})$ at time $t$. The density function $f(\underline{x},\underline{v},t)$ evolves following a proper balance equation, namely the famous Boltzmann equation. The Boltzmann equation is an integro-differential equation whose explicit formulation is based on the analytic description of the effects of collisions among molecules. The study of the Boltzmann equation allows deriving macroscopic properties of a gas, concerning, for instance, the average velocity of its molecules, its temperature, and its pressure. In summary, kinetic theory of gases studies the effects of collisions among molecules from a microscopic point of view and uses such results to derive macroscopic properties of gases.

The same approach can be used to model opinion dynamics in multi-agent systems by means of a proper parallelism between *molecules* and *collisions* among them, and *agents* and *interactions* among them, as proposed, for example, in (Schweitzer and Holyst, 2000) or in (Düring et al., 2009). It is worth noting that, since the effects of collisions among molecules differ from the effects of interactions among agents, analytic results derived to study the dynamics of the opinion differ significantly from well-known results of the kinetic theory of gases. However, the approach of kinetic theory of gases can be effectively generalized to obtain analytic results in the field of opinion dynamics, as follows.

Each agent is associated with a single parameter $v$, which represents its opinion. Note that, in our context, the term opinion is used to denote the level of agreement on a single topic or the level of appreciation of a single item. Therefore, we assume that opinion $v$ is valid only in a closed interval

$$I = [-1, 1] \tag{1}$$

where $-1$ stands for *strong disagree* or *strong dislike*, 1 stands for *strong agree* or *strong like*, and values close to 0 are considered moderate opinions. We remark that the choice of the interval $I$ in (1) is discretional and any other closed interval can be used, provided that the following analytic developments are coherently changed. In agreement with the general ideas of kinetic theory of gases, we also postulate the existence of a density function $f(v,t)$, which represents the number of agents with opinion in $(v, v+\mathrm{d}v)$ at time $t$. The definition of the density function $f(v,t)$ allows computing proper macroscopic properties of the system from an analytic point of view. For instance, the number of agents at time $t$, denoted as $n(t)$, can be computed as

$$n(t) = \int_I f(v,t)\mathrm{d}v. \tag{2}$$

From (2), it is evident that the number of agents at time $t$ is obtained by integrating the density function with respect to all values of $v \in I$. In kinetic theory of gases, a similar integral is used to compute the mass of the considered gas. Similarly, the average opinion of the system at time $t$, denoted as $u(t)$, is obtained by multiplying $f(v,t)$ by $v$, dividing by the number of agents, and integrating with respect to $v$,

$$u(t) = \frac{1}{n(t)} \int_I f(v,t) v \,\mathrm{d}v. \tag{3}$$

Observe that the average opinion $u(t)$ is related to the first momentum of the density function $f(v,t)$. In kinetic theory of gases, a similar integral is used to compute the average velocity of molecules in the considered gas. Finally, the variance of the opinion at time $t$, denoted as $\sigma^2(t)$, can be obtained by multiplying $f(v,t)$ by $(v-u(t))^2$, dividing by the number of agents, and integrating with respect to $v$, as follows

$$\sigma^2(t) = \frac{1}{n(t)} \int_I (v-u(t))^2 f(v,t)\,\mathrm{d}v. \tag{4}$$

Note that the variance of the opinion $\sigma^2(t)$ is related to the second-order momentum of the density function $f(v,t)$. In kinetic theory of gases, similar integrals are used to compute the pressure and the temperature of the considered gas.

The temporal evolution of the density function $f(v,t)$ is governed by a balance equation, whose explicit expression is inspired from that of the Boltzmann equation. For this reason, we adopt the same nomenclature of kinetic theory of gases and we still use the term Boltzmann equation for such a balance equation. The homogeneus formulation of the Boltzmann equation that can be used to study the dynamics of the opinion is

$$\frac{\partial f}{\partial t}(v,t) = Q(f)(v,t) \tag{5}$$

where the left-hand side is related to the temporal evolution of the density function and in the right-hand side $Q$ is an operator meant to account for the effects of interactions among agents. Keeping the same nomenclature of the kinetic theory of gases, $Q$ is called *collisional operator* and it computes a function of variables $v$ and $t$ by using function $f$.

In order to analytically study the temporal evolution of the number of agents $n(t)$, the average opinion $u(t)$, and the variance of the opinion $\sigma^2(t)$, the *weak form* of the Boltzmann equation can be used. In functional analysis, the weak form of a differential equation is obtained by multiplying both sides of the equation by a suitable *test function* and by integrating with respect to one of the variables. Therefore, the weak form of the Boltzmann equation with respect to the generic test function $\phi(v)$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I f(v,t)\phi(v)\,\mathrm{d}v = \int_I Q(f)(v,t)\phi(v)\,\mathrm{d}v \tag{6}$$

where the right-hand side is called weak form of the collisional operator $Q$ with respect to test function $\phi(v)$. Recalling (2), (3), and (4), it is evident that the left-hand side of (6) can be used to compute the time derivative of macroscopic characteristics of the multi-agent system, provided that specific test functions are chosen, as follows:

1. If $\phi(v) = 1$, the left-hand side of (6) represents the time derivative of the number of agents $n(t)$;
2. If $\phi(v) = v$, the left-hand side of (6) is related to the time derivative of the average opinion; and
3. If $\phi(v) = (v - u(t))^2$, the left-hand side of (6) is related to the time derivative of the variance of the opinion.

In next section, the details of the microscopic effects of the interactions among agents on the opinion are outlined and, consequently, the explicit expression of the collisional operator and of its weak form are derived. Such derivations are sufficient to express ordinary differential equations whose unknowns are $n(t)$, $u(t)$, and $\sigma^2(t)$. The solutions of such equations express the dynamics of such quantities and they allow studying analytically their asymptotic behaviours.

# 3 A STOCHASTIC MODEL OF DIFFUSION

In order to derive explicit results using the framework outlined in previous section, the explicit formulation of the interaction among agents is needed. Note that in the proposed framework interactions among agents are not coordinated by any supervising entity, and, in our assumptions, any agent can freely interact with any other agent in the system. Moreover, we assume that interactions among agents are binary, which means that they involve only two agents. This assumption is not restrictive, since interactions involving a larger number of agents can be regarded as sequences of binary interactions. Concerning the sociological phenomenon considered in this paper, as shortly explained in the introduction, we focus on diffusion, according to which agents can change their opinions due to external influence. However, we remark that the proposed approach can be extended to take into account more complex interaction rules, including, e.g., homophily (Tsang and Larson, 2014), which accounts for the fact that agents tend to communicate only with those having similar opinions.

Diffusion can be modeled using the following interaction rules, as proposed, e.g., in (Pareschi and Toscani, 2013; Toscani, 2006). Such interaction rules model how the opinions of two interacting agents change after an interaction if only diffusion is considered relevant to opinion formation and other sociological phenomena are neglected

$$\begin{cases} v' = v + \eta_1 D(v) \\ w' = w + \eta_2 D(w) \end{cases} \tag{7}$$

where

1. $v$ and $w$ represent the opinions of the two agents before the interaction, often called *pre-interaction* opinions;
2. $v'$ and $w'$ represent the opinions of the two agents after the interaction, often called *post-interaction* opinions;
3. $\eta_1$ and $\eta_2$ are two independent random variables that we call *diffusion parameters*; and
4. $D(\cdot)$ models the effects of diffusion on the opinion of the two interacting agents, and it is called *diffusion function*.

In the rest of this paper we assume that diffusion parameters have the same distribution function, which is denoted as $\vartheta(\cdot)$. We denote the support of $\vartheta(\cdot)$ as $S$, and we assume that the average values of the two random variables equal 0

$$\int_S \vartheta(\eta_1)\eta_1 d\eta_1 = \int_S \vartheta(\eta_2)\eta_2 d\eta_2 = 0. \tag{8}$$

Note that according to the proposed model of diffusion the post-interaction opinion of an agent depends only on its pre-interaction opinion, and it does not depend on the pre-interaction opinion of the other agent. This models the idea that the opinion of an agent can change simply because an interaction occurred, and the change of the opinion does not necessarily depend on the opinion of the other agent, which is how diffusion is studied in opinion dynamics (Pareschi and Toscani, 2013; Toscani, 2006).

Observe that from (7) it is evident that the post-interaction opinions are obtained by adding to the pre-interaction opinions an addend which is proportional to the diffusion function evaluated in the pre-interaction opinions according to the values of the parameters $\eta_1$ and $\eta_2$. Some considerations on the choice of the diffusion function and on the choice of the distribution functions of $\eta_1$ and $\eta_2$ are needed. First, let us observe that different choices of the diffusion function may lead to very different models. Following the literature (Toscani, 2006), we assume that diffusion functions are nonzero and symmetric with respect to the central value of the interval where opinions are defined. In our assumptions, since interval $I$ is symmetric with respect to 0, this corresponds to state that the diffusion function is even and, for this reason, from now on we assume that $D(\cdot)$ is a function of the absolute value of the opinion $v$. Moreover, we assume that $D(\cdot)$ is not increasing with respect to $|v|$, in agreement with the idea that agents with opinions close to the bounds of $I$ are conservative. Finally, we also assume that the following inequalities hold to ensure that the magnitude of the effects of diffusion are controlled only by diffusion parameters $\eta_1$ and $\eta_2$

$$0 \le D(|v|) \le 1. \tag{9}$$

We call *admissible* any diffusion function that respects mentioned assumptions, and we consider only admissible diffusion functions in the rest of this paper.

Some considerations need to be made to guarantee that post-interaction opinions $v'$ and $w'$ belong to the interval of interest $I$. To this aim, it is necessary to impose proper conditions on the support of the two random variables $\eta_1$ and $\eta_2$. Observe that the following inequalities hold

$$\begin{aligned} |v+\eta_1 D(|v|)| &\le |v| + |\eta_1| D(|v|) \\ |w+\eta_2 D(|w|)| &\le |w| + |\eta_2| D(|w|). \end{aligned} \tag{10}$$

Hence, possible sufficient conditions to guarantee that $v' \in I$ and $w' \in I$ are

$$\begin{aligned} |\eta_1| &\le \frac{1-|v|}{D(|v|)} \\ |\eta_2| &\le \frac{1-|w|}{D(|w|)}. \end{aligned} \tag{11}$$

If (11) holds, diffusion parameters are said to be *admissible* for the chosen $D(\cdot)$, and note that we consider only admissible diffusion parameters in the rest of this paper. In next section, such conditions will be further discussed in correspondence of a specific choice of an admissible diffusion function.

The microscopic interaction rules (7) can be finally used to derive the explicit expression of the collisional operator, which is the integral operator that can be evaluated as the difference between the gain and the loss of agents whose opinions fall in $(v, v+\mathrm{d}v)$. In order to do so, let us introduce the *transition rate* $W(v,w,v',w')\mathrm{d}v'\mathrm{d}w'$ which represents the probability per unit time that after the interaction among two agents with opinions $v$ and $w$, the opinion of the first agent is in $(v', v'+\mathrm{d}v')$ and the opinion of the second agent is in $(w', w'+\mathrm{d}w')$. Following the derivation in (Toscani, 2000), in which a different microscopic model is considered, the transition rate can be written as

$$W(v,w,v',w')\mathrm{d}v'\mathrm{d}w' = \vartheta(\eta_1)\vartheta(\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2 \tag{12}$$

where, as previously said, $\vartheta(\cdot)$ denotes the distribution function of the random variables $\eta_1$ and $\eta_2$, whose support is $S$. Under all mentioned assumptions, the difference between the gain and the loss terms in the explicit formulation of the weak form of the collisional operator can be written as a unique integral, thus obtaining the following formulation of the right-hand side of the weak form of the Boltzmann equation

$$\begin{aligned} &\int_{S^2}\int_{I^2} \vartheta(\eta_1)\vartheta(\eta_2) f(v,t) f(w,t) \cdot \\ &(\phi(v+\eta_1 D(|v|)) - \phi(v))\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \end{aligned} \tag{13}$$

Given the explicit formulation of the weak form of the Boltzmann equation (13), it is finally possible to analytically study relevant macroscopic properties of the system, as explained at the end of previous section. In the following section we show how the choice of proper test functions in (13) allow the derivation of three first order differential equations whose unknowns are the number of agents, the average opinion, and the variance of the opinion, respectively.

# 4 COLLECTIVE AND ASYMPTOTIC PROPERTIES

In this section we show relevant analytic results concerning collective and asymptotic properties of multi-agents systems in which agents interact following the rules outlined in previous section. In particular, starting from the weak form of the Boltzmann equation (13), we derive proper differential equations whose unknowns are:

1. The number of agents $n(t)$;
2. The average opinion of the system $u(t)$; and
3. The variance of the opinion $\sigma^2(t)$.

In the last part of this section, analytic results in closed form are also derived for a specific diffusion function, which is then used to run independent simulations.

**Proposition 1.** *Given a multi-agent system where agents interact according to (7), the chosen diffusion function $D(\cdot)$ is admissible, and diffusion parameters are admissible for $D(\cdot)$, the number of agents $n(t)$ in the multi-agent system does not depend on time.*

*Proof.* Let us consider the test function

$$\phi(v) = 1 \tag{14}$$

in (13). Since the test function (14) is a constant, the difference

$$\phi(v+\eta_1 D(|v|)) - \phi(v) \tag{15}$$

inside the integral in (13) equals 0. Hence, the weak form of the Boltzmann equation relative to the chosen test function $\phi(v) = 1$ reduces to the following relevant equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I f(v,t)\mathrm{d}v = 0. \tag{16}$$

The left-hand side of (16) represents the time derivative of the number of agents $n(t)$ and, hence, (16) can be reformulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}n(t) = 0, \tag{17}$$

and it can be finally concluded that the number of agents is constant

$$n(t) = n(0), \tag{18}$$

which proves the proposition. □

Proposition 1 allows dropping the dependance on time for $n(t)$. In the rest of this paper, we denote the number of agents as $n$, thus omitting the dependance on time. We remark that analogous considerations can be derived also when considering the Boltzmann equation in the context of kinetic theory of gases. This property corresponds to mass conservation in gases.

Another, much more interesting, collective and asymptotic property of diffusion is captured by the following proposition.

**Proposition 2.** *Given a multi-agent system where agents interact according to (7), the chosen diffusion function $D(\cdot)$ is admissible, and diffusion parameters are admissible for $D(\cdot)$, the average opinion of the multi-agent system $u(t)$ does not depend on time.*

*Proof.* Let us consider the test function

$$\phi(v) = v \tag{19}$$

in (13), so that the right-hand side of the weak form of the Boltzmann equation can be written as

$$\int_{S^2}\int_{I^2} \vartheta(\eta_1)\vartheta(\eta_2) f(v,t) f(w,t)\cdot \\ \eta_1 D(|v|)\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2. \tag{20}$$

From (3) and recalling that, according to (18), the number of agents is constant, the left-hand side of the weak form of the Boltzmann equation equals

$$n\frac{\mathrm{d}}{\mathrm{d}t}u(t) \tag{21}$$

and it is therefore proportional to the derivative of the average opinion with respect to time. Moreover, the integral in (20) can be written as the product of four terms, as follows

$$\int_S \eta_1\vartheta(\eta_1)\,\mathrm{d}\eta_1 \int_S \vartheta(\eta_2)\,\mathrm{d}\eta_2\cdot \\ \int_I f(v,t)D(|v|)\,\mathrm{d}v \int_I f(w,t)\,\mathrm{d}w, \tag{22}$$

and, since $\vartheta(\cdot)$ is a distribution function, the integral on its support $S$ equals 1. Finally, using (2), (22) can be simplified to

$$\bar{\eta}_1 n \int_I f(v,t)D(|v|)\mathrm{d}v \tag{23}$$

where $\bar{\eta}_1$ is the average value of random variable $\eta_1$. Recalling that the average value of random variable $\eta_1$ is assumed to be equal to 0, from (8), it can be concluded that (20) equals 0, regardless of the choice of the diffusion function $D(\cdot)$. Therefore, the weak form of the Boltzmann equation corresponding to the test function $\phi(v) = v$ can finally be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = 0. \tag{24}$$

Equation (24) implies that the average opinion of the system is constant

$$u(t) = u(0), \tag{25}$$

which proves the proposition. □

As already done for the number of agents, from now on we omit the dependance of the average opinion on time and we simply denote it as $u$. An analogous property is found in the kinetic theory of gases and it corresponds to the conservation of momentum.

Note that if the average value of $\eta_1$ is not 0, which equals to dropping the assumption that diffusion parameters are admissible for the chosen $D(\cdot)$, then the following differential equation for the average opinion can be derived

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \bar{\eta}_1 \int_I f(v,t)D(|v|)\mathrm{d}v \tag{26}$$

and the validity of Proposition 2 would depend on the actual choice of the diffusion function $D(\cdot)$.

Regarding the asymptotic properties of the variance of the opinion, the following proposition holds.

**Proposition 3.** *Given a multi-agent system where agents interact according to (7), the chosen diffusion function $D(\cdot)$ is admissible, and diffusion parameters are admissible for $D(\cdot)$, the variance of the opinion of the multi-agent system $\sigma^2(t)$ is not constant.*

*Proof.* Let us consider the test function

$$\phi(v) = (v-u)^2 \tag{27}$$

in (13), so that the right-hand side of the weak form of the Boltzmann equation can be written as

$$\int_{S^2}\int_{I^2} \vartheta(\eta_1)\vartheta(\eta_2) f(v,t) f(w,t)\cdot \\ \left[\eta_1^2 D^2(|v|) + 2\eta_1 D(|v|)(v-u)\right]\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2$$

where we used the following identity that can be easily verified by simple algebraic manipulations

$$(v+\eta_1 D(|v|)-u)^2-(v-u)^2 = \eta_1^2 D^2(|v|)+2\eta_1 D(|v|)(v-u).$$

Further algebraic manipulations, and the assumption that the average value of $\eta_1$ equals 0, show that the right-hand side of the weak form of the Boltzmann equation can be written as

$$n\sigma_{\eta_1}^2 \int_I f(v,t) D^2(|v|)\,\mathrm{d}v \tag{28}$$

where $\sigma_{\eta_1}^2$ denotes the variance of the random variable $\eta_1$. From the definition of $\sigma^2(t)$, and using previously obtained results on the conservation of the number of agents and of the average opinion, the left-hand side of the weak form of the Boltzmann equation can be shown to be proportional to the derivative of the average opinion with respect to time. Actually, it can be written as

$$n\frac{\mathrm{d}}{\mathrm{d}t}\sigma^2(t). \tag{29}$$

Therefore, the weak form of the Boltzmann equation relative to the test function $\phi(v)=(v-u)^2$ can be finally written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^2(t) = \sigma_{\eta_1}^2 \int_I f(v,t) D^2(|v|)\,\mathrm{d}v. \tag{30}$$

Observe that $f(v,t)$ is a density function, and therefore, by definition, it is non-negative. Moreover, $f(v,t)$ cannot be identically 0, since its integral on $I$ equals the number of agent $n$. Similarly, function $D^2(\cdot)$ is also nonnegative and it is supposed not to be identically 0. According to these considerations, the integral in (30) is necessarily strictly positive and it depends on the choice of the diffusion function $D(\cdot)$. Hence, it can be concluded that the variance of opinion $\sigma^2(t)$ is not constant. □

Note that no equivalent form of Proposition 3 is found in the kinetic theory of gases because it follows from the specific assumptions that we took to model the sociological phenomenon of diffusion.

In order to explicitly solve the differential equation (30), and possibly validate obtained results on the asymptotic behaviour of the opinion, a given admissible diffusion function needs to be fixed. We consider the following diffusion function, which is also used in next section

$$D(|v|) = \sqrt{1-v^2}. \tag{31}$$

Observe that the chosen $D(\cdot)$ is admissible because it satisfies the requirements outlined in Section 3. As a matter of fact, it is an even function and it is a decreasing function of $|v|$. Moreover, it also satisfies condition (9). Using this diffusion function, diffusion parameters $\eta_1$ and $\eta_2$ are admissible if the following conditions, derived from (11), are satisfied

$$\begin{aligned} |\eta_1| &\le \frac{1-|v|}{\sqrt{1-v^2}} \\ |\eta_2| &\le \frac{1-|w|}{\sqrt{1-w^2}}. \end{aligned} \tag{32}$$

The rest of this section is dedicated to the proof of the following proposition.

**Proposition 4.** *Given a multi-agent system where agents interact according to (7) with diffusion function (31), and assuming that diffusion parameters are admissible for $D(\cdot)$, the variance of the opinion $\sigma^2(t)$ of the multi-agent system exponentially tends to $1-u^2$ as $t$ tends to $+\infty$.*

*Proof.* Let us start from the differential equation (30) relative to the variance of opinion $\sigma^2(t)$. Considering diffusion function (31), equation (30) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^2(t) = \sigma_{\eta_1}^2 \int_I f(v,t)(1-v^2)\,\mathrm{d}v. \tag{33}$$

The integral at the right-hand side of (33) can be written as the difference

$$\int_I f(v,t)\,\mathrm{d}v - \int_I f(v,t) v^2\,\mathrm{d}v. \tag{34}$$

The first integral in (34) equals the number of agents $n$, while the second integral in (34) can be associated to the definition of the variance of the opinion. As a matter of fact, from the definition of $\sigma^2(t)$, the following equalities can be easily derived

$$n\sigma^2(t) = \int_I f(v,t)(v-u)^2\,\mathrm{d}v = \int_I f(v,t) v^2\,\mathrm{d}v - 2nu^2 + nu^2$$

where we used the definitions of the number of agents $n$ and of the average opinion $u$. Using these equations in (30), the following differential equation for the variance of the opinion $\sigma^2(t)$ can be derived from the weak form of the Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^2(t) = n\sigma_{\eta_1}^2 (1-u^2-\sigma^2(t)). \tag{35}$$

Observe that (35) is a non-homogeneous first-order differential equation whose solution can be found analytically. The solution of (34) is

$$\sigma^2(t) = C\mathrm{e}^{-n\sigma_{\eta_1}^2 t} + (1-u^2) \tag{36}$$

where $C$ is a constant which has to be set in order to satisfy the initial condition. Denoting as $\sigma^2(0)$ the initial value of the variance of the opinion, constant $C$ has to satisfy the following condition

$$\sigma^2(0) = C + (1 - u^2) \tag{37}$$

and, hence, from (36) it can be concluded that the solution of (34) is

$$\sigma^2(t) = [\sigma^2(0) - (1 - u^2)]\mathrm{e}^{-n\sigma^2_{\eta_1} t} + (1 - u^2). \tag{38}$$

Let us now observe that the coefficient of $t$ in the exponential function is negative, since both the number of agents $n$ and the variance of $\eta_1$ are positive. It can then be concluded that the exponential function in (38) tends to 0 as $t$ tends to $+\infty$. As a consequence, it can be finally concluded that

$$\lim_{t \to +\infty} \sigma^2(t) = (1 - u^2), \tag{39}$$

which proves the proposition. $\square$

## 5 VERIFICATION BY SIMULATION

In this section, we show results of independent simulations meant to validate analytic results derived in previous sections. We consider a system composed of $n = 10^3$ agents and, as stated in the introduction, we assume that each agent can interact with any other agent in the system. At each step of the simulation, two agents are randomly chosen and an interaction among them is simulated. This means that both of them change their opinions according to (7) with the chosen diffusion function $D(\cdot)$ in (31). We consider three different scenarios, corresponding to different initial distributions of the opinion and, hence, to different values of the average opinion $u$. In all considered scenarios, $1.5 \cdot 10^5$ binary interactions among randomly chosen agents are simulated, which equals to 300 interactions per agent on average.

We start by considering a multi-agent system where the initial distribution of the opinion is uniform on the entire interval $I$ where the opinion is defined

$$f(v, 0) = \mathcal{U}_{(-1,1)}(v). \tag{40}$$

According to this assumption, the average opinion of the system at time $t = 0$ is $u = 0$ and, as shown in the proof of Proposition 2, it remains constant as agents interact. Under these assumption, from Proposition 4 it is expected that the variance of the opinion tends to 1 as $t$ tends to $+\infty$. Figure 1 shows the values of $\sigma^2(t)$ (solid blue line) for values of $t$ between 0 and $1.5 \cdot 10^5$ and, as expected, the variance tends to 1 as $t$ increases. Note that plotted line follows very closely the expected exponential increase. Let us now consider a multi-agent system where the initial distribution of the opinion is

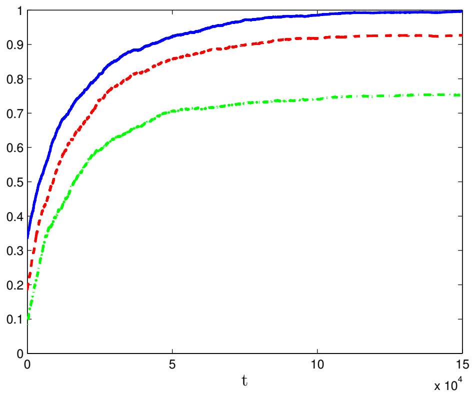


Figure 1: The variance $\sigma^2(t)$ of the opinion as a function of the number of interactions $t$ when $f(v,0) = \mathcal{U}_{(-1,1)}(v)$ (solid blue line), $f(v,0) = \mathcal{U}_{(-\frac{1}{2},1)}(v)$ (dashed red line), and $f(v,0) = \mathcal{U}_{(-1,0)}(v)$ (dash-dotted green line).

$$f(v, 0) = \mathcal{U}_{(-\frac{1}{2},1)}(v) \tag{41}$$

so that the value of the average opinion is equal to $u = 1/4$. In this case, from Proposition 4, it is expected that the variance of the opinion tends to $15/16$ as $t$ tends to $+\infty$. This result is confirmed in Figure 1, where the values of $\sigma^2(t)$ are shown as a function of $t$ (dashed red line). As the number of interactions increases, the variance tends to $15/16 = 0.9375$, as expected. Also in this case, the plot approximates well the expected exponential increase.

Finally, we now assume that the initial distribution of the opinion in the considered multi-agent system is

$$f(v, 0) = \mathcal{U}_{(-1,0)}(v). \tag{42}$$

In this case, the value of the average opinion equals $u = -1/2$ and, hence, it is expected that the variance of the opinion tends to $3/4$ as $t$ tends to $+\infty$. Figure 1 shows that, as expected, the value of the variance tends to $3/4 = 0.75$ (dash-dotted green line), and the plot follows the expected exponential function.

## 6 CONCLUSIONS

This paper presented an analytic model of the social phenomenon of diffusion, which is normally used as one of the ingredients to study the dynamics of the opinion in multi-agent systems. First, the adopted kinetic framework for the study of opinion dynamics was recalled. Then, the framework was completed with the details needed to study diffusion by introducing specific interaction rules. Adopted rules model diffusion in terms of a diffusion function and of two stochastic diffusion parameters. Under the assumption that the diffusion function is admissible and that diffusion parameters are admissible for the diffusion function, interesting asymptotic properties of the multi-agent system were proved. In detail, the paper showed proofs of the fact that diffusion does not change the average opinion of the multi-agent system, but that it influences its variance. In the last part of the paper, a specific diffusion function is considered, and expected properties of the average opinion and of the variance of the opinion for the chosen diffusion function were verified by independent simulations. Note that presented simulations do not depend on the adopted kinetic approach. They are simple implementations of the studied interaction rules.

The work reported in this paper can be extended by considering multi-agent systems made of agents with different propensity to change opinion because of interactions. In (Bergenti and Monica, 2017), we have already investigated the possibility of having different classes of agents in the same multi-agent system, where different classes are associated with different parameters, such as different number of agents and different values of the parameters of compromise. Similar considerations are planned as future work for the study of collective and asymptotic properties of diffusion in multi-agent systems with multiple classes of agents. Finally, the model of diffusion studied in this paper could be coupled with similar models of other sociological phenomena, such as compromise and homophily, to study analytically the collective and asymptotic properties of more complex systems. We have already studied the combined effects of compromise and diffusion under specific assumptions in (Monica and Bergenti, 2015b), and we plan to extend such results by modelling other phenomena.

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